Large Deviations of Cover Times by Random Walk on Tori

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Setup and Previous Results

Setup Consider **cover time** C_N of simple random walk $(X_n)_{n\geq 0}$ on discrete torus $\mathbb{T}_N = (\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 3$.

Previous results

- $d = 2$. For $\gamma \in (0,1)$, $\mathsf{P}(C_N \leq \gamma \cdot \frac{4}{\pi} N^2 \log^2 N) = \exp(-N^{2(1-\sqrt{\gamma})+o(1)})$. (Comets–Gallesco–Popov–Vachkovskaia '13).
- $d \geq 3$. For $\gamma \in (0,1)$, $\mathsf{P}(C_N \leq \gamma \cdot g(0) N^d \log N^d) \leq \exp(-N^{d(1-\gamma)+o(1)})$. A straightforward adaptation of the LDP for the ε-cover time of Brownian motion (Goodman–den Hollander '14) under the discrete setup.
- General Markov chain. *Linear cover time is exponentially unlikely* (Benjamini–Gurel-Gurevich–Morris '13).

Remark Upward deviation is polynomially unlikely. This is an almost trivial consequence of moment bounds.

Main Results: Downward Deviation for $d \geq 3$

- For all $\gamma \in (\frac{d+2}{2d}, 1)$, sharp asymptotics:

$$\mathsf{P}\big(C_N \leq \gamma \cdot g(0) N^d \log N^d\big) = \exp\big(-(\mathbf{1}+\mathbf{o}(\mathbf{1}))\mathbf{N}^{\mathbf{d}(\mathbf{1}-\gamma)}\big). \tag{1}$$

- For all $\gamma \in (0,1)$, correct order of lower bound that matches the upper bound from [GdH14]:

$$\mathsf{P}\big(C_N \leq \gamma \cdot g(0) N^d \log N^d\big) \geq \exp(-A \cdot N^{d(1-\gamma)}) \text{ for some } A > 0. \tag{2}$$

Proof Sketches

Upper bound

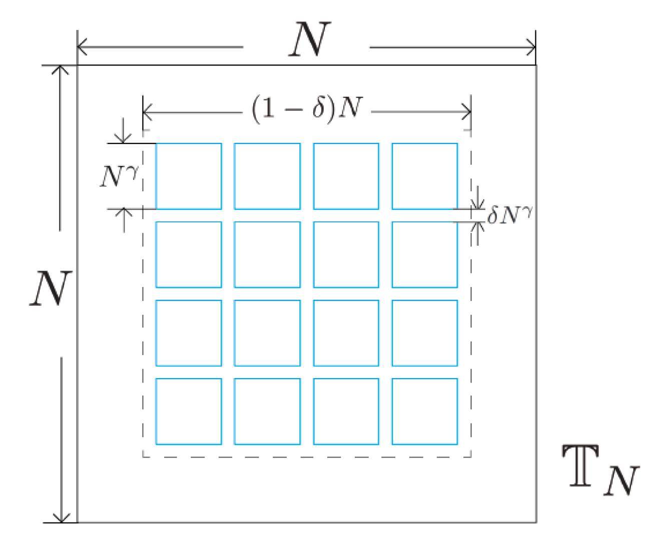
- Localization: Dominate the trace of random walk by disjoint boxes of size N^γ by **independent** random interlacements using a coupling from [PRS23] (see below).
- Cover level of random interlacements: The probability that a box of size N^γ is covered by random interlacements $\mathcal{I}^u$ at level $u \approx \gamma g(0) \log N^d$ converges to e^{-1} as $N \to \infty$ (adapted from Belius '12).

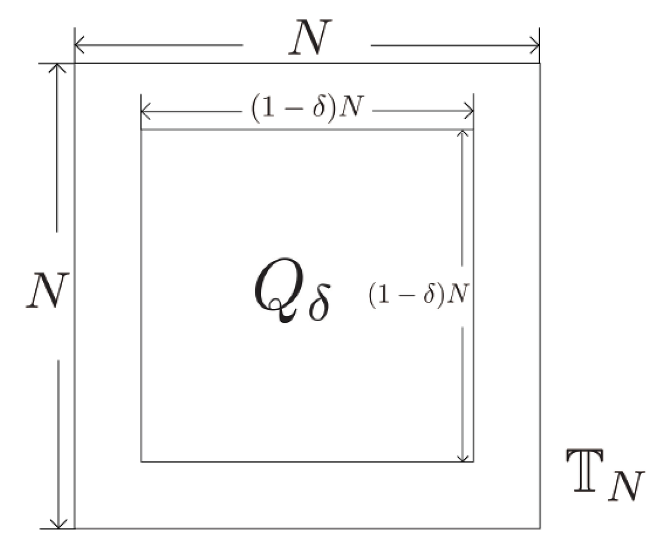
Lower bound

Let the random walk evolve freely until KN^d steps left for some large K and apply different strategies for late points inside and outside Q_δ respectively (see figure to the right).

- Macroscopic coupling (only for (1)): In Q_δ, couple the trace of random walk with random interlacements (using [PRS23]) and apply **FKG** for Poisson point processes.
- Covering via loops: For late points outside Q_δ, insert **meticulously designed loops** into the trace of random walk (applied on the whole $\mathbb{T}_N$ instead for (2)).

Illustrations





A Slightly Improved Coupling from Prévost–Rodriguez–Sousi '23

For $R \asymp N^\eta$ with $\eta \in (1/2, 1]$, write $Q(x,R)$ for the box of side length R around x. Let $(Q(x,R))_{x\in F}$ be a collection of δR-separated boxes contained in Q_δ. There exists a coupling of $(X_n)_{n\geq 0}$ and independent copies of random interlacements $(\mathcal{I}^{(x),u(1\pm\varepsilon)})_{x\in F}$ such that

$$\mathcal{I}^{(x),u(1-\varepsilon)} \cap Q(x,R) \subset X[0, uN^d] \cap Q(x,R) \subset \mathcal{I}^{(x),u(1+\varepsilon)} \cap Q(x,R) \quad \text{simultaneously for all } x \in F,$$

with probability exceeding $1 - CuN^{3d}\exp(-c\varepsilon^2\sqrt{uN^{d-2}})$, where c and C are constants depending only on δ and d.

References

[Bel12] D. Belius. Cover levels and random interlacements. *AAP*, 2012.
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[CGPV13] F. Comets, C. Gallesco, S. Popov, and M. Vachkovskaia. On large deviations for the cover time of two-dimensional torus. *EJP*, 2013.
[GdH14] J. Goodman and F. den Hollander. Extremal geometry of a brownian porous medium. *PTRF*, 2014.
[PRS23] A. Prévost, P.-F. Rodriguez, and P. Sousi. Phase transition for the late points of random walk. *arXiv:2309.03192*, 2023.