Exercises for random matrices (30 points)

Please solve **ONE** of the following three exercises, and send the solution (scanned PDF or photo) to yukun.he@math.uzh.ch by 17:00, Jan 24, 2020. Please also write your name (in Chinese) and student ID in the email.

1 Cumulant expansion

If h is a real-valued random variable with $\mathbb{E}\mathrm{e}^h < \infty$, we denote by $\mathcal{C}_k(h)$ the kth cumulant of h, i.e.

$$\mathcal{C}_k(h) := \big(\partial_\lambda^k \log \mathbb{E}\mathrm{e}^{\lambda h}\big)\big|_{\lambda=0}\,.$$

Let $f : \mathbb{R} \to \mathbb{C}$ be a smooth function, and denote by $f^{(k)}$ its kth derivative. **Show that**, for every fixed $\ell \in \mathbb{N}$, we have

$$\mathbb{E}\big[h \cdot f(h)\big] = \sum_{k=0}^{\ell} \frac{1}{k!}\mathcal{C}_{k+1}(h)\mathbb{E}[f^{(k)}(h)] + \mathcal{R}_{\ell+1}, \tag{1.1}$$

assuming that all expectations in (1.1) exist, where $\mathcal{R}_{\ell+1}$ is a remainder term (depending on f and h), such that for any $t > 0$,

$$\mathcal{R}_{\ell+1} = O(1) \cdot \bigg(\mathbb{E} \sup_{|x|\leqslant|h|} \big|f^{(\ell+1)}(x)\big|^2 \cdot \mathbb{E}\,\big|h^{2\ell+4}\mathbf{1}_{|h|>t}\big|\bigg)^{1/2} + O(1) \cdot \sup_{|x|\leqslant t} \big|f^{(\ell+1)}(x)\big| \cdot \mathbb{E}|h|^{\ell+2}\,.$$

2 Fluctuating averaging

Let $H = H^T \in \mathbb{R}^{N\times N}$ be a Gaussian Orthogonal Ensemble, i.e. the upper triangular entries $(H_{ij} : 1 \leqslant i \leqslant j \leqslant N)$ are independent, and

$$\sqrt{N}H_{ij} \overset{d}{=} \mathcal{N}(0, 1 + \delta_{ij})\,.$$

For fixed $\tau > 0$, we define the spectral domain

$$\mathbf{S} := \{z = E + \mathrm{i}\eta : E \in \mathbb{R}, \eta \geqslant N^{-1+\tau}\}\,.$$

Let $G(z) := (H - z)^{-1}$, $\underline{G} := N^{-1}\operatorname{Tr} G$, and

$$m(z) := \int \frac{\rho(x)}{x - z}\mathrm{d}x\,,$$

where $\rho(x) = \frac{1}{2\pi}\sqrt{(4 - x^2)_+}$. Let $z \in \mathbf{S}$, suppose that we have

$$\max_{i,j} |G_{ij}(z) - m(z)\delta_{ij}| \prec \phi\,.$$

Show that

$$\mathbb{E}|1 + z\underline{G} + \underline{G}^2|^2 \prec (1+\phi)^{12}\Big(\frac{\operatorname{Im} m + \phi}{N\eta}\Big)^2$$

at z.

3 The last resort

Let us adopt the notations in Question 2. Suppose we have

$$\sup_{z\in\mathbf{S}} N\eta|\underline{G}(z) - m(z)| \prec 1\,.$$

Note that this implies

$$\sup_{z\in\mathbf{S}} N\eta|\underline{G}(z) - \mathbb{E}\underline{G}(z)| \prec 1\,.$$

Show that, for any fixed $n \in \mathbb{N}_+$, we have

$$\underline{G^n} \prec \eta^{1-n}$$

as well as

$$\underline{G^n} - \mathbb{E}\underline{G^n} \prec \frac{1}{N\eta^n}$$

uniformly for all $z = E + \mathrm{i}\eta \in \mathbf{S}$. (Be careful that $\underline{G^n} = N^{-1}\operatorname{Tr}(G^n)$.)

Hint: the Helffer-Sjöstrand formula holds for $f : \mathbb{R} \to \mathbb{C}$ smooth satisfying $|f^{(k)}(x)| = O(1 + |x|^{-2})$ whenever $k \in \mathbb{N}$.