Favorite Sites for Random Walk in Two and More Dimensions

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1: Favorite (Most Visited) Sites

- $(S_n)_{n\in\mathbb{N}}$: d-dimensional discrete-time simple random walk. Write
$$\xi(x,n)=\sum_{k=0}^{n}1_{\{S_k=x\}}\quad\text{and}\quad \xi^*(n):=\max_x \xi(x,n)$$
for the local time and maximal local time at time n.
- The set of favorite site(s) at time n:
$$\mathcal{K}^{(d)}(n):=\{x\in\mathbb{Z}^d:\xi(x,n)=\xi^*(n)\}.$$
- First time k sites of local time m:
$$T_m^k:=\inf\big\{n>T_m^{k-1}:\#\{x\in\mathbb{Z}^d:\xi(x,n)\geq m\}=k\big\}.$$
- Location of "favorite sites": $L_m^k:=S_{T_m^k}$.
- Event of k favorite sites of local time m:
$$M_m^k:=\{T_m^k<T_{m+1}^1\}=\big\{L_m^1,\dots,L_m^j\notin S_{(T_m^j,T_m^{j+1})}\text{ for } j=1,\dots,k-1\big\}.$$

3: Main Results

We completely solve the open question of Erdős and Révész:

Theorem. *For $d=2$, with probability 1,*
$$\limsup_{n\to\infty}\#\mathcal{K}^{(2)}(n)=3.$$

We derive **sharp** asymptotics of Erdős-Révész (1991).

Theorem. *For $d\geq 3$, writing $\gamma_d:=P\big(0\notin S_{(0,\infty)}\big)$,*
$$\limsup_{n\to\infty}\frac{\#\mathcal{K}^{(d)}(n)}{\log\log n}=-\frac{1}{\log\gamma_d}\quad a.s.$$

3.5: Improved Bound on 2D Max Local Time

An improvement of Dembo-Peres-Rosen-Zeitouni (2001) by refining arguments from Rosen (2005):

Proposition. *$(d=2)$ For any $\beta>\frac{3}{5}$, w.h.p.,*
$$\xi^*(n)\geq\tfrac{1}{\pi}(\log n)^2-(\log n)^{1+\beta}.$$

Remark. Conjectural precise asymptotics:
$$\sqrt{\xi^*(n)}-\tfrac{1}{\sqrt{\pi}}\log n+\tfrac{1}{\sqrt{\pi}}\log\log n\xrightarrow{\text{law}}\text{Gumbel*}.$$

5: Lower Bound for $d=2$

Large deviation bound on the stopping time T_m^1:
$$\xi^*(n)\sim\frac{(\log n)^2}{\pi}\Rightarrow T_m^1\sim e^{\pi^{1/2}m^{1/2}}\overset{\text{Proposition}}{\Longrightarrow}T_m^1\leq e^{2m^{1/2}}\text{ w.h.p.}$$
Same (if not better) bound applies to $T_m^2-T_m^1$ and $T_m^3-T_m^2$. The escape probability bound $P\big[0\notin S_{[1,e^{2m^{1/2}}]}\big]\gtrsim m^{-1/2}$ heuristically implies
$$P\big(L_m^1,L_m^2\notin S(T_m^2,T_m^3]\,\big|\,\mathcal{F}_{T_m^2}\big)\cdot P\big(L_m^1\notin S(T_m^1,T_m^2]\,\big|\,\mathcal{F}_{T_m^1}\big)\gtrsim m^{-1}.$$
Use generalized Borel-Cantelli to conclude.

6: The $d\geq 3$ Case

- By Csáki-Földes-Révész-Shi (2005), it is much harder to produce a pair of nearby thick points than to produce an isolated one $\Longrightarrow$
$$P\Big(L_m^1,\dots,L_m^j\notin S_{(T_m^j,T_m^{j+1})}\Big)\approx P\Big(L_m^j\notin S_{(T_m^j,T_m^{j+1})}\Big)\approx P\big(0\notin S_{(0,\infty)}\big)=\gamma_d.$$
$$\Rightarrow P\Big(M_m^k\Big)\approx\gamma_d^k\approx m^{-1}\text{ for }k=\frac{\log m}{-\log\gamma_d}.$$
- Again use (generalized) Borel-Cantelli to conclude.

2: Number of Favorite Sites

Easy to deduce for every $d\geq 1$,
$$P(\#\mathcal{K}^{(d)}(n)=2\ \ i.o.)=P(\#\mathcal{K}^{(d)}(n)=1\ \ i.o.)=1.$$
Erdős-Révész (1984):

Can $\#\mathcal{K}^{(d)}(n)=r$ occur infinitely often for $r\geq 3$?

For $d=1$:

- Tóth (2001): a.s., $\limsup_{n\to\infty}\#\mathcal{K}^{(1)}(n)\leq 3$.
- Ding-Shen (2018): a.s., $\limsup_{n\to\infty}\#\mathcal{K}^{(1)}(n)=3$.

For $d\geq 3$:

- Erdős-Révész (1991): a.s., $\limsup_{n\to\infty}\#\mathcal{K}^{(d)}(n)=\infty$.

4: Upper Bound for $d=2$

Claim. Probability of creating one more favorite site with local time m is $O(m^{-\kappa})$ for some $\kappa>1/3$. [We conjecture the optimal exponent is $1/2$.]

- Analysis for 2nd favorite site (same for 3rd, 4th) using escape probability control:
  - For $|L_m^2-L_m^1|$ large, $P\big(M_m^2,|L_m^2-L_m^1|\geq\exp(m^\kappa)\big)\lesssim m^{-\kappa}$.
  - For $|L_m^2-L_m^1|$ small, on $\star:=\big\{M_m^2,|L_m^2-L_m^1|\approx\exp(m^\alpha)\big\}$ for some $\alpha\in(0,\kappa)$, $\triangle:=\{\xi(L_m^2,T_m^1)>m-m^{\alpha+\varepsilon}$ occurs w.h.p.$\}$
    - ⋄ Let $M(m,\alpha):=\big\{x:\xi(x,T_m^1)\in(m-m^\alpha,m)\big\}$ be the set of "near-favorite" sites at time T_m^1.
    - ⋄ By **Lemma** and escape probability control, for all $\alpha\in(0,\kappa)$,
$$P(\star)\approx P(\star,\triangle)\lesssim P(M(m,\alpha+\varepsilon\neq\emptyset))\cdot m^{-\alpha}\lesssim m^{-\kappa+2\varepsilon}.$$

Use **Claim** and Borel-Cantelli to conclude.

Lemma. *For $\kappa\in(\frac{1}{3},\frac{7}{20})$ and all $\alpha\in(0,\kappa)$,*
$$P\big(M(m,\alpha)\neq\emptyset\big)\lesssim\frac{(\log m)^2}{m^{\kappa-\alpha}}.$$

Key ideas to prove **Lemma**:

1. Decomposition of local time into lazy (two-step excursions) and non-lazy parts. The law of the former is "almost" locally determined by the latter.
2. Entropic repulsion by analogy with some urn models.
3. A variant of **Proposition** dictates w.h.p. regular behavior of non-lazy local time of near-favorite sites.

7: References

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