Local semicircle law for Wigner matrices

Time & Content: 6hrs, We will prove the local semicircle law for Wigner matrices, and give some applications on the distribution of eigenvalues and eigenvectors.

Preliminaries: Elementary analysis, linear algebra, basic probability theory, complex variables.

Reference: lecture notes by Florent Benaych-Georges and Antti Knowles. arxiv.org/abs/1601.04055.
We take applications and explainations from the notes, but the proof will be different.

1. Introduction.

Let $H = H^* = (H_{ij} : 1 \le i,j \le N) \in \mathbb{C}^{N\times N}$ be an $N\times N$ random Hermitian matrix. (Hermitian: eigenvalues are real).

We normalize H so that its eigenvalues are typically of order 1, $\|H\| \asymp 1$.

Central Goal: Understand the distribution of eigenvalues $\lambda_1 \le \cdots \le \lambda_N$ and corresponding eigenvectors.

N: Fundamental large parameter, we are interested in the case when N is very large. Most quantities we study shall depend on N, thus we almost always omit the explicit argument N from our notation. Every quantity that is not explicitly a constant (usually denote by C, c) is infact a sequence indexed by N. ⌜ $A = O(B)$ means $|A| \leq CB$ for some constant $C > 0$. ⌟

1.1 Global and Local law.

We define the empirical eigenvalue distribution

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i} .$$

For many matrix models H, one has $\mu \xrightarrow{w} \varrho$ in probability as $N \to +\infty$, where ϱ is a deterministic probability measure that does not depend on N.

⌜ The convergence in probability is wrt the randomness of μ, and "$\xrightarrow{w}$" denotes the weak convergence of probability measures. $\mu \xrightarrow{w} \varrho$ in probability : $\forall f \in C_b(\mathbb{R})$, $\varepsilon > 0$, we have

$$\mathbb{P}\left(\left|\int f \, d\mu - \int f \, d\varrho\right| > \varepsilon\right) \to 0 \quad \text{as} \quad N \to +\infty . \;⌟$$

The convergence is best formulated in terms of Stieltjes transforms.

For the rest of the course we always write $z = E + i\eta$, and assume $\eta > 0$.

The Stieltjes transform of μ, ρ are given by

$$s(z) := \int \frac{\mu(dx)}{x-z} , \qquad m(z) := \int \frac{\rho(dx)}{x-z} .$$

They are well-defined since $\operatorname{Im} z = \eta \neq 0$.

Lemma 1. Let $\mu \equiv \mu_N$ be a probability measure, ρ be a deterministic probability measure. Then

$$\mu \xrightarrow{w} \rho \text{ in probability} \iff$$

$$s(z) \to m(z) \text{ in probability for any } \underset{\substack{\uparrow \\ N\text{-independent}}}{\text{fixed}}\ z \in \mathbb{C}_+ .$$

$$s(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - z} = \frac{1}{N} \sum_{i=1}^{N} \frac{\lambda_i - E + i\eta}{(\lambda_i - E)^2 + \eta^2}$$

$$\operatorname{Im} s(z) = \frac{1}{N\eta} \sum_{i=1}^{N} \frac{1}{\left(\frac{\lambda_i - E}{\eta}\right)^2 + 1}$$

$\Rightarrow$ $\operatorname{Im} s(z)$ is a control of the empirical distribution of μ smoothed on the scale η. η: spectral resolution.

In lemma 1, z is N-independent, $s(z) \to m(z)$ has a spectral resolution of order 1. A result of the form

$$s(z) \to m(z) \quad \text{for all fixed } z \in \mathbb{C}_+$$

is therefore called the global law.

A local law is a result that controls $s(z) - m(z)$ for all η satisfying $\eta >> N^{-1}$, i.e. η depends on N.

- The restriction $\eta >> N^{-1}$ is obvious, since the typical size of the eigenvalues is order 1, the typical separation of eigenvalues is of order N^{-1}.

1.2 The model and some notations

Definition 2. (Wigner matrix).

A Wigner matrix is a Hermitian $N \times N$ matrix whose entries H_{ij} satisfy

(i) The upper triangular entries $(H_{ij}, i \le j)$ are independent

(ii) For all i, j we have $\mathbb{E} H_{ij} = 0$, $\mathbb{E}|H_{ij}|^2 = N^{-1}(1 + O(\delta_{ij}))$

(iii) For any fixed $p \in \mathbb{N}$, $\exists$ $C_p > 0$, s.t.

$$\mathbb{E}|H_{ij}|^p \le C_p N^{-p/2} .$$

$p \geq 2$

$$\|H_{ij}\|_2 \underset{\substack{\uparrow \\ \text{Jensen}}}{\le} \|H_{ij}\|_p \le (C_p)^{\frac{1}{p}} N^{-1/2} \le \tilde{C}_p \|H_{ij}\|_2$$

(H_{ij}) are light-tailed.

By normalization, typical size of the eigenvalues is of order 1.

$$\frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\lambda_i^2 = \frac{1}{N}\mathbb{E}\operatorname{Tr} H^2 = \frac{1}{N}\sum_{i,j=1}^{N} \mathbb{E} H_{ij} H_{ji} = \frac{1}{N}\sum_{i,j=1}^{N} \mathbb{E}|H_{ij}|^2 = 1 + O(N^{-1}).$$

We define the semicircle distribution

$$\varrho(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}\,dx,$$

its Stieltjes transform is given by

$$m(z) = \int \frac{\varrho(dx)}{x-z} = \frac{-z+\sqrt{z^2-4}}{2}.$$

Here we choose the branch-cut of the complex square root at the positive real axis, namely

$$\operatorname{Im}\sqrt{z} > 0 \quad \text{for all } z \in \mathbb{C}\setminus\mathbb{R}$$

We often use that $m(z)$ is the unique solution of

$$1 + zm + m^2 = 0$$

satisfying $\operatorname{Im} m(z) > 0$.

[Thm 3 + lem 1 + ϱ is cont $\Rightarrow$ density of states: $\mu[a,b] \to \varrho[a,b]$ in probability, $\forall$ fixed $[a,b]$]

Theorem 3. For a Wigner matrix H, we have

$$s(z) \to m(z)$$

in probability as $N \to +\infty$, for any fixed $z \in \mathbb{C}_+$.

In order to state the local law, we use the following notion of high-probability bounds.

Definition 4 (Stochastic Domination). let

$$X = \left(X^{(N)}(u) : N \in \mathbb{N},\, u \in U^{(N)}\right), \quad Y = \left(Y^{(N)}(u) : N \in \mathbb{N},\, u \in U^{(N)}\right)$$

be two families of random variables, where $Y^{(N)}(u)$ are nonnegative and $\mathcal{U}^{(N)}$ is a parameter set. We say X is stocastically dominated by Y, uniformly in $\mathcal{U}$, if for all (small) $\varepsilon > 0$ and (big) $D > 0$ we have

$$\sup_{u \in \mathcal{U}^{(N)}} P\left[|X^{(N)}(u)| \geq N^{\varepsilon} Y^{(N)}(u) \right] \leq N^{-D}$$

for large enough $N \geq N_0(\varepsilon, D)$. We also denote this by $X \prec Y$ or $X = O_{\prec}(Y)$.

⌈ In practice, we prove $\mathbb{E}|X|^p \leq C_p Y^p$ for any fixed p. Given $\varepsilon, D > 0$, choose p large s.t.

$$P[\, |X| \geq N^{\varepsilon} Y \,] \leq C_p N^{-\varepsilon p} \leq N^{-D}$$

⌋

Green Function is defined by

$$G(z) := (H - z)^{-1}, \qquad z \in \mathbb{C} \setminus \mathbb{R}.$$

We denote the normalized trace $\underline{G} := \frac{1}{N} \operatorname{Tr} G$.

Local law is a control of $\underline{G} = \frac{1}{N} \sum_i \frac{1}{\lambda_i - z} = s(z)$ as well as G_{ij}.

1.3 Main results

Theorem 5 Let H be a Wigner matrix. Fix $\tau > 0$ and define

$$S \equiv S_N(\tau) = \{ z = E + i\eta : |E| \leq 3, \ N^{-1+\tau} \leq \eta \leq 3 \}$$

Then we have

$$\left| \underline{G}(z) - m(z) \right| \prec \frac{1}{N\eta}$$

and

$$\left| G_{ij}(z) - m(z)\delta_{ij} \right| \prec \frac{1}{N\eta} + \sqrt{\frac{\operatorname{Im} m(z)}{N\eta}}$$

uniformly for $i,j = 1, 2, \cdots, N$ and $z \in S$.

Corollary 6. (Semicircle law on small scales).

For a Wigner matrix we have

$$\mu(I) = \varrho(I) + O_{\prec}(N^{-1})$$

uniformly for all intervals $I \subset \mathbb{R}$.

Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ be the eigenvalues of H. We define the typical location of λ_i as γ_i satisfying

$$N \int_{-2}^{\gamma_i} \varrho(dx) = i .$$

Corollary 7. We have

$$|\lambda_i - \gamma_i| \prec N^{-\frac{2}{3}} \left(i \wedge (N+1-i) \right)^{-\frac{1}{3}}$$

uniformly for $i = 1, 2, \cdots, N$.

$\gamma_k - \gamma_{k+1} \sim N^{-\frac{2}{3}}$ for fixed k

$\gamma_i - \gamma_{i+1} \sim N^{-1}$ for $i \in [\varepsilon N, (1-\varepsilon) N]$.

Corollary 8. Let $\vec{u}_1, \cdots, \vec{u}_N$ be the eigenvectors of H associating with $\lambda_1 \le \cdots \le \lambda_N$. We assume the eigenvectors are normalized, i.e. $\|\vec{u}_i\|_2 = 1$. Then

$$|u_i(k)|^2 \prec \frac{1}{N}$$

uniformly for $i, k = 1, 2, \cdots, N$.

pf: H is Hermitian, thus normal ($HH^* = H^*H$)

H can be decomposed as $\quad H = U \Lambda U^* = \sum_{i=1}^{N} \lambda_i \vec{u}_i \vec{u}_i^*$

$$G = (H - z)^{-1} = U(\Lambda - z)^{-1} U^* = \sum_{i=1}^{N} \frac{\vec{u}_i \vec{u}_i^*}{\lambda_i - z}$$

$$\operatorname{Im} G_{kk}(\lambda_j + i\eta) = \operatorname{Im} \sum_{i=1}^{N} \frac{|u_i(k)|^2}{\lambda_i - z} = \sum_{i=1}^{N} \frac{\eta}{(\lambda_i - \lambda_j)^2 + \eta^2} |u_i(k)|^2 \ge \frac{1}{\eta} |u_j(k)|^2$$

$$\operatorname{Im} G_{kk} \le |G_{kk}| \le |m| + |G_{kk} - m| \le 1 + O_\prec\left(\frac{1}{N\eta} + \sqrt{\frac{\operatorname{Im} m}{N\eta}}\right) \prec 1$$

Thus $\quad |u_j(k)|^2 \prec \eta = N^{-1+\tau}$ for any fixed $\tau > 0$.

Strategy of proving Theorem 5.

i) For any fixed $n \in \mathbb{N}_+$, compute

$$\mathbb{E}\,|1 + z\underline{G} + \underline{G}^2|^n \quad \text{and} \quad \mathbb{E}\,|1 + zG_{ii} + \underline{G} \cdot G_{ii}|^n.$$

and show that

$$1 + z\underline{G} + \underline{G}^2 \prec \mathcal{E}_1(\phi), \qquad \delta_{ij} + zG_{ij} + \underline{G} \cdot G_{ij} \prec \mathcal{E}_2(\phi)$$

whenever $\max_{i,j} |G_{ij}| \prec \phi, \quad \phi \in [N^{-1}, N^{\varepsilon}]$

ii) For $\eta \geq 1$,

$$|G_{ij}| \underset{(*)}{\leq} \|G\| = \max_i |\lambda_i - E - i\eta|^{-1} \leq \eta = 1$$

$$(*) : \quad \|G\| = \sup_{x \in \mathbb{C}^n} \frac{|Gx|_2}{|x|_2} \underset{\substack{x = e_i \\ = (0,\cdots,0,1,0,\cdots,0)}}{\geq} |(G_{i1}, \cdots, G_{in})|_2 \geq |G_{ij}|$$

$\|G\|$ = largese singular value of G.

Thus $1 + z\underline{G} + \underline{G}^2 \prec \mathcal{E}_1(1)$, $\delta_{ij} + zG_{ij} + \underline{G} \cdot G_{ij} \prec \mathcal{E}_2(1)$

$\Rightarrow$ $\underline{G} - m$ = (error term), $G_{ij} - m\delta_{ij}$ = (error term)

i.e. local law at $\eta \geq 1$.

iii) (Boot strapping). Suppose we have local law at η_0

$G_{ij} \prec |m\delta_{ij}| + |G_{ij} - m\delta_{ij}| \prec 1$.

$\Gamma(z) := \max_{ij} |G_{ij}(z)|$

Lemma 9 : For $M > 1$, $z \in \mathbb{C}_+$, $\Gamma(E + i\eta/M) \leq M\Gamma(E + i\eta)$.

Thus for all $\eta_1 \in [\eta_0 \cdot N^{-\varepsilon}, \eta_0)$,

$$\Gamma(E + i\eta_1) \leq \frac{\eta_0}{\eta_1} \Gamma(E + i\eta_0) \leq N^{\varepsilon} \Gamma(E + i\eta_0) \prec N^{\varepsilon}$$

Thus at η_1,

$$1 + z\underline{G} + \underline{G}^2 \prec \mathcal{E}_1(N^{\varepsilon}), \quad \delta_{ij} + zG_{ij} + \underline{G} \cdot G_{ij} \prec \mathcal{E}_2(N^{\varepsilon})$$

$\Rightarrow$ local law at η_1.

2. Proof of Theorem 5.

Let us focus on the case H is real & symmetric, i.e. $H = H^T$, and $H_{ij} \in \mathbb{R}$ $\forall i,j$. The complex Hermitian case works in a similar fashion.

The main step is the following result.

Proposition 10. We define

$$\Pi(G) := 1 + zG + \underline{G}\cdot G \in \mathbb{C}^{N\times N}.$$

Let $z \in S(\tau)$ and suppose that $\max_{i,j} |G_{ij} - m\delta_{ij}| \prec \phi$ for some deterministic $\phi \in [N^{-1}, N^{\tau/10}]$ hold at z. Then the following estimates hold at z.

(i) $\sup_{i,j} |[\Pi(G)]_{ij}| \prec (1+\phi)^3 \sqrt{\frac{\operatorname{Im} m + \phi}{N\eta}}$;

(ii) $\underline{\Pi(G)} \prec (1+\phi)^6 \frac{\operatorname{Im} m + \phi}{N\eta}$.

2.1 Cumulants and cumulant expansion.

Let X be a real random variable s.t. $\mathbb{E}|X|^n < +\infty$ $\forall n \in \mathbb{N}$. The nth moment of X is defined as

$$M_n(X) := \mathbb{E}X^n.$$

Now the nth cumulant of X is defined as

$$C_n(X) := [\partial_t^n \log \mathbb{E} e^{itX}]_{t=0} \cdot (-i)^n.$$

One easily checks that $C_1(X) = \mathbb{E}X$, $C_2(X) = \mathrm{var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$ and $C_3(X) = \mathbb{E}(X-\mathbb{E}X)^3$, ...

Properties: ① let a be deterministic number.

$$\log \mathbb{E} e^{it(X+a)} = \log e^{ita}\mathbb{E}e^{itX} = ita + \log \mathbb{E}e^{itX}$$

$$C_n(X+a) = \partial_t^n \log \mathbb{E}e^{it(X+a)}\Big|_{t=0} \cdot (-i)^n = \begin{cases} C_1(X)+a & n=1 \\ C_n(X) & n\geq 2. \end{cases}$$

② let X, Y be independent random variables with $\mathbb{E}|X^n| < +\infty$, $\mathbb{E}|Y|^n < +\infty$. Then

$$\log \mathbb{E} e^{it(X+Y)} = \log(\mathbb{E}e^{itX} \cdot \mathbb{E}e^{itY}) = \log \mathbb{E}e^{itX} + \log \mathbb{E}e^{itY}$$

$$\Rightarrow \quad C_n(X+Y) = C_n(X) + C_n(Y)$$

③ let $X \sim N(\mu, \sigma^2)$

$$\log \mathbb{E} e^{itX} = \log e^{it\mu - \frac{1}{2}\sigma^2 t^2} = it\mu - \frac{1}{2}\sigma^2 t^2$$

$$C_n(X) \equiv 0 \quad \forall\, n \geq 3.$$

Non of ①-③ are true for moments. ④ $C_n(aX) = a^n C_n(X)$.

Lemma 11. (Cumulant expansion).

Let $f: \mathbb{R} \to \mathbb{C}$ be a smooth function. Then for any fixed $l \in \mathbb{N}$, we have

$$\mathbb{E}[h \cdot f(h)] = \sum_{k=0}^{l} \frac{1}{k!} C_{k+1}(h)\, \mathbb{E} f^{(k)}(h) + R_{l+1},$$

assuming that all expectations in above exist. Here R_{l+1} is a remainder term satisfying

$$R_{l+1} = O(1)\cdot\Big(\mathbb{E}\sup_{|x|\le|h|}|f^{(l+1)}(x)|^2\cdot\mathbb{E}|h^{2l+4}\mathbb{1}_{|h|>t}\Big)^{1/2}$$

$$+\,O(1)\cdot\mathbb{E}|h|^{l+2}\cdot\sup_{|x|\le t}|f^{(l+1)}(x)|$$

for any $t>0$.

proof: Let $\mathcal{X}(t)=\log\mathbb{E}e^{ith}$. For $n\ge 1$, we have

$$\partial_t^{n+1}(e^{\mathcal{X}(t)}) = \partial_t^{n}(\mathcal{X}'(t)e^{\mathcal{X}(t)}) = \sum_{k=0}^{n}\binom{n}{k}\partial_t^k(\mathcal{X}'(t))\,\partial_t^{n-k}(e^{\mathcal{X}(t)})$$

Hence, $\mathbb{E}h^{n+1} = (-i)^{n+1}\partial_t^{n+1}e^{\mathcal{X}(t)}\big|_{t=0} = \sum_{k=0}^{n}\binom{n}{k}C_{k+1}(h)\,\mathbb{E}h^{n-k}$

For $f(h)=h^n$, $n\le l$

$$\mathbb{E}[h\cdot f(h)] = \mathbb{E}h^{n+1} = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}C_{k+1}(h)\,\mathbb{E}h^{n-k}$$

$$= \sum_{k=0}^{n}\frac{1}{k!}C_{k+1}(h)\,\mathbb{E}f^{(k)}(h) = \sum_{k=0}^{l}\frac{1}{k!}C_{k+1}(h)\,\mathbb{E}f^{(k)}(h),$$

Thus the formula holds for any polynomial of degree $\le l$. ($R_{l+1}\equiv 0$).

For general smooth function: use Taylor expansion (exercise).

2.2 Proof of Proposition 9 (i).

Lemma 12. (Ward identity).

Let $G=G(z)$. Then

$$\sum_j |G_{ij}|^2 = \frac{\operatorname{Im} G_{ii}}{\eta}.$$

proof: $\sum_j|G_{ij}|^2 = \sum_j G_{ij}\overline{G_{ij}} = \sum_j G_{ij}G^*_{ji} = (GG^*)_{ii}$

$G=(H-z)^{-1}$, $G^*=(H^*-z^*)^{-1} = (H-\bar{z})^{-1}$.

For any $A, B \in \mathbb{C}^{N\times N}$ invertiable,

$$A^{-1}(B-A)B^{-1} = ABB^{-1} - A^{-1}AB^{-1} = A^{-1} - B^{-1}.$$

let $A = H - z$, $B = H - \bar{z}$.

$$G(H-\bar{z}-(H-z))G^* = G - G^*$$

$$(z-\bar{z})GG^* = G - G^* \Rightarrow GG^* = \frac{G-G^*}{2i\eta}$$

$$\sum_j |G_{ij}|^2 = (GG^*)_{ii} = \frac{G_{ii} - G^*_{ii}}{2i\eta} = \frac{G_{ii} - \overline{G_{ii}}}{2i\eta} = \frac{\operatorname{Im} G_{ii}}{\eta}.$$

Now let us begin the proof of Prop 10 (i).

Pick $i, j \in \{1, 2, \cdots, N\}$. Fix $n \in \mathbb{N}$.

$$\mathbb{E}|\pi(G)_{ij}|^{2n} = \mathbb{E}(\delta_{ij} + z G_{ij} + \underline{G}\cdot G_{ij})[\pi(G)_{ij}]^{n-1}[\overline{\pi(G)_{ij}}]^n$$

$$\underset{\substack{\uparrow \\ (H-z)G = I}}{=} \mathbb{E}((HG)_{ij} + \underline{G}\cdot G_{ij})[\pi(G)_{ij}]^{n-1}[\overline{\pi(G)_{ij}}]^n$$

$$= \mathbb{E}\Big(\sum_k H_{ik}G_{kj} + \underline{G}\cdot G_{ij}\Big)[\pi(G)_{ij}]^{n-1}[\overline{\pi(G)_{ij}}]^n.$$

$$= \sum_k \mathbb{E} H_{ik}G_{kj}[\pi(G)_{ij}]^{n-1}[\overline{\pi(G)_{ij}}]^n + \mathbb{E}[\underline{G}\cdot G_{ij}\cdot \pi_{ij}^{n-1}\overline{\pi}_{ij}^n]$$

$$=: (A) + (B).$$

Now let $h = H_{ik}$, $f \equiv f_k(H) = G_{kj}\pi_{ij}^{n-1}\overline{\pi}_{ij}^n$.

The upper triangular entries of H are independence, think of f is a function of H_{ik} and H_{ki}. By lemma 11

$$(A) = \sum_k \sum_{n=0}^{L} \frac{1}{n!} C_{n+1}(H_{ik})\,\mathbb{E}\,\partial_{ik}^n [G_{kj}\pi_{ij}^{n-1}\overline{\pi}_{ij}^n] + \sum_k R_{L+1}^{(k)}.$$

$$:= \sum_{n=0}^{L} L_n + R_{L+1}$$

Green function is easy to differentiate.

$$0 = \frac{G(H-z)}{\partial H_{ij}} = \frac{\partial G}{\partial H_{ij}}\cdot(H-z) + G\cdot\frac{\partial H}{\partial H_{ij}} = \frac{\partial G}{\partial H_{ij}}\cdot(H-z) + G\cdot(\Delta^{ij} + \Delta^{ji})(1+\delta_{ij})^{-1}.$$

Here $\Delta^{ij} \in \mathbb{C}^{N\times N}$ with $\Delta^{ij}_{kl} = \delta_{ik}\delta_{jl}$.

Thus $$\frac{\partial G}{\partial H_{ij}} = -G\,(\Delta^{ij}+\Delta^{ji})(1+\delta_{ij})^{-1}G \quad \text{in the matrix sense}$$

$$\frac{\partial G_{kl}}{\partial H_{ij}} = -(G_{ki}G_{jl} + G_{kj}G_{il})(1+\delta_{ij})^{-1}.$$

$C_{0+1}(H_{ik}) = \mathbb{E} H_{ik} = 0$. $\quad C_2(H_{ik}) = \mathrm{var}(H_{ik}) = \mathbb{E}|H_{ik}|^2$.

Recall we assume $\mathbb{E}|H_{ik}|^2 = N^{-1}(1+O(\delta_{ik}))$, for simplicity we shall assume $\mathbb{E} H_{ij}^2 = N^{-1}(1+\delta_{ik})$.

Then $$L_1 = \sum_{k=1}^{N} N^{-1}(1+\delta_{ik})\,\mathbb{E}\,\partial_{ik}[G_{kj}\Pi_{ij}^{n-1}\overline{\Pi}_{ij}^{n}]$$

$$= \sum_{k=1}^{N} N^{-1}(1+\delta_{ik})\,\mathbb{E}\left[-(G_{ki}G_{kj}+G_{kk}G_{ij})(1+\delta_{ik})^{-1}\Pi_{ij}^{n-1}\overline{\Pi}_{ij}^{n}\right]$$

$$+ \sum_{k=1}^{N} N^{-1}(1+\delta_{ik})\,\mathbb{E}\left[G_{kj}\,\partial_{ik}(\Pi_{ij}^{n-1}\overline{\Pi}_{ij}^{n})\right] =: L_1^{(1)} + L_1^{(2)}.$$

Now $$L_1^{(1)} = N^{-1}\,\mathbb{E}-((G^2)_{ij} + \mathrm{Tr}\,G\cdot G_{ij})\Pi_{ij}^{n-1}\overline{\Pi}_{ij}^{n}] = -\frac{1}{N}\mathbb{E}(G^2)_{ij}\Pi_{ij}^{n-1}\overline{\Pi}_{ij}^{n} - \mathbb{E}\,\underline{G}\cdot\Pi_{ij}^{n-1}\overline{\Pi}_{ij}^{n}.$$

$$L_1^{(1)} + (B) = -\frac{1}{N}\mathbb{E}(G^2)_{ij}\Pi_{ij}^{n-1}\overline{\Pi}_{ij}^{n} \quad \text{(Key cancellation).}$$

Thus
$$\mathbb{E}|\Pi_{ij}|^{2n} = (A)+(B) = \sum_{n=0}^{l} L_n + R_{l+1} + (B)$$

$$= L_1^{(2)} + \sum_{n=2}^{l} L_n + R_{l+1} - \frac{1}{N}\mathbb{E}(G^2)_{ij}\Pi_{ij}^{n-1}\overline{\Pi}_{ij}^{n}. \quad \cdots (\star).$$

$$\left|\frac{1}{N}\mathbb{E}(G^2)_{ij}\Pi_{ij}^{n-1}\overline{\Pi}_{ij}^{n}\right| \le \frac{1}{N}\mathbb{E}\left[|(G^2)_{ij}|\cdot|\Pi_{ij}|^{2n-1}\right] \le \frac{1}{N}\mathbb{E}\left[\frac{(\mathrm{Im}\,G_{ii}\cdot\mathrm{Im}\,G_{jj})^{\frac{1}{2}}}{\eta}|\Pi_{ij}|^{2n-1}\right]$$

$$|(G^2)_{ij}| \le \sum_k |G_{ik}G_{kj}| \le \left(\sum_k |G_{ik}|^2\right)^{\frac{1}{2}}\left(\sum_k |G_{jk}|^2\right) = \left[\frac{\mathrm{Im}\,G_{ii}}{\eta}\cdot\frac{\mathrm{Im}\,G_{jj}}{\eta}\right]^{\frac{1}{2}}$$

$$|\mathrm{Im}\, G_{ii}| \le |\mathrm{Im}\, m| + |\mathrm{Im}\, G_{ii} - \mathrm{Im}\, m| \le \mathrm{Im}\, m + |G_{ii} - m| \prec \mathrm{Im}\, m + \phi$$

Let $D = 100n$. $\forall \varepsilon > 0$, $\exists N_0$ s.t. $P(\underbrace{|\mathrm{Im}\, G_{ii} \cdot \mathrm{Im}\, G_{jj}|^{\frac{1}{2}} > (\mathrm{Im}\, m + \phi) \cdot N^{\varepsilon}}_{A_\varepsilon}) \le N^{-D}$ $\forall N \ge N_0$

$$\mathbb{E}[\,|\mathrm{Im}\, G_{ii}\, \mathrm{Im}\, G_{jj}|^{\frac{1}{2}} |\Pi_{ij}^{2n-1}|\,] \le \mathbb{E}[\,|\mathrm{Im}\, G_{ii}\, \mathrm{Im}\, G_{jj}|^{\frac{1}{2}} |\Pi_{ij}^{2n-1}| \cdot A_\varepsilon\,]$$
$$+ \mathbb{E}[\,|\mathrm{Im}\, G_{ii}\, \mathrm{Im}\, G_{jj}|^{\frac{1}{2}} |\Pi_{ij}^{2n-1}|\, A_\varepsilon^c\,]$$

$$\le \mathbb{E}[\,|\mathrm{Im}\, G_{ii}\, \mathrm{Im}\, G_{jj}|^{\frac{1}{2}} |\Pi_{ij}^{2n-1}|\, A_\varepsilon\,] + N^{\varepsilon}(\mathrm{Im}\, m + \phi)\, \mathbb{E}[\,|\Pi_{ij}|^{2n-1}\,]$$

$$\underset{\substack{\uparrow \\ |G_{ij}| \le \frac{1}{\eta} < N}}{\le} P(A_\varepsilon) \cdot N^{10n} + N^{\varepsilon}(\mathrm{Im}\, m + \phi)\, \mathbb{E}[\,|\Pi_{ij}|^{2n-1}\,] \le N^{-90n} + N^{\varepsilon}(\mathrm{Im}\, m + \phi)\, \mathbb{E}[\,|\Pi_{ij}|^{2n-1}\,].$$

Since ε is arbitrary, $\mathbb{E}[\,|\mathrm{Im}\, G_{ii} \cdot \mathrm{Im}\, G_{jj}|^{\frac{1}{2}} |\Pi_{ij}^{2n-1}|\,] \prec (\mathrm{Im}\, m + \phi)\, \mathbb{E}[\,|\Pi_{ij}|^{2n-1}\,]$.

Thus $\left|\frac{1}{N} \mathbb{E}(G^2)_{ij}\, \Pi_{ij}^{n-1} \overline{\Pi}_{ij}^{n}\right| \prec \frac{\mathrm{Im}\, m + \phi}{N\eta} \mathbb{E}|\Pi_{ij}|^{2n-1} \le \frac{\mathrm{Im}\, m + \phi}{N\eta} [\mathbb{E}|\Pi_{ij}|^{2n}]^{\frac{2n-1}{2n}}$.

Lemma 13.

$$\mathbb{E}|\Pi_{ij}|^{2n} \prec \sum_{a=1}^{2n} \xi^{a} \cdot [\mathbb{E}|\Pi_{ij}^{2n}|]^{\frac{2n-a}{2n}}, \qquad \xi := (1+\phi)^3 \sqrt{\frac{\mathrm{Im}\, m + \phi}{N\eta}}.$$

Lemma 12 will automatically imply $\mathbb{E}|\Pi_{ij}|^{2n} \prec \xi^{2n}$.

The last term in $(\star)$ is good.

Let us look at $L_1^{(2)}$, we take the term

$$\left| \sum_{k=1}^{N} N^{-1}(1+\delta_{ik})\, \mathbb{E}[\, G_{ky}\, \partial_{ik}(z G_{ij}) \cdot (n-1)\, \Pi_{ij}^{n-2}\, \overline{\Pi}_{ij}^{n}\,] \right|$$

$$\leq C \sum_{k=1}^{N} N^{-1} (1+\delta_{ik}) \mathbb{E} \left| G_{kj} (G_{ii} G_{kj} + G_{ik} G_{ij}) (1+\delta_{ik})^{-1} \Pi_{ij}^{2n-2} \right|$$

$$\leq C \sum_{k=1}^{N} N^{-1} \mathbb{E} \left| (G_{kj}^2 G_{ii} + G_{ik} G_{kj} G_{ij}) \Pi_{ij}^{2n-2} \right|$$

$$\prec \frac{\mathrm{Im}\, m + \phi}{N\eta} \mathbb{E} \left| \Pi_{ij}^{2n-2} \right| \leq \frac{\mathrm{Im}\, m + \phi}{N\eta} \left[\mathbb{E} \left| \Pi_{ij}^{2n} \right| \right]^{\frac{2n-2}{2n}} \leq \xi^2 \left[\mathbb{E} \left| \Pi_{ij}^{2n} \right| \right]^{\frac{2n-2}{2n}}.$$

Other terms in $L_1^{(2)}$ works similarly.

L_1 corresponds to 2nd cumulant terms.

The higher cumulant terms are easier to estimate: recall assumption

$$\mathbb{E} |H_{ij}|^k \leq C_k N^{-k/2} \qquad \mathbb{E} |\sqrt{N} H_{ij}|^k \leq C_k$$

$$C_k(\sqrt{N} H_{ij}) \leq \hat{C}_k \qquad C_k(aX) = a^k C_k(X) \Rightarrow C_k(H_{ij}) \leq \hat{C}_k N^{-\frac{k}{2}}$$

e.g.

$$L_2 = \sum_{k=1}^{N} \frac{1}{2!} C_3(H_{ik}) \mathbb{E}\, \partial_{ik}^2 \left[G_{kj} \Pi_{ij}^{n-1} \overline{\Pi}_{ij}^{n} \right]$$

$$\prec N^{-\frac{3}{2}} \sum_{k=1}^{N} \mathbb{E} \left| \partial_{ik}^2 \left[G_{kj} \Pi_{ij}^{n-1} \overline{\Pi}_{ij}^{n} \right] \right|$$

$$\prec \cdots$$

$$\prec \sum_{a=1}^{3} \xi^a \left[\mathbb{E} \left| \Pi_{ij}^{2n} \right| \right]^{\frac{2n-a}{2n}}.$$

2.3 Idea of proving Proposition 10 (ii)

The summation in $\underline{\Pi(G)} = \frac{1}{N} \mathrm{Tr}\, \Pi(G) = \frac{1}{N} \sum_i \Pi_{ii}$ gives rise to more Ward Identity.

2.4 Proof of the local law. $z = E + i\eta$, fix $E \in [-3, 3]$.

Step 1. $\eta \geq 2$.

$|G_{ij}| \leq \frac{1}{\eta} \leq \frac{1}{2}$, $|m| \leq 1$, $\phi = \max_{i,j} |G_{ij} - m\delta_{ij}| \leq 2$.

By Proposition 10 (i),

$$1 + z\underline{G} + \underline{G}^2 \prec (1+\phi)^6 \frac{\operatorname{Im} m + \phi}{N\eta} \prec \frac{1}{N\eta}$$

Then
$$\underline{G} = -\frac{1}{z+\underline{G}}\left(1 + O_\prec\left(\frac{1}{N\eta}\right)\right). \qquad \operatorname{Im}(z+\underline{G}) \geq \operatorname{Im} z \geq 2$$

Since $1 + zm + m^2 = 0$, $m = -\frac{1}{z+m}$, we have

$$\underline{G} - m = \frac{\underline{G} - m}{(z+\underline{G})(z+m)} + O_\prec\left(\frac{1}{N\eta}\right), \text{ thus } \underline{G} - m \prec \frac{1}{N\eta}$$

Lemma 14. Let $z \in S$. Suppose $\max_{i,j} |G_{ij} - m\delta_{ij}| \prec \phi$, $\phi \in [N^{-1}, N^{\tau/10}]$, and $\underline{G} - m \prec \theta$ for some $\theta \leq N^{-\frac{\tau}{10}}$.
Then we have at z

$$\max_{i,j} |G_{ij} - m\delta_{ij}| \prec \theta + \psi, \qquad \psi := \sqrt{\frac{\operatorname{Im} m}{N\eta}} + \frac{1}{N\eta}.$$

proof: By Proposition 10 (i),

$$\delta_{ij} + zG_{ij} + \underline{G}\cdot G_{ij} \prec \sqrt{\frac{\operatorname{Im} m + \phi}{N\eta}}\,(1+\phi)^3,$$

Note that $-\frac{1}{z+\underline{G}} = -\frac{1}{z+m+\underline{G}-m} = -\frac{1}{z+m} + O_\prec(\theta) = m + O_\prec(\theta)$

Thus
$$G_{ij} = m\delta_{ij} + O_\prec\left(\theta + (1+\phi)^3\sqrt{\frac{\operatorname{Im} m + \phi}{N\eta}}\right)$$

Which shows we have the recursive relation:

$$\max_{i,j} |G_{ij} - m\delta_{ij}| \prec \phi \implies \max_{i,j} |G_{ij} - m\delta_{ij}| \prec \theta + (1+\phi)^3 \sqrt{\frac{\operatorname{Im} m + \phi}{N\eta}}$$

Iterating the above finishes the proof.

Now for $\eta \geq 2$, the conditions of Lemma 13 holds, with $G - m \prec \frac{1}{N\eta} := \phi$. Thus

$$\max_{i,j} |G_{ij} - m\delta_{ij}| \prec \sqrt{\frac{\operatorname{Im} m}{N\eta}} + \frac{1}{N\eta}.$$

Step 2. General $\eta \geq N^{-1+\tau}$.

Recall Lemma 9: $\Gamma(z) = \max_{i,j} |G_{ij}(z)|$. For $M > 1$, $\Gamma(E + i\eta/M) \leq M\Gamma(E+i\eta)$.

Proof of Lemma 9: Fix E and write $\Gamma(\eta) = \Gamma(E+i\eta)$.

$$|\Gamma(\eta+h) - \Gamma(\eta)| \leq \max_{i,j} |G_{ij}(E + i(\eta+h)) - G_{ij}(E+i\eta)|$$

$$\leq |h| \max_{i,j} \sum_k |G_{ik}(E+i(\eta+h)) G_{kj}(E+i\eta)|$$

$$\leq |h| \sqrt{\frac{\Gamma(\eta+h)\Gamma(\eta)}{(\eta+h)\eta}}$$

Γ is locally Lip-cont., and its almost everywhere defined derivative satisfies

$$\left|\frac{d\Gamma}{d\eta}\right| \leq \frac{\Gamma}{\eta}.$$

This implies $\frac{d}{d\eta}(\eta\Gamma(\eta)) \geq 0$, thus the claim follows.

We need one more result.

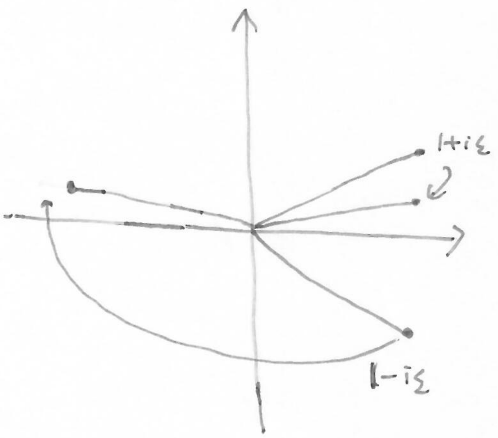
Lemma 15. Let $\Delta : S \to \mathbb{R}$ be a function s.t.

$$\left| 1 + z\underline{G} + \underline{G}^2 \right| \prec \Delta .$$

Suppose $N^{-2} \le \delta \le N^{-\varepsilon}$ for $z \in S$, that Δ is Lipschitz continuous with Lipschitz constant N. Moreover, assume that for each fixed E the function $\eta \to \Delta(E+i\eta)$ is non-increasing for $\eta > 0$. Then

$$\left| \underline{G} - m \right| \prec \frac{\Delta}{\sqrt{\operatorname{Im} m + \Delta}} .$$

$\top$



$$\underline{G} = \frac{-z + \sqrt{z^2 - 4 - O_\prec(\Delta)}}{2}$$

$$m = \frac{-z + \sqrt{z^2 - 4}}{2}$$

Square-root is unstable:

$$\sqrt{1+i\varepsilon} \simeq 1 + i\frac{\varepsilon}{2}, \quad \sqrt{1+i\varepsilon - 2i\varepsilon} \simeq -1 + i\frac{\varepsilon}{2}$$

$\perp$

A proof of Lemma 14 can be found e.g. on Lemma 4.5, arXiv 1308.5729.

Now we write $\{E\} \times [N^{-1+\tau}, 3] = \bigcup_{k=0}^{n} W_k$

$W_0 := \{E\} \times [2,3]$, and $W_k := \{E\} \times [2N^{-\delta k}, 2N^{-\delta(k-1)}]$

Here $\delta = \frac{\tau}{100}$, $n \le \frac{1}{\delta}$.

Lemma 16. For $k = 1, 2, \cdots, n$. Suppose Theorem 5 holds for all $z \in W_{k-1}$, then it also holds for all $z \in W_k$.

Proof: Let $b_k := E + i2N^{-\delta(k-1)}$ be the upper edge of W_k (lower edge of W_{k-1}).

$\max_{i,j} |G_{ij} - \delta_{ij} m| \prec \sqrt{\frac{\operatorname{Im} m}{N\eta}} + \frac{1}{N\eta}$ at b_k $\Rightarrow$ $\max_{i,j} |G_{ij}| \prec 1$ at b_k

$\Rightarrow$ $\max_{i,j} |G_{ij}| \prec N^{\delta}$ for $z \in W_k$ $\Rightarrow$ $\max_{i,j} |G_{ij} - m\delta_{ij}| \prec N^{\delta}$, $z \in W_k$.

By Prop 10 (ii),

$$1 + z\underline{G} + \underline{G}^2 \prec (1+N^{\delta})^6 \, \frac{\operatorname{Im} m + N^{\delta}}{N\eta}, \qquad z \in W_k$$

and with Lemma 15, $\quad \underline{G} - m \prec (N\eta)^{-\frac{1}{4}}$. $\qquad z \in W_k$.

Now suppose $\underline{G} - m \prec \theta$, $\theta \le N^{-\frac{\tau}{10}}$. Lemma 14 shows

$$\max_{i,j} |G_{ij} - m\delta_{ij}| \prec \theta + \psi.$$

By Proposition 10 (ii), $\phi = \theta + \psi$

$$1 + z\underline{G} + \underline{G}^2 \prec \frac{\operatorname{Im} m + \theta + \psi}{N\eta} = \frac{\operatorname{Im} m}{N\eta} + \frac{1}{(N\eta)^2} + \frac{\theta}{N\eta} = \Delta$$

By Lemma 15,

$$\underline{G} - m \prec \frac{\Delta}{\operatorname{Im} m + \sqrt{\Delta}} \prec \frac{1}{N\eta} + \sqrt{\frac{\theta}{N\eta}}$$

This means $\underline{G} - m \prec \theta \Rightarrow \underline{G} - m \prec \frac{1}{N\eta} + \sqrt{\frac{\theta}{N\eta}}$

Thus $\underline{G} - m \prec \frac{1}{N\eta}$.

Using Lemma 13 with $\theta = \frac{1}{N\eta}$, we get

$$\max_{ij} |G_{ij} - m\delta_{ij}| \prec \theta + \psi \prec \sqrt{\frac{\operatorname{Im} m}{N}} + \frac{1}{N\eta}.$$

3. Applications.

3.1 Proof of Corollary 6.

Lemma 17 (Helffer-Strojstrand formula).

Let $f \in C_c^2(\mathbb{R})$, and define the almost analytic extension of f by

$$\tilde{f}(x+iy) = f(x) + iyf'(x).$$

Let $\chi \in C_c^\infty(\mathbb{R})$ be a cut-off function satisfying $\chi(0)=1$, and by a slight abuse of notation we write $\chi(z) \equiv \chi(\operatorname{Im} z)$. Then for any $\lambda \in \mathbb{R}$, we have

$$f(\lambda) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial_{\bar{z}}(\tilde{f}(z)\chi(z))}{\lambda - z} d^2 z,$$

where $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$ is the anti-holomorphic derivative.

pf: Exercise.

Proof of Corollary 6.

Fix $\varepsilon > 0$, let $\eta = N^{-1+\varepsilon}$. For any interval $I \subset [-3, 3]$, choose smooth $f \in C_c^\infty(\mathbb{R})$ satisfying $f(x) = 1$ for $x \in I$, $f(x) = 0$ for $\operatorname{dist}(x, I) \geq \eta$.

We further assume $\|f'\|_\infty \le C\eta^{-1}$, $\|f''\|_\infty \le C\eta^{-2}$.

$$\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial_{\bar z}(\tilde f(z)\chi(z))}{\lambda_i - z}\, d^2z = \frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar z}(\tilde f(z)\chi(z))\, G(z)\, d^2(z).$$

Similarly, $\displaystyle\int f(x)\,\ell(dx) = \frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar z}(\tilde f(z)\chi(z))\, m(z)\, d^2z.$

Thus

$$\int f(x)\,m(dx) - \int f(x)\,\ell(dx) = \frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar z}(\tilde f(z)\chi(z))[G(z) - m(z)]\, d^2z.$$

$$= \frac{1}{2\pi}\int_{\mathbb{C}} [(f(x) + iyf'(x))\, i\chi'(y) + iyf''(x)\chi(y)]\,(G(z) - m(z))\, d^2z$$

Here we choose $\chi \in C_c^\infty$ s.t. $\chi(y) = 1$ for $|y| \le 1$, $\chi(y) = 0$ for $|y| \ge 2$.

① $\displaystyle\frac{1}{2\pi}\int_{\mathbb{C}_+} f(x)\, i\chi'(y)\,(G(z) - m(z))\, d^2z \prec \int_{\mathbb{C}_+} |f(x)\chi'(y)| \cdot \frac{1}{Ny}\, d^2z$

$$\le \int_{\mathbb{R}}\int_1^2 |f(x)| \cdot \frac{1}{N}\, d^2z \le \frac{C}{N}.$$

② Similarly, $\displaystyle\frac{1}{2\pi}\int_{\mathbb{C}_+} iyf'(x)\, i\chi'(y)\,(G(z) - m(z))\, d^2z \prec \frac{1}{N}.$

(By using $|f'|_\infty = O(1)$).

③ To estimate the term concerning $yf''(x)$, we need the following result.

Lemma 18. We have

$$\underline{G}(z) \prec \frac{N^{\varepsilon}}{Ny}$$

for all $0 < y \le \eta = N^{-1+\varepsilon}$.

Proof: $G_{ij}(x+i\eta) - m(x+i\eta)\delta_{ij} \prec \frac{1}{N\eta}$, $|m| \le 1$

$$\Rightarrow \max_{i,j} |G_{ij}(x+i\eta)| \prec 1$$

By Lemma 9,

$$\max_{i,j} |G_{ij}(x+iy)| = \Gamma(x+iy) \le \frac{\eta}{y}\Gamma(x+i\eta) = \frac{N^{\varepsilon}}{Ny}\Gamma(x+i\eta) \prec \frac{N^{\varepsilon}}{Ny}.$$

By Lemma 18, $|m| \le 1$, we have

$$\underline{G}(z) - m(z) \prec \frac{N^{\varepsilon}}{Ny}$$

for all $y \le \eta$. Thus

$$\frac{1}{2\pi}\int_{0<y\le\eta} iy f''(x)\chi(y)(\underline{G}(z)-m(z))\, d^2z$$

$$\prec \int_{\mathbb{R}}\int_0^{\eta} \left| y f''(x) \cdot \frac{N^{\varepsilon}}{Ny} \right| dy\, dx \prec \frac{N^{\varepsilon}}{N} \cdot \eta \cdot \|f''\|_1 = \frac{N^{\varepsilon}}{N}.$$

On the other hand,

$$\frac{1}{2\pi}\int_{y>\eta} iy f''(x)\chi(y)(\underline{G}(z)-m(z))\, d^2z$$

$$= -\frac{1}{2\pi}\int_{y>\eta} iy f'(x)\chi(y)\,\partial_x(\underline{G}(z)-m(z))\, d^2z$$

$$= -\frac{1}{2\pi}\int_{y>\eta} iy f'(x)\chi(y)\,\partial_y(\underline{G}(z)-m(z))\, d^2z$$

$$= -\frac{i}{2\pi}\int iy f'(x)\chi(y)\,(\mathbb{E}(x+iy)-m(x+iy))\,dx$$

$$+\frac{i}{2\pi}\int_{y>y}[if'(x)\chi(y)+iyf'(x)\chi'(y)]\,(\mathbb{E}(z)-m(z))\,d^2z \prec \frac{N^{\varepsilon}}{N}.$$

By ①-③ we have

$$\int f(x)\mu(dx)-\int f(x)\,\rho(dx) \prec \frac{N^{\varepsilon}}{N}.$$

- Let $I\subset[-3,3]$.

$$\mu(I)\le\int f_{I,\eta}(x)\mu(dx)\le\int f_{I,\eta}(x)\,\rho(dx)+O_{\prec}(N^{-1+\varepsilon})$$

$$\le \rho(I)+O_{\prec}(N^{-1+\varepsilon}).$$

$I'=\{x:\operatorname{dist}(x,I^c)\ge\eta\}$. Then $\mu(I)\ge\int f_{I',\eta}(x)\mu(dx)\ge\cdots\ge\rho(I)+O_{\prec}(N^{-1+\varepsilon})$.

Then $\mu(I)-\rho(I)\prec N^{-1}$, since ε is arbitrary

- Let $I\not\subset[-3,3]$.

$$\mu(\mathbb{R}\setminus[-2,2])=1-\mu([-2,2])=1-\rho([-2,2])+O_{\prec}(N^{-1})=O_{\prec}(N^{-1})$$

$$\mu(I)=\mu(I\cap[-2,2])+\mu(I\setminus[-2,2])=\rho(I\cap[-2,2])+O_{\prec}(N^{-1})$$

$$=\rho(I)+O_{\prec}(N^{-1}).$$

3.2. Proof of Corollary 8.

Let us order the eigenvalues by $\lambda_1\le\lambda_2\le\cdots\le\lambda_N$, and define the eigenvalue counting function $\Sigma:\mathbb{R}\to\{1,2,\cdots,N\}$ by

$$\Sigma(E):=|\{i:\lambda_i\le E\}|.$$

We shall prove that

$$|\lambda_i - \gamma_i| \prec N^{-1}$$

for all $i \in [cN, (1-c)N]$.

Fix $\varepsilon > 0$. We have the duality

$$\{\lambda_i \leq \gamma_i + N^{-1+\varepsilon}\} = \{\Sigma(\gamma_i + N^{-1+\varepsilon}) \geq i\}.$$

$$\frac{1}{N}\Sigma(\gamma_i + N^{-1+\varepsilon}) = \mu((-\infty, \gamma_i + N^{-1+\varepsilon}])$$

$$= \mu((-\infty, \gamma_i + N^{-1+\varepsilon}]) - \varrho((-\infty, \gamma_i + N^{-1+\varepsilon}]) + \varrho((-\infty, \gamma_i] \cup (\gamma_i, \gamma_i + N^{-1+\varepsilon}])$$

$$= O_\prec(N^{-1}) + \frac{i}{N} + \varrho((\gamma_i, \gamma_i + N^{-1+\varepsilon}]) = O_\prec(N^{-1}) + \frac{i}{N} + C_i N^{-1+\varepsilon}, \quad C_i > 0.$$

Thus $\Sigma(\gamma_i + N^{-1+\varepsilon}) \geq i$ with high probability, which implies

$$(\lambda_i - \gamma_i)_+ \prec N^{-1}.$$

Similarly, we can show that $(\lambda_i - \gamma_i)_- \prec N^{-1}$, which implies the desired result.

Some final remarks:

- The estimate $|G - m| \prec \frac{1}{N\eta}$ is optimal, in the sense that for $E \in (-2, 2)$, $\eta \in [N^{-1+\varepsilon}, N^{-\varepsilon}]$, we have

$$N\eta(G - m) \xrightarrow{d} N_{\mathbb{C}}(0, \tfrac{1}{2}).$$

Here $N_{\mathbb{C}}(0, \frac{1}{2}) \overset{d}{=} X + iY$, where X, Y are independent Gaussians, with $\mathrm{Var}(X) = \mathrm{Var}(Y) = \frac{1}{2}$. (So that $\mathrm{Var}(X + iY) = \frac{1}{2}$).

Recall that

$$\operatorname{Im} G(E+i\eta) = \frac{1}{N\eta}\sum_i \frac{1}{\left(\frac{\lambda_i - E}{\eta}\right)^2+1} = \frac{1}{N\eta}\operatorname{Tr} f\left(\frac{H-E}{\eta}\right), \quad f(x) := \frac{1}{x^2+1}.$$

For general $f \in C_c^2(\mathbb{R})$,

$$\operatorname{Tr} f\left(\frac{H-E}{\eta}\right) - \mathbb{E}\operatorname{Tr} f\left(\frac{H-E}{\eta}\right) \to \mathcal{N}\left(0, \frac{1}{2\pi}\int\left(\frac{f(x)-f(y)}{x-y}\right)^2 dx\,dy\right).$$

The above happens regardless of the specific η.

Let $\tilde{H} = cH$, $\operatorname{Tr} f\left(\frac{\tilde{H}-E}{\eta}\right) = \operatorname{Tr}\tilde{f}\left(\frac{H-\frac{E}{c}}{\eta}\right)$ $\quad \tilde{f}(x) = f(cx)$.

$$\operatorname{Tr} f\left(\frac{\tilde{H}-E}{\eta}\right) - \mathbb{E}\operatorname{Tr} f\left(\frac{\tilde{H}-E}{\eta}\right) \to \mathcal{N}\left(0, \frac{1}{2\pi}\int\left(\frac{\tilde{f}(x)-\tilde{f}(y)}{x-y}\right)^2 dx\,dy\right).$$

$$\int\left(\frac{\tilde{f}(x)-\tilde{f}(y)}{x-y}\right)^2 dx\,dy = \int\left(\frac{f(cx)-f(cy)}{x-y}\right)^2 dx\,dy$$

$$\underset{\substack{a=cx\\ b=cy}}{\overset{\uparrow}{=}} \int\left(\frac{f(a)-f(b)}{a-b}\right)^2 da\,db.$$

- By far we have only consider the real symmetric Wigner matrix, using cumulant expansion to expand $\mathbb{E}hf(h)$.

For a complex Hermitian Wigner matrix, one has

$$\mathbb{E}\, h f(h,\bar{h}) = \sum_{p+q=0}^{l} \frac{1}{p!q!} C^{(p+1,q)}(h)\, \mathbb{E} f^{(p,q)}(h,\bar{h}) + R_{l+1},$$

where the (p,q)-cumulant of h is defined as

$$C^{(p,q)}(h) = (-i)^{p+q} \cdot \left(\frac{\partial^{p+q}}{\partial s^p \partial t^p} \log \mathbb{E}\, e^{ish+it\bar{h}} \right) \Big|_{s=t=0}.$$