A lower bound for disconnection by simple random walk

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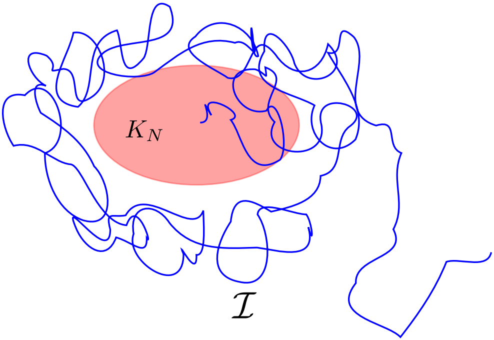
What we talk about when we talk about disconnection

- Consider (continuous-time) simple random walk $(X_t)_{t \geq 0}$ on $\mathbb{Z}^d$, $d \geq 3$, started from 0. Denote its law by P_0.
- Trace: $\mathcal{I} = X_{[0,\infty)}$. Vacant set: $\mathcal{V} = \mathbb{Z}^d \setminus \mathcal{I}$.
- For compact $K \subset \mathbb{R}^d$ and $N \in \mathbb{N}$, look at its discrete blow-up

$$K_N \stackrel{\triangle}{=} \{x \in \mathbb{Z}^d ; d_\infty(NK, x) \leq 1\}.$$

- Event of interest $A_N \stackrel{\triangle}{=} \left\{ K_N \stackrel{\mathcal{V}}{\not\leftrightarrow} \infty \right\}$: "no path in $\mathcal{V}$ connects K_N with infinity".



The asymptotic lower bound

- Question is trivial if $d = 1$ or $d = 2$ (recurrence).

Theorem (L. 14')

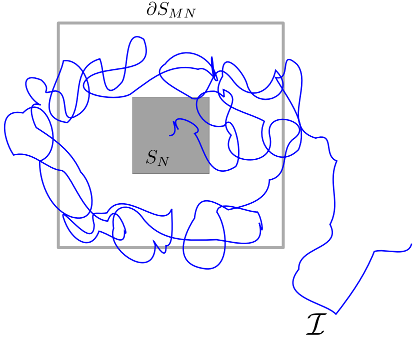
One has the following asymptotic lower bound:

$$\liminf_{N \to \infty} \frac{1}{N^{d-2}} \log P_0[A_N] \geq -\frac{u_{**}}{d} \mathrm{cap}_{\mathbb{R}^d}(K).$$

- $\mathrm{cap}_{\mathbb{R}^d}(K)$: the Brownian capacity of K.
  $= \inf \mathcal{E}(g, g)$, where infimum runs over all compactly supported non-negative $g \in H^1$ such that $g \geq 1$ on K.
- $u_{**} \in (0, \infty)$ is one of the critical thresholds for the percolation on the vacant set of *random interlacements*.
- Is the lower bound tight?

The same lower bound for a slightly different setup

- Let $S_N \overset{\triangle}{=} B_\infty(0, N) \cap \mathbb{Z}^d$.
- Fix $M > 1$ and consider $B_N \overset{\triangle}{=} \{S_N \overset{\mathcal{V}}{\nleftrightarrow} \partial S_{MN}\}$.



- Proof for the lower bound of $P_0[A_N]$ also works for $P_0[B_N]$.

Theorem (L. 14')

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log P_0[B_N] \geq -\frac{u_{**}}{d} \mathrm{cap}_{\mathbb{R}^d}([-1,1]^d).$$

An accompanying upper bound

Theorem (L. 14')

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log P_0[B_N] \geq -\frac{u_{**}}{d} \mathrm{cap}_{\mathbb{R}^d}([-1,1]^d).$$

Theorem (Sznitman 14')

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log P_0[B_N] \leq -\frac{\overline{u}}{d} \mathrm{cap}_{\mathbb{R}^d}([-1,1]^d).$$

- $\overline{u}$: another percolation threshold for the vacant set of random interlacements.
- $0 < \overline{u} \leq u_{**} < \infty$.
- Conjecture: $\overline{u}\,(= u_*) = u_{**}$?

Change of measure

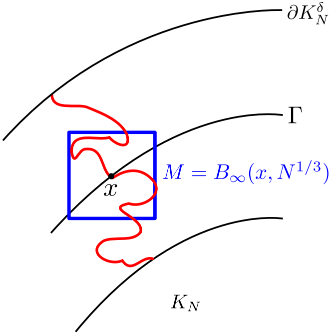
- A useful classical inequality: for $\widetilde{\mathbb{P}} \ll \mathbb{P}$ and A s.t. $\widetilde{\mathbb{P}}[A] \neq 0$,

$$\mathbb{P}[A] \geq \widetilde{\mathbb{P}}[A] \exp(-(H(\widetilde{\mathbb{P}}|\mathbb{P}) + c)/\widetilde{\mathbb{P}}[A]).$$

- If $\widetilde{\mathbb{P}}[A] \approx 1$, then $\mathbb{P}[A] \gtrapprox \exp(-H(\widetilde{\mathbb{P}}|\mathbb{P}))$.
- Hence, we need a family of tilted probability measures $\widetilde{P}_N$, corresponding to (possibly non-homogeneous) Markov chains, such that
  - $\widetilde{P}_N[A_N] \approx 1$;
  - $H(\widetilde{P}_N|P_0)$ is minimized.

A la recherche de la Connectivity Decay

- Let $K^\delta \stackrel{\triangle}{=} B(K,\delta)$ and write K_N^δ for its discrete blow-up.



- Let $\Gamma \stackrel{\triangle}{=} \partial K_N^{\delta/2}$ be a "strip" between ∂K_N and ∂K_N^δ.
- Note that $A_N^c = \{K_N \stackrel{\mathcal{V}}{\leftrightarrow} \infty\} \implies \{K_N \stackrel{\mathcal{V}}{\leftrightarrow} \partial K_N^\delta\}$

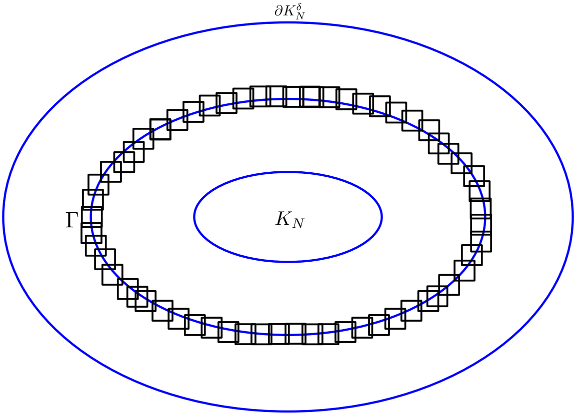
$$\implies \bigcup_{x\in\Gamma}\{x \stackrel{\mathcal{V}}{\leftrightarrow} \partial B(x, N^{1/3})\}.$$

- $|\Gamma| = O(N^{d-1})$.
- Hence, $\widetilde{P}_N[\{x \stackrel{\mathcal{V}}{\leftrightarrow} \partial M\}] \le e^{-N^c} \implies \widetilde{P}_N[A_N^c] \le e^{-N^{c'}}$.

Deus ex machina: Random Interlacements

- How "much" tilting to ensure $\widetilde{P}_N[A_N] \geq 1 - \exp(-N^{c'})$?
- Random interlacements: The natural model for the study of traces left by random walk.
- Random subset $\mathcal{I}^u \subset \mathbb{Z}^d$, $d \geq 3$: the traces of a Poissonian random collection of infinite trajectories, with intensity parameter $u > 0$.
- The connectivity decay we need: for $d \geq 3$, $\exists u_{**}(d) \in (0, \infty)$ such that
  - for all $u > u_{**}$, the connectivity of the vacant set $\mathcal{V}^u \triangleq \mathbb{Z}^d \backslash \mathcal{I}^u$ decays stretched-exponentially fast, i.e., $\exists c > 0$, s.t.
$$\mathbb{P}^u[0 \stackrel{\mathcal{V}^u}{\leftrightarrow} \partial B(0, N)] \leq e^{-N^c};$$
  - and for all $u \in (0, u_{**})$ the connectivity decays slower than any stretched exponential function, i.e., such c does not exist.

Only local couplings needed



- Only need to find couplings of $\widetilde{P}_N$ and $\mathbb{P}^{u_{**}+\epsilon}$ on all $M = B_\infty(x, N^{1/3})$ for $x \in \Gamma$,i.e., on M, with high probability
  - $\mathcal{I} \cap M \supset \mathcal{I}^{u_{**}+\epsilon} \cap M$, or
  - $\mathcal{V} \cap M \subset \mathcal{V}^{u_{**}+\epsilon} \cap M$, equivalently.

The tilted random walk

- Generator of a (possibly non-homogeneous) MC

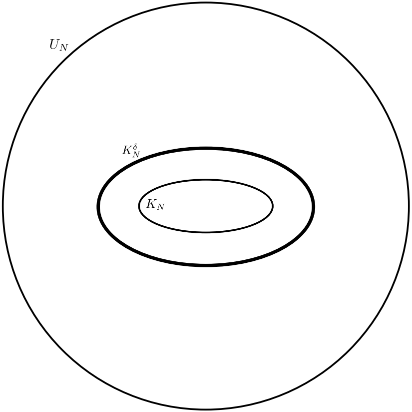
$$\overline{L}h(x) = \frac{1}{2d}\sum_{|e|=1}\frac{f(x+e)}{f(x)}\big(h(x+e)-h(x)\big),$$

  where f is to be chosen to our advantage.
  - If $f \equiv C$, $\overline{L}$ is the generator of simple random walk.
  - Transition rate from x to $x+e$: $f(x+e)/f(x)$.
  - MC lands more often on the vertices where f is larger.
  - Reversibility measure: $\pi(x) = f^2(x)$.
- Construction of the tilted random walk $\widetilde{P}_N$:
  - Let MC run with the generator $\overline{L}$ up to some T_N deterministic;
  - After T_N it is "released" and runs as SRW happily ever after.
- $H(\widetilde{P}_N|P_0)$ is proportional to T_N and the discrete Dirichlet form of f.
- Need to minimize both T_N and the Dirichlet form!

Mount Roraima: NOT the best choice of f

The story of f

- Let $U \stackrel{\triangle}{=} B(0, R)$ for large R and U_N be its discrete blow-up.

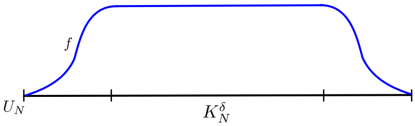


- Let $f = 1$ on K_N^δ, $f = 0$ outside U_N.
- In $U_N \backslash K_N^\delta$, choose f as discretized Brownian potential of K^δ w.r.t U: best choice to minimize Dirichlet form!

Mount Fuji, the optimal shape of f

The story of f reloaded; denouement

- Let $f = 1$ on K_N^δ, $f = 0$ outside U_N.
- In $U_N \backslash K_N^\delta$, choose f as discretized Brownian potential of K^δ w.r.t U.



- Choose T_N such that: on K_N^δ, the expected occupation time of the tilted walk equals that of random interlacements with intensity $u_{**} + \epsilon$.
- This ensures the feasibility of the couplings!
- In the asymptotic lower bound $\exp(-u_{**}\mathrm{cap}_{\mathbb{R}^d}(K)N^{d-2}/d)$,
  - u_{**} will appear from the choice of T_N, taking $\epsilon \to 0$;
  - $\mathrm{cap}_{\mathbb{R}^d}(K)$ and d will appear from the limit of the Dirichlet form of f, taking $R \to \infty$ and $\delta \to 0$.
  - N^{d-2} comes from both terms.

Comparison with results for Random Interlacements

Theorem (L.-Sznitman 13', Sznitman 14')

Consider $u \in (0, u_{**})$. *Similar to the definition of* A_N *and* B_N, *let* $A_N^u \overset{\triangle}{=} \{K_N \overset{\mathcal{V}^u}{\nleftrightarrow} \infty\}$ *and* $B_N^u \overset{\triangle}{=} \{S_N \overset{\mathcal{V}^u}{\nleftrightarrow} \partial S_{MN}\}$. *Then,*

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}^u[A_N^u] \geq -\frac{1}{d}(\sqrt{u_{**}} - \sqrt{u})^2 \mathrm{cap}_{\mathbb{R}^d}(K),$$

and

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}^u[B_N^u] \leq -\frac{1}{d}(\sqrt{\overline{u}} - \sqrt{u})^2 \mathrm{cap}_{\mathbb{R}^d}([-1,1]^d).$$

- If $u > u_{**}$, $\mathbb{P}^u[A_N] \to 1$ as $N \to \infty$. Hence the problem is non-trivial only when $u < u_{**}$.
- Intuitively, the case for SRW is a limiting case ($u \to 0$) of the disconnection problem for random interlacements.
- Proof of lower bound includes similar tiltings, but with different scheme.

Thanks for your attention!