# A lower bound for disconnection by random interlacements

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# PRELIMINARIES

We consider (continuous-time) simple random walk on $\mathbb{Z}^d$, $d \geq 3$.
For $M \subset\subset \mathbb{Z}^d$, we denote

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- $\mathrm{cap}(B(0,N)) = O(N^{d-2})$.
- Alternative definition of capacity:

$$\mathrm{cap}(M) = \inf\{D(f,f);\ f \geq 1 \text{ on } M \text{ and } f \text{ has finite support}\}.$$

# RANDOM INTERLACEMENTS, LOCAL PICTURE

Random interlacements can be regarded as a random subset of $\mathbb{Z}^d$, governed by a non-negative parameter $u$, which we denote by $\mathcal{I}^u$, and the complement (i.e. the VACANT SET) by $\mathcal{V}^u = \mathbb{Z}^d \backslash \mathcal{I}^u$.

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Characterisation of $\mathbb{P}$, the law of $\mathcal{I}^u$:

$$\mathbb{P}[\mathcal{I}^u \cap M = \emptyset] = e^{-u\mathrm{cap}(M)}.$$

# RANDOM INTERLACEMENTS, GLOBAL PICTURE

We denote the space of continuous-time doubly-infinite nearest-neighbour paths tending to infinity at both sides by

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Random interlacements at level $u$, are a Poisson point process on $W^*$, with intensity measure $u\nu$, where $\nu$ is the unique ergodic and translation-invariant measure on $W^*$ such that the trace of this PPP on $\mathbb{Z}^d$ has the same distribution as $\mathcal{I}^u$ defined above.

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Theorem (Sznitman 07', Sidoravicius-Sznitman 08', Teixeira 08', Sznitman 09')

*Let*

$$u_{**} = \inf\{u \geq 0; \exists k < \infty, \ s.t. \ \forall L \geq 0, \ \mathbb{P}[0 \overset{\mathcal{V}^u}{\leftrightarrow} B(0, L)] \leq \kappa \cdot e^{-L^{1/k}}\},$$

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Conjecture

*Do the two critical parameters actually coincide, i.e.,*

$$u_{**} = u_*?$$

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Theorem (L.-Sznitman 13')

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}[A_N] \geq -\frac{1}{d}(\sqrt{u_{**}} - \sqrt{u})^2 \mathrm{cap}_{\mathbb{R}^d}(K),$$

*where* $\mathrm{cap}_{\mathbb{R}^d}(K)$ *denotes the Brownian capacity of* $K$.

# IDEA OF PROOF

- We need to find a law $\widetilde{\mathbb{P}}$ of "tilted random interlacements" (which are Poissonian "clouds" of tilted random walks) such that $\widetilde{\mathbb{P}}[A_N] \to 1$ as $N \to \infty$ and need to minimise the relative entropy $H(\widetilde{\mathbb{P}}|\mathbb{P})$.

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- To this end, we take a tilted random walk with generator

$$\widetilde{L}h(x) = \frac{1}{2d} \sum_{|e|=1} \frac{f(x+e)}{f(x)} (h(x+e) - h(x)),$$

and reversibility measure $\pi(x) = f^2(x)$, where $f$ is to be chosen carefully in order to minimise the relative entropy.

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Thanks for your attention!