Algebraic Dynamics, Canonical Heights and Arakelov Geometry

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Abstract. This expository article introduces some recent applications of Arakelov geometry to algebraic dynamics. We present two results in algebraic dynamics, and introduce their proofs assuming two results in Arakelov geometry.

Contents

1. Introduction

Algebraic dynamics studies iterations of algebraic endomorphisms on algebraic varieties. To clarify the object of this expository article, we make an early definition.

Definition 1.1. A *polarized algebraic dynamical system over* a field K consists of a triple (X, f, L) where:

- X is a projective variety over K,
- $f : X \to X$ is an algebraic morphism over K,
- L is an ample line bundle on X polarizing f in the sense that $f^*L \cong L^{\otimes q}$ for some integer $q > 1$.

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In this article, we may abbreviate "polarized algebraic dynamical system" as "algebraic dynamical system" or "dynamical system".

The canonical height $h_f = h_{L,f}$ on $X(\overline{K})$, a function naturally defined using Arakelov geometry, describes concisely many important properties of an algebraic dynamical system. For example, preperiodic points are exactly the points of canonical height zero. Note that the canonical height can be defined even K is not a global field following the work of Moriwaki and Yuan–Zhang. But we will mainly focus on the number field case in this article.

Hence, it is possible to study algebraic dynamics using Arakelov geometry. The aim of this article is to introduce the relation between these two seemingly different subjects. To illustrate the idea, we are going to introduce mainly the following four theorems:

(1) Rigidity of preperiodic points (Theorem 2.9),
(2) Equidistribution of small points (Theorem 6.2),
(3) A non-archimedean Calabi–Yau theorem, uniqueness part (Theorem 3.3),
(4) An arithmetic bigness theorem (Theorem 8.7).

The first two results lie in algebraic dynamics, and the last two lie in Arakelov geometry. The logic order of the implications is:

$$(2) + (3) \Longrightarrow (1), \quad (4) \Longrightarrow (2).$$

For more details on the arithmetic of algebraic dynamics, we refer to [**Zh06, Sil07, BR10, CL10a**]. For introductions to complex dynamics, we refer to [**Mil06, Sib99, DS10**].

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2. Algebraic theory of algebraic dynamics

Here we make some basic definitions for an algebraic dynamical system, and state the rigidity of preperiodic points.

2.1. Algebraic dynamical systems. Recall that an algebraic dynamical system is a triple (X, f, L) over a field K as in Definition 1.1. The ampleness of L implies that the morphism f is finite, and the degree is $q^{\dim X}$. Denote by $\overline{K}$ the algebraic closure of K as always.

Definition 2.1. A point $x \in X(\overline{K})$ is said to be *preperiodic* if there exist two integers $a > b \geq 0$ such that $f^a(x) = f^b(x)$. It is said to be *periodic* if we can further take $b = 0$ in the equality. Denote by $\mathrm{Prep}(f)$ (resp. $\mathrm{Per}(f)$) the set of preperiodic (resp. periodic) points of $X(\overline{K})$. Here we take the convention that $f^0(x) = x$.

Remark 2.2. The sets $\mathrm{Prep}(f)$ and $\mathrm{Per}(f)$ are stable under base change. Namely, for any field extension M of K, we have $\mathrm{Prep}(f_M) = \mathrm{Prep}(f)$ and $\mathrm{Per}(f_M) = \mathrm{Per}(f)$. Here f_M represents the dynamical system (X_M, f_M, L_M) over M. More generally, the set $\mathrm{Prep}(f)_{a,b}$ of points $x \in X(\overline{K})$ with $f^a(x) = f^b(x)$ is stable under base change. We leave it as an exercise.

The following basic theorem is due to Fakhruddin [**Fak03**]. In the case $X = \mathbb{P}^n$ and $\mathrm{char}(K) = 0$, it is a consequence of Theorem 4.8, an equidistribution theorem due to Briend–Duval [**BD99**].

THEOREM 2.3 (Fakhruddin). *The set* $\mathrm{Per}(f)$ *is Zariski dense in* X.

It implies the more elementary result that $\mathrm{Prep}(f)$ is Zariski dense in X. Now we give some examples of dynamical systems.

EXAMPLE 2.4. **Abelian variety.** Let $X = A$ be an abelian variety over K, $f = [2]$ be the multiplication by 2, and L be a symmetric and ample line bundle on A. Recall that L is called symmetric if $[-1]^*L \cong L$, which implies $[2]^*L \cong L^{\otimes 4}$. It is obvious that L is not unique. It is easy to see that $\mathrm{Prep}(f) = A(\overline{K})_{\mathrm{tor}}$ is exactly the set of torsion points.

EXAMPLE 2.5. **Projective space.** Let $X = \mathbb{P}^n$, and $f : \mathbb{P}^n \to \mathbb{P}^n$ be any finite morphism of degree greater than one. Then $f = (f_0, f_1, \cdots, f_n)$, where $f_0, f_1, \cdots, f_n$ are homogeneous polynomials of degree $q > 1$ without non-trivial common zeros. The polarization is automatically given by $L = O(1)$ since $f^*O(1) = O(q)$. We usually omit mentioning the polarization in this case. The set $\mathrm{Prep}(f)$ could be very complicated in general, except for those related to the square map, the Lattès map, and the Chebyshev polynomials introduced below.

EXAMPLE 2.6. **Square map.** Let $X = \mathbb{P}^n$ and $f(x_0, \cdots, x_n) = (x_0^2, \cdots, x_n^2)$. Set $L = O(1)$ since $f^*O(1) = O(2)$. Then $\mathrm{Prep}(f)$ is the set of points $(x_0, \cdots, x_n)$ with each coordinate x_j equal to zero or a root of unity.

EXAMPLE 2.7. **Lattès map.** Let E be an elliptic curve over K, and $\pi : E \to \mathbb{P}^1$ be any finite separable morphism. For example, π can be the natural double cover ramified exactly at the 2-torsion points. Then any endomorphism of E descends to an endomorphism f on $\mathbb{P}^1$, and thus gives a dynamical system on $\mathbb{P}^1$ if the degree is greater than 1. In that case, $\mathrm{Prep}(f) = \pi(E(\overline{K})_{tor})$.

EXAMPLE 2.8. **Chebyshev polynomial.** Let d be a positive integer. The d-th Chebyshev polynomial T_d be the unique polynomial satisfying

$$\cos(d\theta) = T_d(\cos(\theta)), \quad \forall \theta.$$

Equivalently, T_d is the unique polynomial satisfying

$$\frac{t^d + t^{-d}}{2} = T_d\left(\frac{t + t^{-1}}{2}\right), \quad \forall t.$$

Then T_d has degree d, integer coefficients and leading term 2^{d-1}. For each $d > 1$, $T_d : \mathbb{P}^1 \to \mathbb{P}^1$ gives an algebraic dynamical system. It is easy to see that the iteration $T_d^n = T_{d^n}$, and

$$\mathrm{Prep}(T_d) = \{\cos(r\pi) : r \in \mathbb{Q}\}.$$

2.2. Rigidity of preperiodic points. Let X be a projective variety over a field K, and L be an ample line bundle on X. Denote

$$\mathcal{H}(L) = \{f : X \to X \mid f^*L \cong L^{\otimes q} \text{ for some integer } q > 1\}.$$

Obviously $\mathcal{H}(L)$ is a semigroup in the sense that it is closed under composition of morphisms. For any $f \in \mathcal{H}(L)$, the triple (X, f, L) gives a polarized algebraic dynamical system.

Theorem 2.9 ([**BDM09**], [**YZ10a**]). *Let X be a projective variety over any field K, and L be an ample line bundle on X. For any $f, g \in \mathcal{H}(L)$, the following are equivalent:*

(a) $\mathrm{Prep}(f) = \mathrm{Prep}(g)$*;*
(b) $g\mathrm{Prep}(f) \subset \mathrm{Prep}(f)$*;*
(c) $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *is Zariski dense in* X*.*

The theorem is proved independently by Baker–DeMarco [**BDM09**] in the case $X = \mathbb{P}^1$, and by Yuan–Zhang [**YZ10a**] in the general case. The proofs of these two papers in the number field case are actually the same, but the extensions to arbitrary fields are quite different. See §7.2 for some details on the difference. The major extra work for the high-dimensional case in [**YZ10a**] is the Calabi–Yau theorem, which is essentially trivial in the one-dimensional case.

Remark 2.10. Assuming that K is a number field, the following are some previously known results:

(1) Mimar [**Mim97**] proved the theorem in the case $X = \mathbb{P}^1$.
(2) Kawaguchi–Silverman [**KS07**] obtains the result in the case that f is a square map on $\mathbb{P}^n$. They also obtain similar results on Lattès maps. Their treatment gives explicit classification of g.

A result obtained in the proof of Theorem 2.9 is the following analytic version:

Theorem 2.11 ([**YZ10a**]). *Let K be either $\mathbb{C}$ or $\mathbb{C}_p$ for some prime number p. Let X be a projective variety over K, L be an ample line bundle on X, and $f, g \in \mathcal{H}(L)$ be two polarized endomorphisms with* $\mathrm{Prep}(f) = \mathrm{Prep}(g)$*. Then the equilibrium measures $\mu_f = \mu_g$ on X^{an}. In particular, the Julia sets of f and g are the same in the complex case.*

Here μ_f denotes the equilibrium measure of (X, f, L) on X^{an}. See §4.2 for the definition. For the notion of Julia sets, we refer to [**DS10**].

2.3. Morphisms with the same preperiodic points. The conditions of Theorem 2.9 are actually equivalent to the equality of the canonical heights of those two dynamical systems. We will prove that in the number field case. Hence, two endomorphisms satisfying the theorem have the same arithmetic properties. It leads us to the following result:

Corollary 2.12. *Let (X, f, L) be a dynamical system over any field K. Then the set*

$$\mathcal{H}(f) = \{g \in \mathcal{H}(L) : \mathrm{Prep}(g) = \mathrm{Prep}(f)\}$$

is a semigroup, i.e., $g \circ h \in \mathcal{H}(f)$ for any $g, h \in \mathcal{H}(f)$.

Proof. By Theorem 2.9, we can write:

$$\mathcal{H}(f) = \{g \in \mathcal{H}(L) \mid g\mathrm{Prep}(f) \subset \mathrm{Prep}(f)\}.$$

Then it is easy to see that $\mathcal{H}(f)$ is a semigroup. □

Example 2.13. **Centralizer.** For any (X, f, L), denote

$$C(f) = \{g \in \mathcal{H}(L) \mid g \circ f = f \circ g\}.$$

Then $C(f)$ is a sub-semigroup of $\mathcal{H}(f)$. Take any $x \in \mathrm{Prep}(f)$, and we are going to prove $x \in \mathrm{Prep}(g)$. By definition, there exist $a > b \geq 0$ such that x lies in

$$\mathrm{Prep}(f)_{a,b} = \{y \in X(\overline{K}) : f^a(y) = f^b(y)\}.$$

The commutativity implies $g^c(x) \in \mathrm{Prep}(f)_{a,b}$ for any integer $c \geq 0$. It is easy to see that $\mathrm{Prep}(f)_{a,b}$ is a finite set. It follows that $g^c(x) = g^d(x)$ for some $c \neq d$, and thus $x \in \mathrm{Prep}(g)$.

We refer to [**Er90, DS02**] for a classification of commuting morphisms on $\mathbb{P}^n$. In fact, this was a reason for Fatou and Julia to study complex dynamical systems.

EXAMPLE 2.14. **Abelian variety.** In the abelian case $(A, [2], L)$, the semigroup $\mathcal{H}(f) = \mathcal{H}(L)$ is the set of all endomorphisms on A polarized by L.

There is a more precise description. Assume that the base field K is algebraically closed for simplicity. Denote by $\mathrm{End}(A)$ the ring of all endomorphisms on A in the category of algebraic varieties, and by $\mathrm{End}_0(A)$ the ring of all endomorphisms on A in the category of group varieties. Then $\mathrm{End}_0(A)$ consists of elements of $\mathrm{End}(A)$ fixing the origin 0 of A, and $\mathrm{End}(A) = \mathrm{End}_0(A) \rtimes A(K)$ naturally. One can check that $\mathcal{H}(L) = \mathcal{H}_0(L) \rtimes A(K)_{\mathrm{tor}}$ where $\mathcal{H}_0(L)$ is semigroup of elements of $\mathrm{End}_0(A)$ polarized by L.

The following list is more or less known in the literature. See [**KS07**] for the case of number fields. All the results can be verified by Theorem 2.11.

EXAMPLE 2.15. **Square map.** Let $X = \mathbb{P}^n_{\mathbb{C}}$ and f be the square map. Then

$$\mathcal{H}(f) = \{g(x_0, \cdots, x_n) = \sigma(\zeta_0 x_0^r, \cdots, \zeta_n x_n^r) : \\ r > 1,\ \zeta_j \text{ root of unity},\ \sigma \in S_{n+1}\}.$$

Here S_{n+1} denotes the permutation group acting on the coordinates.

EXAMPLE 2.16. **Lattès map.** If f is a Lattès map on $\mathbb{P}^1_{\mathbb{C}}$ descended from an elliptic curve, then $\mathcal{H}(f)$ consists of all the Lattès maps of degree greater than one descended from the same elliptic curve.

EXAMPLE 2.17. **Chebyshev polynomial.** For the Chebyshev polynomial T_d (with $d > 1$) on $\mathbb{P}^1_{\mathbb{C}}$, we have

$$\mathcal{H}(f) = \{\pm T_e :\ e \in \mathbb{Z}_{\geq 2}\}.$$

If f is not of the above special cases, we expect that $\mathcal{H}(f)$ is much "smaller".

CONJECTURE 2.18. *Fix two integers $n \geq 1$ and $d > 1$, and let $M_{n,d}$ be the moduli space of finite morphisms $f : \mathbb{P}^n \to \mathbb{P}^n$ of degree d. Then*

$$\mathcal{H}(f) = \{f, f^2, f^3, \cdots\}$$

for generic $f \in M_{n,d}(\mathbb{C})$.

The conjecture is more or less known in the literature for $n = 1$. For example, it is known for polynomial maps on $\mathbb{P}^1$ by the classification of Schmidt–Steinmetz [**SS95**]. In general, we may expect that the techniques of [**Er90, DS02**] can have some consequences on this conjecture.

3. Analytic metrics and measures

Let K be a complete valuation field, and denote by $|\cdot|$ its absolute value. If K is archimedean, then it is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ by Ostrowski's theorem. If K is non-archimedean, the usual examples are:

- p-adic field $\mathbb{Q}_p$,
- finite extensions of $\mathbb{Q}_p$,
- the completion $\mathbb{C}_p$ of algebraic closure $\overline{\mathbb{Q}}_p$,
- the field $k((t))$ of Laurent series over a field k.

3.1. Metrics on line bundles. Let K be as above, X be a projective variety over K, and L be a line bundle on X. A *K-metric* $\|\cdot\|$ on L is a collection of a $\overline{K}$-norm $\|\cdot\|$ over the fiber $L(x) = L_{\overline{K}}(x)$ of each algebraic point $x \in X(\overline{K})$ satisfying the following two conditions:

- The metric is *continuous* in the sense that, for any section s of L on a Zariski open subset U of X, the function $\|s(x)\|$ is continuous in $x \in U(\overline{K})$.
- The metric is *Galois invariant* in the sense that, for any section s as above, one has $\|s(x^\sigma)\| = \|s(x)\|$ for any $\sigma \in \mathrm{Gal}(\overline{K}/K)$.

If $K = \mathbb{C}$, then the metric is just the usual continuous metric of $L(\mathbb{C})$ on $X(\mathbb{C})$ in complex geometry. If $K = \mathbb{R}$, it is a continuous metric of $L(\mathbb{C})$ on $X(\mathbb{C})$ invariant under complex conjugation.

In both cases, we say that the metric is *semipositive* if its Chern form

$$c_1(L, \|\cdot\|) = \frac{\partial\overline{\partial}}{\pi i} \log \|s\| + \delta_{\mathrm{div}(s)}$$

is semipositive in the sense of currents. Here s is any non-zero meromorphic section of $L(\mathbb{C})$.

Next, assume that K is non-archimedean. We are going to define the notion of algebraic metrics and semipositive metrics.

Suppose $(\mathcal{X}, \mathcal{M})$ is an integral model of (X, L^e) for some positive integer e, i.e.,

- $\mathcal{X}$ is an integral scheme projective and flat over O_K with generic fiber $\mathcal{X}_K$ isomorphic to X,
- $\mathcal{M}$ is a line bundle over $\mathcal{X}$ such that the generic fiber $(\mathcal{X}_K, \mathcal{M}_K)$ is isomorphic to (X, L^e).

Any point x of $X(\overline{K})$ extends to a point $\bar{x}$ of $\mathcal{X}(O_{\overline{K}})$ by taking Zariski closure. Then the fiber $\mathcal{M}(\overline{x})$ is an $O_{\overline{K}}$-lattice of the one-dimensional $\overline{K}$-vector space $L^e(x)$. It induces a $\overline{K}$-norm $\|\cdot\|'$ on $L^e(x)$ by the standard rule that

$$\mathcal{M}(\overline{x}) = \{s \in L^e(x) : \|s\|' \leq 1\}.$$

It thus gives a $\overline{K}$-norm $\|\cdot\| = \|\cdot\|'^{1/e}$ on $L(x)$. Patching together, we obtain a continuous K-metric on L, and we call it *the algebraic metric induced by* $(\mathcal{X}, \mathcal{M})$.

This algebraic metric is called *semipositive* if $\mathcal{M}$ is relatively nef in the sense that $\mathcal{M}$ has a non-negative degree on any complete vertical curve on the special fiber of $\mathcal{X}$. In that case, L is necessarily nef on X.

The induced metrics are compatible with integral models. Namely, if we further have an integral model $(\mathcal{X}', \mathcal{M}')$ of (X, L^e) dominating $(\mathcal{X}, \mathcal{M})$, then they induce the same algebraic metric on L. Here we say that $(\mathcal{X}', \mathcal{M}')$ dominates $(\mathcal{X}, \mathcal{M})$ if there is a morphism $\pi : \mathcal{X}' \to \mathcal{X}$, extending the identity map on the generic fiber, such that $\pi^*\mathcal{M} = \mathcal{M}'$.

A K-metric $\|\cdot\|$ on L is called *bounded* if there exists one algebraic metric $\|\cdot\|_0$ on L such that the continuous function $\|\cdot\|/\|\cdot\|_0$ on (the non-compact space) $X(\overline{K})$ is bounded. If it is true for one algebraic metric, then it is true for all algebraic metrics. We leave it as an exercise. For convenience, in the archimedean case we say that all continuous metrics are *bounded*.

Now we are ready to introduce the notion of semipositive metrics in the sense of Zhang [**Zh95b**].

DEFINITION 3.1. Let K be a non-archimedean and (X, L) be as above. A continuous K-metric $\|\cdot\|$ on L is said to be *semipositive* if it is a uniform limit of semipositive algebraic metrics on L, i.e., there exists a sequence of semipositive algebraic K-metrics $\|\cdot\|_m$ on L such that the continuous function $\|\cdot\|_m/\|\cdot\|$ on $X(\overline{K})$ converges uniformly to 1.

In the following, we will introduce a measure $c_1(L, \|\cdot\|)^n$ on the analytic space X^{an} associated to X. We will start with review the definition of the Berkovich analytification in the non-archimedean case.

3.2. Berkovich analytification. If $K = \mathbb{R}, \mathbb{C}$, then for any K-variety X, define X^{an} to be the complex analytic space $X(\mathbb{C})$. It is a complex manifold if X is regular.

In the following, let K be a complete non-archimedean field with absolute value $|\cdot|$. For any K-variety X, denote by X^{an} the Berkovich space associated to X. It has many good topological properties in spite of its complicated structure. It is a Hausdorff and locally path-connected topological space with $\pi_0(X^{\mathrm{an}}) = \pi_0(X)$. Moreover, it is compact if X is proper.

We will briefly recall the definition and some basic properties here. For more details, we refer to Berkovich's book [**Be90**]. We also refer to the book Baker–Rumely [**BR10**] for an explicit description of $(\mathbb{P}^1)^{\mathrm{an}}$.

3.2.1. *Construction of the Berkovich analytification.* We first consider the affine case. Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over K, where A is a finitely generated ring over K. Then U^{an} is defined to be the set of multiplicative semi-norms on A extending the absolute value of K. Namely, U^{an} is the set of maps $\rho : A \to \mathbb{R}_{\geq 0}$ satisfying:

- (compatibility) $\rho|_K = |\cdot|$,
- (triangle inequality) $\rho(a+b) \leq \rho(a) + \rho(b),\ \forall a, b \in A$,
- (multiplicativity) $\rho(ab) = \rho(a)\rho(b),\ \forall a, b \in A$.

Any $f \in A$ defines a map

$$|f| : U^{\mathrm{an}} \longrightarrow \mathbb{R}, \quad \rho \longmapsto |f|_\rho := \rho(f)$$

Endow U^{an} with the coarsest topology such that $|f|$ is continuous for all $f \in A$.

For an general K-variety X, cover it by affine open subvarieties U. Then X^{an} is obtained by patching the corresponding U^{an} in the natural way. Each U^{an} is an open subspace of X^{an} by definition.

3.2.2. *Maps to the variety.* Denote by $|X|$ the underlying space of the scheme X, and by $|X|_c$ the subset of closed points of $|X|$. There is a natural surjective map $X^{\mathrm{an}} \to |X|$. Every point of $|X|_c$ has a unique preimage, and thus there is a natural inclusion $|X|_c \hookrightarrow X^{\mathrm{an}}$.

It suffices to describe these maps for the affine case $U = \mathrm{Spec}(A)$. Then the map $U^{\mathrm{an}} \to |U|$ is just $\rho \mapsto \ker(\rho)$. The kernel of $\rho : A \to \mathbb{R}_{\geq 0}$ is a prime ideal of A

by the multiplicativity. Let $x \in |U|_c$ be a closed point corresponding to a maximal idea m_x of A. Then A/m_x is a finite field extension of K, and thus has a unique valuation extending the valuation of K. This extension gives the unique preimage of x in $U^{\rm an}$.

3.2.3. *Shilov boundary.* Let $\mathcal{X}$ be an integral model of X over O_K. That is, $\mathcal{X}$ is an integral scheme, projective and flat over O_K with generic fiber $\mathcal{X}_K = X$. Further assume that $\mathcal{X}$ is normal. Then this integral model will determine some special points on $X^{\rm an}$.

Let η be an irreducible component of the special fiber $\overline{\mathcal{X}}$ of $\mathcal{X}$. It is a Weil divisor on $\mathcal{X}$ and thus gives an order function $v_\eta : K(X)^\times \to \mathbb{Z}$ by vanishing order on η. Here $K(X)$ denotes the function field of X, which is also the function field of $\mathcal{X}$. It defines a point in $X^{\rm an}$, locally represented by the semi-norm $\rho_\eta = e^{-av_\eta(\cdot)}$. Here $a > 0$ is the unique constant such that $\rho_\eta|_K$ agrees with the absolute value $|\cdot|$ of K. We call the point $\rho_\eta \in X^{\rm an}$ *the Shilov boundary corresponding to* η following [**Be90**].

Alternatively, one can define a (surjective) reduction map $r : X^{\rm an} \to |\overline{\mathcal{X}}|$ by the integral model. Then ρ_η is the unique preimage of the generic point of η in $X^{\rm an}$ under the reduction map.

3.3. Monge–Ampère measure and Chambert-Loir measure. Let K be any valuation field, X be a projective variety over K of dimension n, L be a line bundle on X, and $\|\cdot\|$ be a semipositive metric on L. There is a semipositive measure $c_1(L, \|\cdot\|)^n$ on the analytic space $X^{\rm an}$ with total volume

$$\int_{X^{\rm an}} c_1(L, \|\cdot\|)^n = \deg_L(X).$$

When $K = \mathbb{R}$ or $K = \mathbb{C}$, then $X^{\rm an}$ is just $X(\mathbb{C})$. If both X and $\|\cdot\|$ are smooth, then

$$c_1(L, \|\cdot\|)^n = \wedge^n c_1(L, \|\cdot\|)$$

is just the usual Monge–Ampère measure on $X^{\rm an}$ in complex analysis. In general, the wedge product can still be regularized to define a semipositive measure on $X^{\rm an}$.

In the following, assume that K is non-archimedean. We are going to introduce the measure $c_1(L, \|\cdot\|)^n$ on $X^{\rm an}$ constructed by Chambert-Loir [**CL06**].

First consider the case that algebraic case that the metric $(L, \|\cdot\|)$ was induced by a single normal integral model $(\mathcal{X}, \mathcal{M})$ of $(X, L^{\otimes e})$ over O_K. Denote by $|\overline{\mathcal{X}}|_g$ the set of irreducible components of $\overline{\mathcal{X}}$. Recall that any $\eta \in |\overline{\mathcal{X}}|_g$ corresponds to a Shilov boundary $\rho_\eta \in X^{\rm an}$. Define

$$c_1(L, \|\cdot\|)^n = \frac{1}{e^n} \sum_{\eta \in |\overline{\mathcal{X}}|_g} \deg_{\mathcal{M}|_\eta}(\eta)\ \delta_{\rho_\eta}$$

Here δ_{ρ_η} is the Dirac measure supported at ρ_η.

Now let $\|\cdot\|$ be a general semipositive metric. It is a uniform limit of semipositive metrics $\|\cdot\|_m$ induced by integral models. We can assume that these models are normal by passing to normalizations. Then we define

$$c_1(L, \|\cdot\|)^n = \lim_m c_1(L, \|\cdot\|_m)^n.$$

Chambert-Loir checked that the sequence converges weakly if K contains a countable and dense subfield. The general case was due to Gubler [**Gu08**].

EXAMPLE 3.2. **Standard metric.** Assume that $(X, L) = (\mathbb{P}^n_K, O(1))$. *The standard metric on* $O(1)$ is given by

$$\|s(z_0, \cdots, z_n)\| = \frac{|\sum_{i=0}^n a_i z_i|}{\max\{|z_0|, \cdots, |z_n|\}}, \quad s \in H^0(\mathbb{P}^n_{K_v}, O(1)).$$

Here $\sum_i a_i x_i$ is the linear form representing the section s. The quotient does not depend on the choice of the homogeneous coordinate $(z_0, \cdots, z_n)$. The metric is semipositive.

If K is archimedean, the metric is continuous but not smooth. The Monge–Ampère measure $c_1(L, \|\cdot\|)^n$ on $\mathbb{P}^n(\mathbb{C})$ is the push-forward of the probability Haar measure on the standard torus

$$S^n = \{(z_0, \cdots, z_n) \in \mathbb{P}^n(\mathbb{C}) : |z_0| = \cdots = |z_n|\}.$$

If K is non-archimedean, then the metric is the algebraic metric induced by the standard integral model $(\mathbb{P}^n_{O_K}, \mathcal{O}(1))$ over O_K. It follows that the Chambert-Loir measure $c_1(L, \|\cdot\|)^n = \delta_\xi$ where ξ is the Gauss point, i.e., the Shilov boundary corresponding to the (irreducible) special fiber of the integral model $\mathbb{P}^n_{O_K}$.

3.4. Calabi–Yau Theorem. Calabi [**Ca54, Ca57**] made the following famous conjecture:

Let $\omega \in H^{1,1}(X, \mathbb{C})$ be a Kähler class on a compact complex manifold X of dimension n, and Ω be a positive smooth (n, n)-form on X such that $\int_X \omega^n = \int_X \Omega$. Then there exists a Kähler form $\widetilde{\omega}$ in the class ω such that $\widetilde{\omega}^n = \Omega$.

Calabi also proved that the Kähler form is unique if it exists. The existence of the Kähler form, essentially a highly non-linear PDE, is much deeper, and was finally solved by the seminal work of S. T. Yau [**Yau78**]. Now the whole result is called the Calabi–Yau theorem.

We can write the theorem in a more algebraic way in the case that the Kähler class ω is algebraic, i.e., it is the cohomology class of a line bundle L. Then L is necessarily ample. There is a bijection

$$\begin{aligned} & \{\text{positive smooth metrics on } L\}/\mathbb{R}^\times_{>0} \\ \longrightarrow \quad & \{\text{Kähler forms in the class } \omega\} \end{aligned}$$

given by

$$\|\cdot\| \longmapsto c_1(L, \|\cdot\|).$$

Thus the Calabi–Yau theorem becomes:

Let L be an ample line bundle on a projective manifold X of dimension n, and Ω be a positive smooth (n, n)-form on X such that $\int_X \Omega = \deg_L(X)$. Then there exists a positive smooth metric $\|\cdot\|$ on L such that $\Omega = c_1(L, \|\cdot\|)^n$. Furthermore, the metric is unique up to scalar multiples.

Now it is easy to translate the uniqueness part to the non-archimedean case.

THEOREM 3.3 (Calabi–Yau theorem, uniqueness part). *Let K be a valuation field, X be a projective variety over K, and L be an ample line bundle over X. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semipositive metrics on L. Then*

$$c_1(L, \|\cdot\|_1)^{\dim X} = c_1(L, \|\cdot\|_2)^{\dim X}$$

if and only if $\dfrac{\|\cdot\|_1}{\|\cdot\|_2}$ is a constant.

REMARK 3.4. The "if" part of the theorem is trivial by definition.

For archimedean K, the positive smooth case is due to Calabi as we mentioned above, and the continuous semipositive case is due to Kolodziej [**Ko03**]. Afterwards Blocki [**Bl03**] provided a very simple proof of Kolodziej's result. The non-archimedean case was formulated and proved by Yuan–Zhang [**YZ10a**] following Blocki's idea.

The non-archimedean version of the existence part is widely open, though the one-dimensional case is trivial. In high-dimensions, we only know the case of totally degenerate abelian varieties by the work of Liu [**Liu10**]. In general, the difficulty lies in how to formulate the problem, where the complication is made by the big difference between positivity of a metric and positivity of its Chambert-Loir measures.

4. Analytic theory of algebraic dynamics

Let K be a complete valuation field as in §3, and let (X, f, L) be a dynamical system over K. In this section, we introduce invariant metrics, equilibrium measures, and some equidistribution in this setting.

4.1. Invariant metrics. Fix an isomorphism $f^*L = L^{\otimes q}$. Then there is a unique continuous K-metric $\|\cdot\|_f$ on L invariant under f such that

$$f^*(L, \|\cdot\|_f) = (L, \|\cdot\|_f)^{\otimes q}$$

is an isometry.

We can construct the metric by Tate's limiting argument as in §5.2. In fact, take any bounded K-metric $\|\cdot\|_0$ on L. Set

$$\|\cdot\|_f = \lim_{m\to\infty} \left((f^m)^*\|\cdot\|_0\right)^{1/q^m}.$$

PROPOSITION 4.1. *The limit is uniformly convergent, and independent of the choice of $\|\cdot\|_0$. Moreover, the invariant metric $\|\cdot\|_f$ is semipositive.*

PROOF. The archimedean case is classical, and the non-archimedean case is due to [**Zh95b**]. Denote

$$\|\cdot\|_m = \left((f^m)^*\|\cdot\|_0\right)^{1/q^m}.$$

Since both $\|\cdot\|_1$ and $\|\cdot\|_0$ are metrics on L, their quotient is a continuous function on $X(\overline{K})$. There exist a constant $C > 1$ such that

$$C^{-1} < \|\cdot\|_1/\|\cdot\|_0 < C.$$

In the non-archimedean case, it follows from the boundedness of the metrics; in the archimedean case, it follows from the continuity of the quotient and the compactness of $X(\overline{K})$. By

$$\|\cdot\|_{m+1}/\|\cdot\|_m = \left((f^m)^*(\|\cdot\|_1/\|\cdot\|_0)\right)^{1/q^m},$$

we have

$$C^{-1/q^m} < \|\cdot\|_{m+1}/\|\cdot\|_m < C^{1/q^m}.$$

Thus $\|\cdot\|_m$ is a uniform Cauchy sequence. Similar argument can prove the independence on $\|\cdot\|_0$. Furthermore, if we take $\|\cdot\|_0$ to be semipositive (and algebraic), then each $\|\cdot\|_m$ is semipositive, and thus the limit is semipositive.

□

4.2. Equilibrium measure. Recall that we have an invariant semipositive metric $\|\cdot\|_f$ on L. By §3, one has a semipositive measure $c_1(L, \|\cdot\|_f)^{\dim X}$ on the analytic space X^{an}. It is the Monge–Ampère measure in the archimedean case, and the Chambert-Loir measure in the non-archimedean case.

The total volume of this measure is $\deg_L(X)$, so we introduce the normalization

$$\mu_f = \frac{1}{\deg_L(X)} c_1(L, \|\cdot\|_f)^{\dim X}.$$

It is called *the equilibrium measure associated to* (X, f, L). It is f-invariant in the sense that

$$f^*\mu_f = q^{\dim X}\mu_f, \quad f_*\mu_f = \mu_f.$$

The first property follows from the invariance of $\|\cdot\|_f$, and the second property follows from the first one.

An alternative construction is to use Tate's limiting argument from any initial probability measure on X^{an} induced from a semipositive metric on L.

REMARK 4.2. The invariant metric $\|\cdot\|_f$ depends on the choice of isomorphism $f^*L = L^{\otimes q}$. Different isomorphisms may change the metric by a scalar. However, it does not change the Chambert-Loir measure. So the equilibrium measure μ_f does not depend on the choice of the isomorphism.

EXAMPLE 4.3. **Square map.** The metric and the measure introduced in Example 3.2 are the invariant metric and measure for the square map on $\mathbb{P}^n_K$.

EXAMPLE 4.4. **Abelian variety.** Let $X = A$ be an abelian variety over K, $f = [2]$ be the multiplication by 2, and L be a symmetric and ample line bundle on A. We will see that the equilibrium measure μ_f does not depend on L.

(1) If $K = \mathbb{C}$, then μ_f is exactly the probability Haar measure on $A(\mathbb{C})$.
(2) Assume that K is non-archimedean. For the general case, we refer to [**Gu10**]. Here we give a simple description of the case that K is a discrete valuation field and A has a split semi-stable reduction, i.e., the identity component of the special fiber of the Néron model is the extension of an abelian variety by $\mathbb{G}_m^r$ for some $0 \leq r \leq \dim(A)$. Then A^{an} contains the real torus $S^r = \mathbb{R}^r/\mathbb{Z}^r$ naturally as its skeleton in the sense that A^{an} deformation retracts to S^r. The measure μ_f is the push-forward to A^{an} of the probability Haar measure on S^r.

EXAMPLE 4.5. **Good reduction.** Assume that K is non-archimedean, and (X, f, L) has good reduction over K. Namely, there is a triple $(\mathcal{X}, \tilde{f}, \mathcal{L})$ over O_K extending (X, f, L) with $\mathcal{X}$ projective and smooth over O_K. Then μ_f is the Dirac measure of the Shilov boundary corresponding to the (irreducible) special fiber of $\mathcal{X}$.

4.3. Some equidistribution results. Here we introduce two equidistribution results in the complex setting. We will see that the equidistribution of small points almost subsumes these two theorems into one setting if the dynamical system is defined over a number field.

We first introduce the equidistribution of backward orbits. Let (X, f, L) be a dynamical system of dimension n over $\mathbb{C}$. For any point $x \in X(\mathbb{C})$, consider the pull-back

$$q^{-nm}(f^m)^*\delta_x = \frac{1}{q^{nm}} \sum_{z \in f^{-m}(x)} \delta_z.$$

Here δ_z denotes the Dirac measure, and the summation over $f^{-m}(x)$ are counted with multiplicities. Dinh and Sibony proved the following result:

THEOREM 4.6 ([**DS03, DS10**]). *Assume that X is normal. Then there is a proper Zariski closed subset E of X such that, for any $x \in X(\mathbb{C})$, the sequence $\{q^{-nm}(f^m)^*\delta_x\}_{m\geq 1}$ converges weakly to μ_f if and only if $x \notin E$.*

REMARK 4.7. By a result of Fakhruddin [**Fak03**], all dynamical systems (X, f, L) can be embedded to a projective space. So the above statement is equivalent to [**DS10**, Theorem 1.56]. We refer to [**Br65, FLM83, Ly83**] for the case $X = \mathbb{P}^1$.

Now we introduce equidistribution of periodic points. Let (X, f, L) over $\mathbb{C}$ be as above. For any $m \geq 1$, denote by P_m the set of periodic points in $X(\mathbb{C})$ of period m counted with multiplicity. Consider

$$\delta_{P_m} = \frac{1}{|P_m|}\sum_{z\in P_m}\delta_z.$$

The following result is due to Briend and Duval.

THEOREM 4.8 ([**BD99**]). *In the case $X = \mathbb{P}^n$, the sequence $\{\delta_{P_m}\}_{m\geq 1}$ converges weakly to μ_f.*

We refer to [**FLM83, Ly83**] for historical works on the more classical case of $X = \mathbb{P}^1$. As remarked in [**DS10**], one can replace P_m by the same set without multiplicity, or by the subset of repelling periodic points of period m.

In the non-archimedean case, both theorems are proved over $\mathbb{P}^1$ by Favre and Rivera-Letelier in [**FR06**]. The high-dimensional case is not known in the literature, but it is closely related to the equidistribution of small points if the dynamical system is defined over number fields. See §6.2 for more details.

5. Canonical height on algebraic dynamical systems

In this section we introduce the canonical height of dynamical systems over number fields using Tate's limiting argument. We will also see an expression in the spirit of Néron in Theorem 5.7. Here we restrict to number field, but all results are true for function fields of curves over finite fields.

For any number field K, denote by M_K the set of all places of K and normalize the absolute values as follows:

- If v is a real place, take $|\cdot|_v$ to be the usual absolute value on $K_v = \mathbb{R}$;
- If v is a complex place, take $|\cdot|_v$ to be the square of the usual absolute value on $K_v \simeq \mathbb{C}$;
- If v is a non-archimedean place, the absolute value $|\cdot|_v$ on K_v is normalized such that $|a|_v = (\#O_{K_v}/a)^{-1}$ for any $a \in O_{K_v}$.

In that way, the valuation satisfies the product formula

$$\prod_{v\in M_K}|a|_v = 1, \quad \forall\ a \in K^\times.$$

5.1. Weil's height machine. We briefly recall the definition of Weil heights. For a detailed introduction we refer to [**Se89**].

Let K be a number field, and $\mathbb{P}^n$ be the projective space over K. The *standard height function* $h : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$ is defined to be

$$h(x_0, x_1, \cdots, x_n) = \frac{1}{[E:K]} \sum_{w \in M_E} \log \max\{|x_0|_w, |x_1|_w, \cdots, |x_m|_w\},$$

where E is a finite extension of K containing all the coordinates x_i, and the summation is over all places w of E. By the product formula, the definition is independent of both the choice of the homogeneous coordinate and the choice of E. Then we have a well-defined function $h : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$.

EXAMPLE 5.1. **Height of algebraic numbers.** We introduce a height function $h : \overline{\mathbb{Q}} \to \mathbb{R}$ by

$$h(x) = \frac{1}{\deg(x)} \left(\log |a_0| + \sum_{z \in \mathbb{C} \text{ conj to } x} \log \max\{|z|, 1\} \right).$$

Here $a_0 \in \mathbb{Z}_{>0}$ is the coefficient of the highest term of the minimal polynomial of x in $\mathbb{Z}[t]$, $\deg(x)$ is the degree of the minimal polynomial, and the summation is over all roots of the minimal polynomial.

In particular, if $x = a/b \in \mathbb{Q}$ represented by two coprime integers $a, b \in \mathbb{Z}$, then

$$h(x) = \max\{\log |a|, \log |b|\}.$$

It is easy to verify that the height is compatible with the standard height on $\mathbb{P}^1_{\mathbb{Q}}$ under the identification $\mathbb{P}^1(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}} \cup \{\infty\}$.

Let X be a projective variety over K, and L be any ample line bundle on X. Let $i : X \to \mathbb{P}^n$ be any morphism such that $i^*O(1) = L^{\otimes d}$ for some $d \geq 1$. We obtain a height function $h_{L,i} = \frac{1}{d} h \circ i : X(\overline{K}) \to \mathbb{R}$ as the composition of $i : X(\overline{K}) \to \mathbb{P}^n(\overline{K})$ and $\frac{1}{d} h : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$. It depends on the choices of d and i.

More generally, let L be any line bundle on X. We can always write $L = A_1 \times A_2^{\otimes(-1)}$ for two ample line bundles A_1 and A_2 on X. For $k = 1, 2$, let $i_k : X \to \mathbb{P}^{n_k}$ be any two morphisms such that $i_k^* O(1) = A_k^{\otimes d_k}$ for some $d_k \geq 1$. We obtain a height function $h_{L,i_1,i_2} = h_{A_1,i_1} - h_{A_2,i_2} : X(\overline{K}) \to \mathbb{R}$. It depends on the choices of (A_1, A_2, i_1, i_2). However, the following result asserts that it is unique up to bounded functions.

THEOREM 5.2 (Weil's height machine). *The above construction $L \mapsto h_{L,i_1,i_2}$ gives a group homomorphism*

$$H : \mathrm{Pic}(X) \longrightarrow \frac{\{\text{functions } \phi : X(\overline{K}) \to \mathbb{R}\}}{\{\text{bounded functions } \phi : X(\overline{K}) \to \mathbb{R}\}}.$$

Here the group structure on the right-hand side is the addition. In particular, the coset of h_{L,i_1,i_2} on the right-hand side depends only on L.

We call a function $h_L : X(\overline{K}) \to \mathbb{R}$ a *Weil height corresponding to L* if it lies in the class of $H(L)$ on the right-hand side of the homorphism in the theorem. It follows that h_{L,i_1,i_2} is a Weil height corresponding to L. A basic and important property of the Weil height is the following Northcott property.

THEOREM 5.3 (Northcott property). *Let K be a number field, X be a projective variety over K, and L be an ample line bundle over X. Let $h_L : X(\overline{K}) \to \mathbb{R}$ be any Weil height corresponding to L. Then for any real numbers A, B, the set*

$$\{x \in X(\overline{K}) : \deg(x) < A,\ h_L(x) < B\}$$

is finite.

Here $\deg(x)$ denotes the degree of the residue field $K(x)$ of x over K. The theorem is easily reduced to the original height $h : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$, for which the property can be obtained explicitly.

5.2. Canonical height. Let (X, f, L) be a dynamical system over a number field K. We are introducing the canonical height following [**CS93**].

Let $h_L : X(\overline{K}) \to \mathbb{R}$ be any Weil height corresponding to L. The *canonical height* $h_f = h_{L,f} : X(\overline{K}) \to \mathbb{R}$ with respect to f is defined by Tate's limit argument

$$h_{L,f}(x) = \lim_{N\to\infty} \frac{1}{q^N} h_L(f^N(x)).$$

THEOREM 5.4. *The limit $h_f(x)$ always exists and is independent of the choice of the Weil height h_L. It has the following basic properties:*

(1) *$h_f(f(x)) = q h_f(x)$ for any $x \in X(\overline{K})$,*
(2) *$h_f(x) \geq 0$ for any $x \in X(\overline{K})$, and the equality holds if and only if x is preperiodic.*

The function $h_f : X(\overline{K}) \to \mathbb{R}$ is the unique Weil height corresponding to L and satisfying (1).

PROOF. The theorem easily follows from Theorem 5.2 and Theorem 5.3. □

EXAMPLE 5.5. **Square map.** For the square map on $\mathbb{P}^n$, the canonical height is exactly the standard height function $h : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$. To prove this, it suffices to verify that the standard height function satisfies Theorem 5.4 (1) under the square map.

EXAMPLE 5.6. **Néron–Tate height.** The prototype of canonical height is the case of abelian variety $(A, [2], L)$ as in Example 2.4. In that case, the canonical height $h_{L,[2]}$, usually denoted $\hat{h}_L$ and called the Néron–Tate height, is quadratic with respect to the group law. See [**Se89**] for example.

5.3. Decomposition to local heights. Let (X, f, L) be a dynamical system over a number field K as above. Fix an isomorphism $f^*L = L^{\otimes q}$ with $q > 1$. By last section, for each v of K, we have an f-invariant semipositive K_v-metric $\|\cdot\|_{f,v}$ on the line bundle L_{K_v} over X_{K_v}.

The data $\overline{L}_f = (L, \{\|\cdot\|_{f,v}\}_{v\in M_K})$ give an *adelic line bundle* in the sense of Zhang [**Zh95b**]. We will review the notion of adelic line bundles in §9.1, but we do not need it here for the moment.

THEOREM 5.7. *For any $x \in X(\overline{K})$, one has*

$$h_f(x) = -\frac{1}{\deg(x)} \sum_{v\in M_K} \sum_{z\in O(x)} \log \|s(z)\|_{f,v}.$$

Here s is any rational section of L regular and non-vanishing at x, and $O(x) = \mathrm{Gal}(\overline{K}/K)x$ is the Galois orbit of x naturally viewed as a subset of $X(\overline{K}_v)$ for every v.

PROOF. We will see that it is a consequence of Lemma 9.6 and Theorem 9.8. □

REMARK 5.8. The result is a decomposition of global height into a sum of local heights $-\sum_{z\in O(x)} \log \|s(z)\|_{f,v}$. It is essentially a generalization of the historical work of Néron [**Ne65**].

6. Equidistribution of small points

6.1. The equidistribution theorem. Let (X, f, L) be a polarized algebraic dynamical system over a number field K.

DEFINITION 6.1. Let $\{x_m\}_{m\geq 1}$ be an infinite sequence of $X(\overline{K})$.

(1) The sequence is called *generic* if any infinite subsequence is Zariski dense in X.
(2) The sequence is called h_f*-small* if $h_f(x_m) \to 0$ as $m \to \infty$.

Fix a place v of K. For any $x \in X(\overline{K})$, the Galois orbit $O(x) = \mathrm{Gal}(\overline{K}/K)x \subset X(\overline{K})$ is a natural subset of $X(\mathbb{C}_v)$. Denote its image under the natural inclusion $X(\mathbb{C}_v) \hookrightarrow X^{\mathrm{an}}_{\mathbb{C}_v}$ by $O_v(x)$. Define the probability measure on $X^{\mathrm{an}}_{\mathbb{C}_v}$ associated to the Galois orbit of x by

$$\mu_{x,v} := \frac{1}{\deg(x)} \sum_{z\in O_v(x)} \delta_z.$$

Recall that, associated to the dynamical system $(X_{\mathbb{C}_v}, f_{\mathbb{C}_v}, L_{\mathbb{C}_v})$, we have the equilibrium measure

$$\mu_{f,v} = \frac{1}{\deg_L(X)} c_1(L_{\mathbb{C}_v}, \|\cdot\|_{f,v})^{\dim X}$$

on $X^{\mathrm{an}}_{\mathbb{C}_v}$. Here $\|\cdot\|_{f,v}$ is the f-invariant metric depending on an isomorphism $f^*L = L^{\otimes q}$.

THEOREM 6.2. *Let (X, f, L) be a polarized algebraic dynamical system over a number field K. Let $\{x_m\}$ be a generic and h_f-small sequence of $X(\overline{K})$. Then for any place v of K, the probability measure $\mu_{x_m,v}$ converges weakly to the equilibrium measure $\mu_{f,v}$ on $X^{\mathrm{an}}_{\mathbb{C}_v}$.*

The theorem was first proved for abelian varieties at archimedean places v by Szpiro–Ullmo–Zhang [**SUZ97**], and was proved by Yuan [**Yu08**] in the full case. The function field analogue was due to Faber [**Fa09**] and Gubler [**Gu08**] independently. For a brief history, we refer to §6.3.

6.2. Comparison with the results in complex case. Theorem 6.2 is closely related to the equidistribution results in §4.3. Let (X, f, L) be a dynamical system of dimension n over a number field K. Fix an embedding $K \subset \mathbb{C}$ so that we can also view it as a complex dynamical system by base change.

We first look at the equidistribution of periodic points in Theorem 4.8. Recall that P_m is the set of periodic points in $X(\overline{K})$ of period m. It is stable under the action of $\mathrm{Gal}(\overline{K}/K)$, and thus splits into finitely many Galois orbits. Take one $x_m \in P_m$ for each m. Then $\{x_m\}_m$ is h_f-small automatically. Once $\{x_m\}_m$ is generic, the Galois orbit of x_m is equidistributed. A good control of the proportion of the "non-generic" Galois orbits in P_m would recover the equidistribution of P_m by Theorem 6.2.

Now we consider the equidistribution of backward orbits in Theorem 4.6. The situation is similar. For fixed $x \in X(\overline{K})$, the preimage $f^{-m}(x)$ is stable under the action of $\mathrm{Gal}(\overline{K}/K)$, and thus splits into finitely many Galois orbits. Take one $x_m \in f^{-m}(x)$ for each m. Then $\{x_m\}_m$ is h_f-small since $h_f(x_m) = q^{-m}h_f(x)$ by the invariant property of the canonical height. Once $\{x_m\}_m$ is generic, its Galois orbit is equidistributed. To recover the equidistribution of $f^{-m}(x)$, one needs to control the "non-generic" Galois orbits in $f^{-m}(x)$. For that, we would need to exclude x from the exceptional set E.

6.3. Brief history on equidistribution of small points. In the following we briefly review the major historical works related to the equidistribution. Due to the limitation of the knowledge of the author and the space of this article, the following list is far from being complete.

The study of the arithmetic of algebraic dynamics was started with the introduction of the canonical height on abelian varieties over number fields, also known as the Néron–Tate height, by Néron [**Ne65**] and an unpublished work of Tate. The generalizations to polarized algebraic dynamical systems was treated by Call–Silverman [**CS93**] using Tate's limiting argument and by Zhang [**Zh95b**] in the framework of Arakelov geometry. In Zhang's treatment, canonical heights of subvarieties are also defined.

The equidistribution of small points originated in the landmark work of Szpiro–Ullmo–Zhang [**SUZ97**]. One major case of their result is for dynamical systems on abelian varieties. The work finally lead to the solution of the Bogomolov conjecture by Ullmo [**Ul98**] and Zhang [**Zh98a**]. See [**Zh98b**] for an exposition of the history of the Bogomolov conjecture.

After the work of [**SUZ97**], the equidistribution was soon generalized to the square map on projective spaces by Bilu [**Bi97**], to almost split semi-abelian varieties by Chambert-Loir [**CL00**], and to all one-dimensional dynamical systems by Autissier [**Au01**].

All these works are equidistribution at complex analytic spaces with respect to embeddings $\sigma : K \hookrightarrow \mathbb{C}$. From the viewpoint of number theory, the equidistribution is a type of theorem that a global condition (about height) implies a local result (at σ). The embedding σ is just an archimedean place of K. Hence, it is natural to seek a parallel theory for equidistribution at non-archimedean places assuming the same global condition.

The desired non-archimedean analogue on $\mathbb{P}^1$, after the polynomial case considered by Baker–Hsia [**BH05**], was accomplished by three independent works: Baker–Rumely [**BR06**], Chambert-Loir [**CL06**], and Favre–Rivera-Letelier [**FR06**]. In the non-archimedean case, the v-adic manifold $X(\overline{K}_v)$ is totally disconnected and obviously not a good space to study measures. The Berkovich analytic space turns out the be the right one. In fact, all these three works considered equidistribution over the Berkovich space, though the proofs are different.

Chambert-Loir [**CL06**] actually introduced equilibrium measures on Berkovich spaces of any dimension, and his proof worked for general dimensions under a positivity assumption which we will mention below. Following the line, Gubler [**Gu07**] deduced the equidistribution on totally degenerate abelian varieties, and proved the Bogomolov conjecture for totally degenerate abelian varieties over function fields as a trophy. Gubler's equidistribution is on the tropical variety, which is just the skeleton of the Berkovich space.

Another problem raised right after [**SUZ97**] was to generalize the results to arbitrary polarized algebraic dynamical systems (X, f, L). The proof in [**SUZ97**] uses a variational principle, and the key is to apply the arithmetic Hilbert–Samuel formula due to Gillet–Soule [**GS90**] to the perturbations of the invariant line bundle. For abelian varieties, the invariant metrics are strictly positive and the perturbations are still positive, so the arithmetic Hilbert–Samuel formula is applicable. However, for a general dynamical systems including the square map, the invariant metrics are only semipositive and the perturbations may cause negative curvatures. In that case, one can not apply the arithmetic Hilbert–Samuel formula. The proof of Bilu [**Bi97**] takes full advantage of the explicit form of the square map. The works [**Au01, CL06**] still use the variational principle, where the key is to use a volume estimate on arithmetic surfaces obtained in [**Au01**] to replace the arithmetic Hilbert–Samuel formula.

Finally, Yuan [**Yu08**] proved an arithmetic bigness theorem (cf. Theorem 8.7), naturally viewed as an extension of the arithmetic Hilbert–Samuel formula. The theorem is inspired by a basic result of Siu [**Siu93**] in algebraic geometry. In particular, it makes the variational principle work in the general case. Hence, the full result of equidistribution of small points, for all polarized algebraic dynamical systems and at all places, was proved in [**Yu08**].

The function field analogue of the result in [**Yu08**] was obtained by Faber [**Fa09**] and Gubler [**Gu08**] independently in slightly different settings. It is also worth mentioning that Moriwaki [**Mo00**] treated equidistribution over finitely generated fields over number fields based on his arithmetic height.

7. Proof of the rigidity

Now we are ready to treat Theorem 2.9. We will give a complete proof for the case of number fields, which is easily generalized to global fields. The general case is quite technical, and we only give a sketch.

7.1. Case of number field. Assume that K is a number field. We are going to prove the following stronger result:

THEOREM 7.1. *Let X be a projective variety over a number field K, and L be an ample line bundle on X. For any $f, g \in \mathcal{H}(L)$, the following are equivalent:*

(a) $\mathrm{Prep}(f) = \mathrm{Prep}(g)$.
(b) $g\mathrm{Prep}(f) \subset \mathrm{Prep}(f)$.
(c) $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *is Zariski dense in X.*
(d) $\overline{L}_f \simeq \overline{L}_g$ *in the sense that, at every place v of K, there is a constant $c_v > 0$ such that $\|\cdot\|_{f,v} = c_v \|\cdot\|_{g,v}$. Furthermore, $\prod_v c_v = 1$, and $c_v = 1$ for all but finitely many v.*
(e) $h_f(x) = h_g(x)$ *for any $x \in X(\overline{K})$.*

REMARK 7.2. Kawaguchi–Silverman [**KS07**] proves that (d) $\Leftrightarrow$ (e) for $X = \mathbb{P}^n$.

REMARK 7.3. In (d) we do not get the simple equality $\overline{L}_f = \overline{L}_g$ mainly because the invariant metrics depend on the choice of isomorphisms $f^*L = L^{\otimes q}$ and $g^*L = L^{\otimes q'}$. One may pose a rigidification on L to remove this ambiguity.

7.1.1. *Easy implications.* Here we show

$$(\mathrm{d}) \Rightarrow (\mathrm{e}) \Rightarrow (\mathrm{a}) \Rightarrow (\mathrm{b}) \Rightarrow (\mathrm{c}).$$

First, (d) $\Rightarrow$ (e) follows from the formula

$$h_f(x) = -\frac{1}{\deg(x)} \sum_{v \in M_K} \sum_{z \in O(x)} \log \|s(z)\|_{f,v}, \quad x \in X(\overline{K}).$$

See Theorem 5.7.

Second, (e) $\Rightarrow$ (a) since preperiodic points are exactly points with canonical height equal to zero.

Third, (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) since $\mathrm{Prep}(f)$ is Zariski dense by Theorem 2.3.

It is also not hard to show (b) $\Rightarrow$ (c). Take any point $x \in \mathrm{Prep}(f)$. Consider the orbit $A = \{x, g(x), g^2(x), \cdots\}$. All points of A are defined over the residue field $K(x)$ of x. Condition (b) implies $A \subset \mathrm{Prep}(f)$, and thus points of A has bounded height. By the Northcott property, A is a finite set. In another word, $x \in \mathrm{Prep}(g)$. It follows that

$$\mathrm{Prep}(f) \subset \mathrm{Prep}(g).$$

Then (c) is true since $\mathrm{Prep}(f)$ is Zariski dense in X.

7.1.2. *From (c) to (d).* Assume that $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is Zariski dense in X. We can choose a generic sequence $\{x_m\}$ in $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$. The sequence is small with respect to both f and g since the canonical heights $h_f(x_m) = h_g(x_m) = 0$ for each m.

By Theorem 6.2, for any place $v \in M_K$, we have

$$\mu_{x_m,v} \to d\mu_{f,v}, \quad \mu_{x_m,v} \to d\mu_{g,v}.$$

It follows that $d\mu_{f,v} = d\mu_{g,v}$ as measures on $X^{\mathrm{an}}_{\mathbb{C}_v}$, or equivalently

$$c_1(L, \|\cdot\|_{f,v})^{\dim X} = c_1(L, \|\cdot\|_{g,v})^{\dim X}.$$

By the Calabi–Yau theorem in Theorem 3.3, we obtain constants $c_v > 0$ such that

$$\|\cdot\|_{f,v} = c_v \|\cdot\|_{g,v}.$$

Here $c_v = 1$ for almost all v. To finish the proof, we only need to check that the product $c = \prod_v c_v$ is 1.

In fact, by the height formula in Theorem 5.7 we have

$$h_f(x) = h_g(x) - \log c, \quad \forall\ x \in X(\overline{K}).$$

Then $c = 1$ by considering any x in $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$. It finishes the proof.

7.2. Case of general field. Now assume that K is an arbitrary field, and we will explain some ideas of extending the above proof. Readers who care more about number fields may skip the rest of this section.

It is standard to use Lefshetz principle to reduce K to a finitely generated field. In fact, Theorem 2.9 depends only on the data (X, f, g, L). Here X is defined by finitely many homogeneous polynomials, f and g are also defined by finitely many polynomials, and L is determined by a Cech cocycle in $H^1(X, O_X^*)$ which is still given by finitely many polynomials. Let K_0 be the field generated over the prime field ($\mathbb{Q}$ or $\mathbb{F}_p$) by the coefficients of all these polynomials. Then (X, f, g, L) is defined over K_0, and we can replace K by K_0 in Theorem 2.9.

In the following, assume that K is a finitely generated field over a prime field $k = \mathbb{Q}$ or $\mathbb{F}_p$. The key of the proof is a good notion of canonical height for dynamical systems over K.

7.2.1. *Geometric height.* Let (X, f, L) be a dynamical system over K as above. Fix a projective variety B over k with function field K, and fix an ample line bundle H on B. The geometric canonical height will be a function $h_f^H : X(\overline{K}) \to \mathbb{R}$.

Let $(\pi : \mathfrak{X} \to B, \mathcal{L})$ be an integral model of (X, L), i.e., $\pi : \mathfrak{X} \to B$ is a morphism of projective varieties over k with generic fiber X, and $\mathcal{L}$ is a line bundle on $\mathfrak{X}$ with generic fiber L. It gives a "Weil height" $h_{\mathcal{L}}^H : X(\overline{K}) \to \mathbb{R}$ by

$$h_{\mathcal{L}}^H(x) = \frac{1}{\deg(x)} (\pi^* H)^{\dim B - 1} \cdot \mathcal{L} \cdot \overline{x}.$$

Here $\overline{x}$ denotes the Zariski closure of x in $\mathfrak{X}$. Take the convention that $h_f^H(x) = 0$ if $d = 0$. The canonical height h_f^H is defined using Tate's limiting argument

$$h_f^H(x) = \lim_{N \to \infty} \frac{1}{q^N} h_{\mathcal{L}}^H(f^N(x)), \quad x \in X(\overline{K}).$$

It converges and does not depend on the choice of the integral model.

If $k = \mathbb{F}_p$, then h_f^H satisfies Theorem 5.4 (1) (2). The proof of Theorem 2.9 is very similar to the case of number fields.

If $k = \mathbb{Q}$, then $h_f^H(x) = 0$ does not imply $x \in \mathrm{Prep}(f)$ due to the failure of Northcott's property for this height function. However, it is proved to be true by Baker [**Ba09**] in the case that f is non-isotrivial on $X = \mathbb{P}^1$. With this result, Baker–DeMarco [**BDM09**] proves Theorem 2.9 using the same idea of number field case.

7.2.2. *Arithmetic height.* Assume that K is finitely generated over $k = \mathbb{Q}$. We have seen that the geometric height is not strong enough to satisfy Northcott's property. Moriwaki [**Mo00, Mo01**] introduced a new class of height functions on $X(\overline{K})$ which satisfies Northcott's property. The notion is further refined by Yuan–Zhang [**YZ10b**]. In the following, we will introduce these height functions, and we refer to §8.1 for some basic definitions in arithmetic intersection theory.

Fix an arithmetic variety $\mathcal{B}$ over $\mathbb{Z}$ with function field K, and fix an ample hermitian line bundle $\overline{\mathcal{H}}$ on $\mathcal{B}$. Moriwaki's canonical height is a function $h_f^{\overline{\mathcal{H}}} : X(\overline{K}) \to \mathbb{R}$ defined as follows.

Let $(\pi : \mathcal{X} \to \mathcal{B}, \overline{\mathcal{L}})$ be an integral model of (X, L), i.e., $\pi : \mathcal{X} \to \mathcal{B}$ is a morphism of arithmetic varieties over $\mathbb{Z}$ with generic fiber X, and $\overline{\mathcal{L}}$ is a hermtian line bundle on $\mathcal{X}$ with generic fiber L. It gives a "Weil height" $h_{\overline{\mathcal{L}}}^{\overline{\mathcal{H}}} : X(\overline{K}) \to \mathbb{R}$ by

$$h_{\overline{\mathcal{L}}}^{\overline{\mathcal{H}}}(x) = \frac{1}{\deg(x)} (\pi^* \overline{\mathcal{H}})^{\dim \mathcal{B} - 1} \cdot \overline{\mathcal{L}} \cdot \overline{x}$$

Here $\overline{x}$ denotes the Zariski closure of x in $\mathcal{X}$. The the canonical height $h_f^{\overline{\mathcal{H}}}$ is still defined using Tate's limiting argument

$$h_f^{\overline{\mathcal{H}}}(x) = \lim_{N \to \infty} \frac{1}{q^N} h_{\overline{\mathcal{L}}}^{\overline{\mathcal{H}}}(f^N(x)), \quad x \in X(\overline{K}).$$

It converges and does not depend on the choice of the integral model $(\mathcal{X}, \overline{\mathcal{L}})$. The height function $h_f^{\overline{\mathcal{H}}}$ satisfies Northcott's property and Theorem 5.4 (1) (2).

With this height function, Yuan–Zhang [**YZ10a**] proves Theorem 2.9. The proof uses the corresponding results of number field case by considering the fibers of $\mathcal{X}_{\mathbb{Q}} \to \mathcal{B}_{\mathbb{Q}}$. The proof is very techinical and we refer to [**YZ10b**].

In the end, we briefly introduce the arithmetic height function of [**YZ10b**]. Let $(\pi : \mathcal{X} \to \mathcal{B}, \overline{\mathcal{L}})$ and $x \in X(\overline{K})$ be as above. Still denote by $\overline{f^N(x)}$ the Zariski closure of $f^N(x)$ in $\mathcal{X}$. Consider the generically finite map $\pi : \overline{f^N(x)} \to \mathcal{B}$. If it is flat, we obtain a hermitian line bundle $\pi_*(\overline{\mathcal{L}}|_{\overline{f^N(x)}}) \in \widehat{\mathrm{Pic}}(\mathcal{B})$ as the norm of $\overline{\mathcal{L}}|_{\overline{f^N(x)}}$. Normalize it by $\overline{\mathcal{L}}_N = \frac{1}{q^N}\pi_*(\overline{\mathcal{L}}|_{\overline{f^N(x)}})$. The limiting behavior of $\overline{\mathcal{L}}_N$ gives information on the height of x.

The idea of [**YZ10b**] is to introduce a canonical topological group $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$, containing $\widehat{\mathrm{Pic}}(\mathcal{B})_{\mathbb{Q}}$ for all integral models $\mathcal{B}$ of K, such that $\{\overline{\mathcal{L}}_N\}_N$ converges in $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$. Denote the limit by $\mathfrak{h}_f(x)$. If any of $\pi : \overline{f^N(x)} \to \mathcal{B}$ is not flat, we may need to blow-up $\mathcal{B}$ to get flatness. Then $\overline{\mathcal{L}}_N$ is still defined, and the limit $\mathfrak{h}_f(x)$ still exists in $\widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$. In summary, we get a well-defined canonical height function

$$\mathfrak{h}_f : X(\overline{K}) \to \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}.$$

The height is independent of the integral models $\mathcal{B}$ and $(\mathcal{X}, \overline{\mathcal{L}})$. It is vector-valued, but still satisfies Northcott's property and Theorem 5.4 (1) (2) by a natural interpretation. It refines $h_f^{\overline{\mathcal{H}}}$ in that

$$h_f^{\overline{\mathcal{H}}}(x) = \mathfrak{h}_f(x) \cdot \overline{\mathcal{H}}^{\dim \mathcal{B} - 1}, \quad x \in X(\overline{K}).$$

The advantage is that it does not require the polarization $\overline{\mathcal{H}}$.

8. Positivity in arithmetic intersection theory

In algebraic geometry, the linear series of a line bundle is closely related to positivity properties of the line bundle in intersection theory. The involved positivity notions of line bundles include effectivity, ampleness, nefness and bigness. We refer to Lazarsfeld [**La04**] for a thorough introduction in this area.

Most of these positivity results can be proved (by much more effort) in the setting of Arakelov geometry, an intersection theory over arithmetic varieties developed by Arakelov [**Ar74**] and Gillet–Soulé [**GS90**].

In this section, we are going to review some of these arithmetic analogues. More precisely, we are going to introduce the intersection numbers of hermitian line bundles following [**De87, GS90**], the notion of ample hermitian line bundles following Zhang [**Zh95a**], and some results on the arithmetic volumes by Yuan [**Yu08, Yu09**]. The prototypes of most of these results can be found in [**La04**].

8.1. Top intersections of hermitian line bundles. Let $\mathcal{X}$ be an *arithmetic variety* of dimension $n+1$, i.e., an integral scheme, projective and flat over $\mathrm{Spec}(\mathbb{Z})$ of relative dimension n.

A *metrized line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$ is an invertible sheaf $\mathcal{L}$ over $\mathcal{X}$ together with an $\mathbb{R}$-metric $\|\cdot\|$ on $\mathcal{L}_{\mathbb{R}}$ in the sense of §3. Namely, it is a continuous metric of $\mathcal{L}(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$ invariant under the complex conjugation. We call $\overline{\mathcal{L}}$ a *hermitian line bundle* if the metric is smooth. Denote by $\widehat{\mathrm{Pic}}(\mathcal{X})$ the group of isometry classes of hermitian line bundles on $\mathcal{X}$.

The smoothness of the metric is clear if the generic fiber $\mathcal{X}_{\mathbb{Q}}$ is smooth. In general, the metric $\|\cdot\|$ is called *smooth* if the pull-back metric $f^*\|\cdot\|$ on $f^*\mathcal{L}_{\mathbb{C}}$

under any analytic map $f : \{z \in \mathbb{C}^n : |z| < 1\} \to \mathcal{X}(\mathbb{C})$ is smooth in the usual sense.

Now we introduce the top intersection

$$\widehat{\mathrm{Pic}}(\mathcal{X})^{n+1} \longrightarrow \mathbb{R}.$$

The intersection number is defined to be compatible with birational morphisms of arithmetic varieties, so we can assume that $\mathcal{X}$ is normal and $\mathcal{X}_{\mathbb{Q}}$ is smooth by pull-back to a generic desingularization of $\mathcal{X}$.

Let $\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \cdots, \overline{\mathcal{L}}_{n+1}$ be $n+1$ hermitian line bundles on $\mathcal{X}$, and let s_{n+1} be any non-zero rational section of $\mathcal{L}$ on $\mathcal{X}$. The intersection number is defined inductively by

$$\begin{aligned} & \overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_{n+1} \\ = \quad & \overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_n \cdot \mathrm{div}(s_{n+1}) - \int_{\mathrm{div}(s_{n+1})(\mathbb{C})} \log \|s_{n+1}\| c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_n). \end{aligned}$$

The right-hand side depends on $\mathrm{div}(s_{n+1})$ linearly, so it suffices to explain the case that $D = \mathrm{div}(s_{n+1})$ is irreducible.

If D is horizontal in the sense that it is flat over $\mathbb{Z}$, then

$$\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_n \cdot D = \overline{\mathcal{L}}_1|_D \cdot \overline{\mathcal{L}}_2|_D \cdots \overline{\mathcal{L}}_n|_D$$

is an arithmetic intersection on D. The integration on the right-hand side needs regularization if $D(\mathbb{C})$ is not smooth.

If D is a vertical divisor in the sense that it is a variety over $\mathbb{F}_p$ for some prime p, then the integration is zero, and

$$\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_n \cdot D = (\overline{\mathcal{L}}_1|_D \cdot \overline{\mathcal{L}}_2|_D \cdots \overline{\mathcal{L}}_n|_D) \log p.$$

Here the intersection is the usual intersection on the projective variety D.

One can check that the definition does not depend on the choice of s_{n+1} and is symmetric and multi-linear. Similarly, we have a multilinear intersection product

$$\widehat{\mathrm{Pic}}(\mathcal{X})^d \times Z_d(\mathcal{X}) \longrightarrow \mathbb{R}.$$

Here $Z_d(\mathcal{X})$ denotes the group of Chow cycles of codimension $n+1-d$ on $\mathcal{X}$ (before linear equivalence). For $D \in Z_d(\mathcal{X})$, the intersection

$$\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_d \cdot D = \overline{\mathcal{L}}_1|_D \cdot \overline{\mathcal{L}}_2|_D \cdots \overline{\mathcal{L}}_d|_D$$

is interpreted as above.

EXAMPLE 8.1. If $\dim \mathcal{X} = 1$, the intersection is just a degree map

$$\widehat{\deg} : \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \mathbb{R}.$$

If $\mathcal{X}$ is normal, then it is isomorphic to $\mathrm{Spec}(O_K)$ for some number field K. Then $\mathcal{L}_1$ is just an O_K-module locally free of rank one. The Hermitian metric $\|\cdot\|$ on $\mathcal{L}_1(\mathbb{C}) = \oplus_{\sigma: K \to \mathbb{C}} \mathcal{L}_\sigma(\mathbb{C})$ is a collection $\{\|\cdot\|_\sigma\}_\sigma$ of metrics on the complex line $\mathcal{L}_\sigma(\mathbb{C})$. The degree is just

$$\widehat{\deg}(\overline{\mathcal{L}}_1) = \log \#(\mathcal{L}_1 / O_K s) - \sum_{\sigma: K \to \mathbb{C}} \log \|s\|_\sigma.$$

Here s is any non-zero element of $\mathcal{L}_1$.

8.2. Arithmetic linear series and arithmetic volumes. Let $\overline{\mathcal{L}}$ be a hermitian line bundle on an arithmetic variety $\mathcal{X}$ of dimension $n+1$. The corresponding arithmetic linear series is the finite set

$$H^0(\mathcal{X}, \overline{\mathcal{L}}) = \left\{ s \in H^0(\mathcal{X}, \mathcal{L}) : \|s\|_{\sup} \leq 1 \right\}.$$

Here

$$\|s\|_{\sup} = \sup_{z \in \mathcal{X}(\mathbb{C})} \|s(z)\|$$

is the usual supremum norm on $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{C}}$. A non-zero element of $H^0(\mathcal{X}, \overline{\mathcal{L}})$ is called *an effective section of* $\overline{\mathcal{L}}$ on $\mathcal{X}$, and $\overline{\mathcal{L}}$ is called *effective* if $H^0(\mathcal{X}, \overline{\mathcal{L}})$ is nonzero.

Denote

$$h^0(\overline{\mathcal{L}}) = \log \# H^0(\mathcal{X}, \overline{\mathcal{L}}).$$

The *volume of* $\overline{\mathcal{L}}$ is defined as

$$\mathrm{vol}(\overline{\mathcal{L}}) = \limsup_{N \to \infty} \frac{h^0(N\overline{\mathcal{L}})}{N^{n+1}/(n+1)!}.$$

Here $N\overline{\mathcal{L}}$ denotes $\overline{\mathcal{L}}^{\otimes N}$ by the convention that we write tensor product additively.

To see the finiteness of $H^0(\mathcal{X}, \overline{\mathcal{L}})$, denote

$$B(\overline{\mathcal{L}}) = \{ s \in H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}} : \|s\|_{\sup} \leq 1 \}.$$

It is the corresponding unit ball in $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$ of the supremum norm. Then

$$H^0(\mathcal{X}, \overline{\mathcal{L}}) = H^0(\mathcal{X}, \mathcal{L}) \cap B(\overline{\mathcal{L}})$$

is the intersection of a lattice with a bounded set in $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$, so it must be finite.

To better study h^0 and vol, we introduce their "approximations" χ and vol_χ. Pick any Haar measure on $H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}$. Define

$$\chi(\overline{\mathcal{L}}) = \log \frac{\mathrm{vol}(B(\overline{\mathcal{L}}))}{\mathrm{vol}(H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}} / H^0(\mathcal{X}, \mathcal{L}))},$$

It is easy to see that the quotient is independent of the choice of the Haar measure. We further introduce

$$\mathrm{vol}_\chi(\overline{\mathcal{L}}) = \limsup_{N \to \infty} \frac{\chi(N\overline{\mathcal{L}})}{N^{n+1}/(n+1)!}.$$

By Minkowski's theorem on lattice points, we immediately have

$$h^0(\overline{\mathcal{L}}) \geq \chi(\overline{\mathcal{L}}) - h^0(\mathcal{L}_{\mathbb{Q}}) \log 2.$$

Here $h^0(\mathcal{L}_{\mathbb{Q}}) = \dim H^0(\mathcal{X}_{\mathbb{Q}}, \mathcal{L}_{\mathbb{Q}})$ is the usual one. It follows that

$$\mathrm{vol}(\overline{\mathcal{L}}) \geq \mathrm{vol}_\chi(\overline{\mathcal{L}}).$$

We will see that the other direction is true in the ample case.

One can verify that both vol and vol_χ are homogeneous of degree $n+1$ in the sense that $\mathrm{vol}(m\overline{\mathcal{L}}) = m^{n+1}\mathrm{vol}(\overline{\mathcal{L}})$ and $\mathrm{vol}_\chi(m\overline{\mathcal{L}}) = m^{n+1}\mathrm{vol}_\chi(\overline{\mathcal{L}})$ for any positive integer m. Hence, vol and vol_χ extend naturally to functions on $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{Q}}$ by the homogeneity. See [**Mo09, Ik10**].

Remark 8.2. Chinburg [**Chi91**] considered a similar limit defined from an adelic set. He called the counterpart of $\exp(-\mathrm{vol}_\chi(\overline{\mathcal{L}}))$ the (inner) sectional capacity. It measures "how many" algebraic points the adelic set can contain. See also [**RLV00, CLR03**].

8.3. Ample hermitian line bundles. As in the algebro-geometric case, $H^0(\mathcal{X}, N\overline{\mathcal{L}})$ generates $N\mathcal{L}$ under strong positivity conditions on $\overline{\mathcal{L}}$.

Following Zhang [**Zh95a**], a hermitian line bundle $\overline{\mathcal{L}}$ is called *ample* if it satisfies the following three conditions:

- $\mathcal{L}_{\mathbb{Q}}$ is ample on $\mathcal{X}_{\mathbb{Q}}$;
- $\overline{\mathcal{L}}$ is *relatively semipositive*, i.e., the Chern form $c_1(\overline{\mathcal{L}})$ of $\overline{\mathcal{L}}$ is semipositive and $\mathcal{L}$ is relatively nef in the sense that $\deg(\mathcal{L}|_C) \geq 0$ for any closed curve C in any special fiber of $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{Z})$;
- $\overline{\mathcal{L}}$ is *horizontally positive*, i.e., the intersection number $\overline{\mathcal{L}}^{\dim \mathcal{Y}} \cdot \mathcal{Y} > 0$ for any horizontal irreducible closed subvariety $\mathcal{Y}$ of $\mathcal{X}$.

One version of the main result of [**Zh95a**] is the following arithmetic Nakai–Moishezon theorem:

THEOREM 8.3 (Arithmetic Nakai–Moishezon, Zhang). *Let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on an arithmetic variety $\mathcal{X}$ with a smooth generic fiber $\mathcal{X}_{\mathbb{Q}}$. Then for any hermitian line bundle $\overline{\mathcal{E}}$ over $\mathcal{X}$, the $\mathbb{Z}$-module $H^0(\mathcal{X}, \mathcal{E}+N\mathcal{L})$ has a $\mathbb{Z}$-basis contained in $H^0(\mathcal{X}, \overline{\mathcal{E}} + N\overline{\mathcal{L}})$ for N large enough.*

We also have the following arithmetic version of the Hilbert–Samuel formula.

THEOREM 8.4 (Arithmetic Hilbert–Samuel, Gillet–Soulé, Zhang). *Let $\overline{\mathcal{L}}$ be a hermitian line bundle on an arithmetic variety of dimension $n+1$.*

(1) *If $\mathcal{L}_{\mathbb{Q}}$ is ample and $\overline{\mathcal{L}}$ is relatively semipositive, then* $\mathrm{vol}_\chi(\overline{\mathcal{L}}) = \overline{\mathcal{L}}^{n+1}$.
(2) *If $\overline{\mathcal{L}}$ is ample, then* $\mathrm{vol}(\overline{\mathcal{L}}) = \overline{\mathcal{L}}^{n+1}$.

In both cases, the "$\limsup$*" defining* $\mathrm{vol}(\overline{\mathcal{L}})$ *and* $\mathrm{vol}_\chi(\overline{\mathcal{L}})$ *are actually limits.*

The result for vol_χ is a consequence of the arithmetic Riemann–Roch theorem of Gillet–Soulé [**GS92**] and an estimate of analytic torsions of Bismut–Vasserot [**BV89**]. It is further refined by Zhang [**Zh95a**]. The result for vol is obtained from that for vol_χ by the Riemann–Roch theorem on lattice points in Gillet–Soulé [**GS92**] and the arithmetic Nakai–Moishezon theorem by Zhang [**Zh95a**] described above. See [**Yu08**, Corollary 2.7] for more details.

REMARK 8.5. In both cases, we can remove the condition that $\mathcal{L}_{\mathbb{Q}}$ is ample. Note that $\mathcal{L}_{\mathbb{Q}}$ is necessarily nef under the assumption that $\overline{\mathcal{L}}$ is relatively semipositive. The extensions can be obtained easily by Theorem 8.7 below.

There are three conditions in the definition of ampleness, where the third one is the most subtle. The following basic result asserts that the third one can be obtained from the first two after rescaling the norm.

LEMMA 8.6. *Let $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ be a hermitian line bundle on an arithmetic variety $\mathcal{X}$ with $\mathcal{L}_{\mathbb{Q}}$ ample and $\overline{\mathcal{L}}$ relatively semipositive. Then the hermitian line bundle $\overline{\mathcal{L}}(\alpha) = (\mathcal{L}, \|\cdot\|e^{-\alpha})$ is ample for sufficiently large real number α.*

PROOF. Since $\mathcal{L}_{\mathbb{Q}}$ is ample, we can assume there exist sections $s_1, \cdots, s_r \in H^0(\mathcal{X}, \mathcal{L})$ which are base-point free over the generic fiber. Let α be such that $\|s_i\|_{\sup} e^{-\alpha} < 1$ for all $i = 1, \cdots, r$. We claim that $\overline{\mathcal{L}}(\alpha)$ is ample.

We need to show that $(\overline{\mathcal{L}}(\alpha)|_{\mathcal{Y}})^{\dim \mathcal{Y}} > 0$ for any horizontal irreducible closed subvariety $\mathcal{Y}$. Assume that $\mathcal{X}$ is normal by normalization. We can find an s_j such that $\mathrm{div}(s_j)$ does not contain $\mathcal{Y}$. Then

$$\begin{aligned}(\overline{\mathcal{L}}(\alpha)|_{\mathcal{Y}})^{\dim \mathcal{Y}} &= (\overline{\mathcal{L}}(\alpha)|_{\mathrm{div}(s_j)|_{\mathcal{Y}}})^{\dim \mathcal{Y}-1} - \int_{\mathcal{Y}(\mathbb{C})} \log(\|s_j\| e^{-\alpha})\ c_1(\overline{\mathcal{L}})^{\dim \mathcal{Y}-1} \\ &> (\overline{\mathcal{L}}(\alpha)|_{\mathrm{div}(s_j)|_{\mathcal{Y}}})^{\dim \mathcal{Y}-1}.\end{aligned}$$

Now the proof can be finished by induction on $\dim \mathcal{Y}$. □

8.4. Volumes of hermitian line bundles. The following arithmetic version of a theorem of [**Siu93**] implies the equidistribution of small points in algebraic dynamics.

THEOREM 8.7 (Yuan [**Yu08**]). *Let $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ be two ample hermitian line bundles over an arithmetic variety $\mathcal{X}$ of dimension $n+1$. Then*

$$\mathrm{vol}_\chi(\overline{\mathcal{L}} - \overline{\mathcal{M}}) \geq \overline{\mathcal{L}}^{n+1} - (n+1)\ \overline{\mathcal{L}}^n \cdot \overline{\mathcal{M}}.$$

Note that the difference of two ample line bundles can be any line bundle, so the theorem is applicable to any line bundle. It is sharp when $\overline{\mathcal{M}}$ is "small". Another interesting property of the volume function is the log-concavity property.

THEOREM 8.8 (Log-concavity, Yuan [**Yu09**]). *Let $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ be two effective hermitian line bundles over an arithmetic variety $\mathcal{X}$ of dimension $n+1$. Then*

$$\mathrm{vol}(\overline{\mathcal{L}} + \overline{\mathcal{M}})^{\frac{1}{n+1}} \geq \mathrm{vol}(\overline{\mathcal{L}})^{\frac{1}{n+1}} + \mathrm{vol}(\overline{\mathcal{M}})^{\frac{1}{n+1}}.$$

The following consequence is the form we will use to prove the equidistribution theorem.

COROLLARY 8.9 (Differentiability). *Let $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ be two hermitian line bundle on an arithmetic variety of dimension $n+1$.*

(1) *If $\mathcal{L}_\mathbb{Q}$ is ample and $\overline{\mathcal{L}}$ is relatively semipositive, then*

$$\mathrm{vol}_\chi(\overline{\mathcal{L}} + t\overline{\mathcal{M}}) = (\overline{\mathcal{L}} + t\overline{\mathcal{M}})^{n+1} + O(t^2), \quad t \to 0.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \mathrm{vol}_\chi(\overline{\mathcal{L}} + t\overline{\mathcal{M}}) = (n+1)\overline{\mathcal{L}}^n \cdot \overline{\mathcal{M}}.$$

(2) *If $\overline{\mathcal{L}}$ is ample, then*

$$\mathrm{vol}(\overline{\mathcal{L}} + t\overline{\mathcal{M}}) = (\overline{\mathcal{L}} + t\overline{\mathcal{M}})^{n+1} + O(t^2), \quad t \to 0.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \mathrm{vol}(\overline{\mathcal{L}} + t\overline{\mathcal{M}}) = (n+1)\overline{\mathcal{L}}^n \cdot \overline{\mathcal{M}}.$$

PROOF. We prove the results by the following steps.

(a) Changing $\overline{\mathcal{L}}$ to $\overline{\mathcal{L}}(\alpha) = (\mathcal{L}, \|\cdot\| e^{-\alpha})$ for any real number α does not affect the result in (1). It follows from the simple facts

$$\begin{aligned}\mathrm{vol}_\chi(\overline{\mathcal{L}}(\alpha) + t\overline{\mathcal{M}}) &= \mathrm{vol}_\chi(\overline{\mathcal{L}} + t\overline{\mathcal{M}}) + \alpha(n+1)\mathrm{vol}(\mathcal{L}_\mathbb{Q} + t\mathcal{M}_\mathbb{Q}), \\ (\overline{\mathcal{L}}(\alpha) + t\overline{\mathcal{M}})^{n+1} &= (\overline{\mathcal{L}} + t\overline{\mathcal{M}})^{n+1} + \alpha(n+1)(\mathcal{L}_\mathbb{Q} + t\mathcal{M}_\mathbb{Q})^n.\end{aligned}$$

Here the classical volume $\mathrm{vol}(\mathcal{L}_\mathbb{Q} + t\mathcal{M}_\mathbb{Q})$ on the generic fiber equals the intersection number $(\mathcal{L}_\mathbb{Q} + t\mathcal{M}_\mathbb{Q})^n$ since $\mathcal{L}_\mathbb{Q} + t\mathcal{M}_\mathbb{Q}$ is ample for small $|t|$.

(b) It suffices to prove (1) for $\overline{\mathcal{L}}$ ample. In fact, assume that $\mathcal{L}_{\mathbb{Q}}$ is ample and $\overline{\mathcal{L}}$ is relatively semipositive. By Lemma 8.6, $\overline{\mathcal{L}}(\alpha)$ is ample for some real number α. Then it follows from (a).

(c) Assuming that $\overline{\mathcal{L}}$ is ample, then

$$\mathrm{vol}(\overline{\mathcal{L}}+t\overline{\mathcal{M}}) \geq \mathrm{vol}_{\chi}(\overline{\mathcal{L}}+t\overline{\mathcal{M}}) \geq (\overline{\mathcal{L}}+t\overline{\mathcal{M}})^{n+1}+O(t^2).$$

It suffices to consider the case $t>0$ by replacing $\overline{\mathcal{M}}$ by $-\overline{\mathcal{M}}$. As in the classical case, any hermitian line bundle can be written as the tensor quotient of two ample hermitian line bundles. Thus $\overline{\mathcal{M}}=\overline{\mathcal{A}}_1-\overline{\mathcal{A}}_2$ for ample hermitian line bundles $\overline{\mathcal{A}}_1$ and $\overline{\mathcal{A}}_2$. It follows that $\overline{\mathcal{L}}+t\overline{\mathcal{M}} = (\overline{\mathcal{L}}+t\overline{\mathcal{A}}_1)-t\overline{\mathcal{A}}_2$ is the difference of two ample $\mathbb{Q}$-line bundles. By Theorem 8.7,

$$\begin{aligned}\mathrm{vol}_{\chi}(\overline{\mathcal{L}}+t\overline{\mathcal{M}}) &\geq (\overline{\mathcal{L}}+t\overline{\mathcal{A}}_1)^{n+1}-(n+1)\,(\overline{\mathcal{L}}+t\overline{\mathcal{A}}_1)^{n}\cdot t\overline{\mathcal{A}}_2\\ &= (\overline{\mathcal{L}}+t\overline{\mathcal{M}})^{n+1}+O(t^2).\end{aligned}$$

(d) Assuming that $\overline{\mathcal{L}}$ is ample, then

$$\mathrm{vol}(\overline{\mathcal{L}}+t\overline{\mathcal{M}})^{\frac{1}{n+1}}+\mathrm{vol}(\overline{\mathcal{L}}-t\overline{\mathcal{M}})^{\frac{1}{n+1}} \geq 2\ \mathrm{vol}(\overline{\mathcal{L}})^{\frac{1}{n+1}}+O(t^2).$$

In fact, by (c),

$$\begin{aligned}\mathrm{vol}(\overline{\mathcal{L}}+t\overline{\mathcal{M}})^{\frac{1}{n+1}} &\geq \left((\overline{\mathcal{L}}+t\overline{\mathcal{M}})^{n+1}+O(t^2)\right)^{\frac{1}{n+1}}\\ &= (\overline{\mathcal{L}}^{n+1})^{\frac{1}{n+1}}\left(1+t\frac{\overline{\mathcal{L}}^{n}\cdot\overline{\mathcal{M}}}{\overline{\mathcal{L}}^{n+1}}+O(t^2)\right).\end{aligned}$$

The result remains true if we replace t by $-t$. Take the sum of the estimates for $\mathrm{vol}(\overline{\mathcal{L}}+t\overline{\mathcal{M}})$ and $\mathrm{vol}(\overline{\mathcal{L}}-t\overline{\mathcal{M}})$.

(e) Assuming that $\overline{\mathcal{L}}$ is ample, then

$$\mathrm{vol}(\overline{\mathcal{L}}+t\overline{\mathcal{M}})^{\frac{1}{n+1}}+\mathrm{vol}(\overline{\mathcal{L}}-t\overline{\mathcal{M}})^{\frac{1}{n+1}} \leq 2\ \mathrm{vol}(\overline{\mathcal{L}})^{\frac{1}{n+1}}.$$

Note that (c) implies that both $\overline{\mathcal{L}}+t\overline{\mathcal{M}}$ and $\overline{\mathcal{L}}-t\overline{\mathcal{M}}$ are effective when $|t|$ is small. Apply Theorem 8.8 to $\overline{\mathcal{L}}+t\overline{\mathcal{M}}$ and $\overline{\mathcal{L}}-t\overline{\mathcal{M}}$.

(f) The equalities in (d) and (e) are in opposite directions. They forces the equality

$$\mathrm{vol}(\overline{\mathcal{L}}+t\overline{\mathcal{M}}) = (\overline{\mathcal{L}}+t\overline{\mathcal{M}})^{n+1}+O(t^2).$$

It follows that

$$\mathrm{vol}_{\chi}(\overline{\mathcal{L}}+t\overline{\mathcal{M}}) = (\overline{\mathcal{L}}+t\overline{\mathcal{M}})^{n+1}+O(t^2).$$

□

Remark 8.10. The inequality in (c) is due to [**Yu08**], which is sufficient for the equidistribution. The proof of the other direction using the log-concavity is due to [**Che08c**]. The abstract idea is that $\mathrm{vol}(\overline{\mathcal{L}}+t\overline{\mathcal{M}})^{n+1}$ is a concave function in t by the log-concavity, and (c) asserts that $l(t)=t(n+1)\overline{\mathcal{L}}^{n}\cdot\overline{\mathcal{M}}$ is a line supporting the graph of the concave function, so the line must be a tangent line.

These results are naturally viewed in the theory of arithmetic big line bundles. A hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ is called *big* if $\mathrm{vol}(\overline{\mathcal{L}})>0$. To end this section, we mention some important properties of the volume function:

- Rumely–Lau–Varley [**RLV00**] proved that "lim sup" in the definition of $\mathrm{vol}_\chi(\overline{\mathcal{L}})$ is a limit if $\mathcal{L}_\mathbb{Q}$ is ample. They actually treated the more general setting proposed by Chinburg [**Chi91**].
- Moriwaki [**Mo09**] proved the continuity of "vol" at big line bundles.
- Chen [**Che08a**] proved that "lim sup" in the definition of $\mathrm{vol}(\overline{\mathcal{L}})$ is always a limit for any $\overline{\mathcal{L}}$. See [**Yu09**] for a different proof using Okounkov bodies.
- Chen [**Che08b**] and Yuan [**Yu09**] independently proved the arithmetic Fujita approximation theorem for any big $\overline{\mathcal{L}}$.
- Chen [**Che08c**] proved the differentiability of "vol" at big line bundles. Corollary 8.9 is just a special case, but it extends to the general case by the above arithmetic Fujita approximation theorem.
- Ikoma [**Ik10**] claimed a proof of the continuity of "vol_χ".

9. Adelic line bundles

The terminology of hermitian line bundles is not enough for the proof of Theorem 6.2. The reason is that the dynamical system can rarely be extended to an endomorphism of a single integral model over O_K. The problem is naturally solved by the notion of adelic line bundles introduced by Zhang [**Zh95a, Zh95b**].

9.1. Basic definitions. Let K be a number field, X be a projective variety of dimension n over K, and L be a line bundle on X. Recall that we have considered local analytic metrics in §3.

An *adelic metric* on L is a *coherent* collection $\{\|\cdot\|_v\}_v$ of bounded K_v-metrics $\|\cdot\|_v$ on L_{K_v} over X_{K_v} over all places v of K. That the collection $\{\|\cdot\|_v\}_v$ is *coherent* means that, there exist a finite set S of non-archimedean places of K and a (projective and flat) integral model $(\mathcal{X}, \mathcal{L})$ of (X, L) over $\mathrm{Spec}(O_K) - S$, such that the K_v-norm $\|\cdot\|_v$ is induced by $(\mathcal{X}_{O_{K_v}}, \mathcal{L}_{O_{K_v}})$ for all $v \in \mathrm{Spec}(O_K) - S$.

In the above situation, we write $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ and call it an *adelic line bundle on* X. We further call L *the generic fiber of* $\overline{L}$.

Let $(\mathcal{X}, \overline{\mathcal{M}})$ be a *hermitian integral model of* $(X, L^{\otimes e})$ for some positive integer e, i.e., $\mathcal{X}$ is an arithmetic variety over O_K, and $\overline{\mathcal{M}} = (\mathcal{M}, \|\cdot\|)$ is a hermitian line bundle on $\mathcal{X}$, such that the generic fiber $(\mathcal{X}_K, \mathcal{M}_K) = (X, L^{\otimes e})$. Then $(\mathcal{X}, \overline{\mathcal{M}})$ induces an adelic metric $\{\|\cdot\|_v\}_v$ on L. The metrics at non-archimedean places are clear from §3. The metrics at archimedean places are just given by the hermitian metric. In fact, we have

$$\mathcal{X}(\mathbb{C}) = \coprod_{\sigma: K \hookrightarrow \mathbb{C}} \mathcal{X}_\sigma(\mathbb{C}), \quad \mathcal{M}(\mathbb{C}) = \coprod_{\sigma: K \hookrightarrow \mathbb{C}} \mathcal{M}_\sigma(\mathbb{C}).$$

The left-hand sides denote $\mathbb{C}$-points over $\mathbb{Z}$, and the right-hand sides denote $\mathbb{C}$-points over O_K via $\sigma : O_K \to \mathbb{C}$. Then the hermitian metric on $\mathcal{M}(\mathbb{C})$ becomes a collection of metrics $\|\cdot\|'_\sigma$ on $\mathcal{M}_\sigma(\mathbb{C}) = L_\sigma^{\otimes e}(\mathbb{C})$ over all σ. Then $\|\cdot\|'^{1/e}_\sigma$ gives the archimedean part of the adelic metric. Such an induced metric is called *algebraic*, and it is called *relatively semipositive* if $\overline{\mathcal{L}}$ is relatively semipositive.

An adelic metric $\{\|\cdot\|_v\}_v$ on an ample line bundle L over X is called *relatively semipositive* if the adelic metric is a uniform limit of relatively semipositive algebraic adelic metrics. Namely, there exists a sequence $\{\{\|\cdot\|_{m,v}\}_v\}_m$ of relatively semipositive algebraic adelic metrics on L, and a finite set S of non-archimedean places of K, such that $\|\cdot\|_{m,v} = \|\cdot\|_v$ for any $v \in \mathrm{Spec}(O_K) - S$ and any m, and such that $\|\cdot\|_{m,v}/\|\cdot\|_v$ converges uniformly to 1 at all places v. In that case, we also say that the adelic line bundle $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ is *relatively semipositive*. It implies that the K_v-metric $\|\cdot\|_v$ on L_{K_v} is semipositive in the sense of §3.

An adelic line bundle is called *integrable* if it is isometric to the tensor quotient of two relatively semipositive adelic line bundles. Denote by $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ the group of isometry classes of integrable adelic line bundles.

EXAMPLE 9.1. Let (X, f, L) be a dynamical system over a number field K. Fix an isomorphism $f^*L = L^{\otimes q}$ as in the definition. For any place v of K, let $\|\cdot\|_{f,v}$ be the invariant K_v-metric of L_{K_v} on X_{K_v}. Then $\overline{L}_f = (L, \{\|\cdot\|_{f,v}\}_v)$ is a relatively semipositive adelic line bundle. It is f-invariant in the sense that $f^*\overline{L}_f = \overline{L}_f^{\otimes q}$.

9.2. Extension of the arithmetic intersection theory. Let X be a projective variety of dimension n over a number field K. By Zhang [**Zh95b**], the intersection of hermitian line bundles in §8.1 extends to a symmetric multi-linear form

$$\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}^{n+1} \longrightarrow \mathbb{R}.$$

Let $\overline{L}_1, \cdots, \overline{L}_{n+1}$ be integrable adelic line bundles on X, and we are going to define the number $\overline{L}_1 \cdot \overline{L}_2 \cdots \overline{L}_{n+1}$. It suffices to assume that every $\overline{L}_j$ is semipositive by linearity. For each $j = 1, \cdots, n+1$, assume that the metric of $\overline{L}_j$ on L_j is the uniform limit of adelic metrics $\{\{\|\cdot\|_{j,m,v}\}_v\}_m$, and each $\{\|\cdot\|_{j,m,v}\}_v$ is induced by a semipositive integral model $(\mathcal{X}_{j,m}, \overline{\mathcal{M}}_{j,m})$ of $(X, L^{\otimes e_{j,m}})$.

We may assume that $\mathcal{X}_{1,m} = \cdots = \mathcal{X}_{n+1,m}$ for each m. In fact, let $\mathcal{X}_m$ be an integral model of X dominating $\mathcal{X}_{j,m}$ for all j in the sense that there are birational morphisms $\pi_{j,m} : \mathcal{X}_m \to \mathcal{X}_{j,m}$ inducing the identity map on the generic fiber. Such a model always exists. For example, we can take $\mathcal{X}_m$ be the the Zariski closure of the image of the composition

$$X \hookrightarrow X^{n+1} \hookrightarrow \mathcal{X}_{1,m} \times_{O_K} \mathcal{X}_{2,m} \times_{O_K} \cdots \times_{O_K} \mathcal{X}_{n+1,m}.$$

Here the first map is the diagonal embedding, and the second map is the injection onto the generic fiber. Then replace $(\mathcal{X}_{j,m}, \overline{\mathcal{M}}_{j,m})$ by $(\mathcal{X}_m, \pi_{j,m}^*\overline{\mathcal{M}}_{j,m})$. It is easy to check that they induce the same adelic metrics.

Go back to the sequence of metrics $\{\|\cdot\|_{j,m,v}\}_v$ induced by $(\mathcal{X}_m, \overline{\mathcal{M}}_{j,m})$. Define

$$\overline{L}_1 \cdot \overline{L}_2 \cdots \overline{L}_{n+1} = \lim_{m\to\infty} \frac{1}{e_{1,m} \cdots e_{n+1,m}} \overline{\mathcal{M}}_{1,m} \cdot \overline{\mathcal{M}}_{2,m} \cdots \overline{\mathcal{M}}_{n+1,m}.$$

It is easy to verify that the limit exists and depends only on $\overline{L}_1, \cdots, \overline{L}_{n+1}$. See Zhang [**Zh95b**].

Similarly, we get an intersection pairing

$$\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}^{d+1} \times Z_d(X) \longrightarrow \mathbb{R}$$

by setting

$$\overline{L}_1 \cdot \overline{L}_2 \cdots \overline{L}_{d+1} \cdot D = \overline{L}_1|_D \cdot \overline{L}_2|_D \cdots \overline{L}_{d+1}|_D.$$

Here $Z_d(X)$ denotes the group of d-dimensional Chow cycles.

EXAMPLE 9.2. Let $\overline{L}$ be an integrable adelic line bundle on X. Fix a place v_0 and a constant $\alpha \in \mathbb{R}$. Change the metric $\|\cdot\|_{v_0}$ of $\overline{L}$ to $e^{-\alpha}\|\cdot\|_{v_0}$. Denote the new adelic line bundle by $\overline{L}(\alpha)$. Then it is easy to check that

$$(\overline{L}(\alpha))^{n+1} = \overline{L}^{n+1} + (n+1)\deg_L(X)\alpha.$$

Example 9.3. If $X = \mathrm{Spec}(K)$, then the line bundle L_1 on X is just a vector space over K of dimension one. We simply have

$$\widehat{\deg}(\overline{L}_1) = -\sum_v \log \|s\|_v.$$

Here s is any non-zero element of L_1, and the degree is independent of the choice of s by the product formula. One may compare it with example 8.1.

9.3. Extension of the volume functions. It is routine to generalize the definitions in §8.2 to the adelic line bundles.

Let K be a number field, X be a projective variety of dimension n over K, and $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ be an adelic line bundle on X. Define

$$\begin{aligned} H^0(X, \overline{L}) &= \{s \in H^0(X, L) : \|s\|_{v,\sup} \leq 1, \ \forall v\}, \\ h^0(\overline{L}) &= \log \# H^0(X, \overline{L}), \\ \mathrm{vol}(\overline{L}) &= \limsup_{N \to \infty} \frac{h^0(N\overline{L})}{N^{n+1}/(n+1)!}. \end{aligned}$$

Here for any place v of K,

$$\|s\|_{v,\sup} = \sup_{z \in X(\overline{K}_v)} \|s(z)\|_v$$

is the usual supremum norm on $H^0(X, L)_{K_v}$. Elements of $H^0(X, \overline{L})$ are called *effective sections* of $\overline{L}$.

Similarly, for any place v of K, denote

$$B_v(\overline{L}) = \{s \in H^0(X, L)_{K_v} : \|s\|_{v,\sup} \leq 1\}.$$

Further denote

$$\mathbb{B}(\overline{L}) = \prod_v B_v(\overline{L}) \ \subset H^0(X, L)_{\mathbb{A}_K}.$$

Here $\mathbb{A}_K$ is the adele ring of K. Then we introduce

$$\begin{aligned} \chi(\overline{L}) &= \log \frac{\mathrm{vol}(\mathbb{B}(\overline{L}))}{\mathrm{vol}(H^0(X, L)_{\mathbb{A}_K}/H^0(X, L))} + \frac{1}{2} h^0(L) \log |d_K|, \\ \mathrm{vol}_\chi(\overline{L}) &= \limsup_{N \to \infty} \frac{\chi(N\overline{L})}{N^{n+1}/(n+1)!}. \end{aligned}$$

Here d_K is the discriminant of K. The definition of $\chi(\overline{L})$ does not depend on the choice of a Haar measure on $H^0(X, L)_{\mathbb{A}_K}$.

One easily checks that the definitions are compatible with the hermitian case if the adelic line bundle is algebraic. Furthermore, it is easy to see that $h^0, \chi, \mathrm{vol}_\chi$ commute with uniform limit of adelic line bundles. The same is true for vol with some extra argument using an adelic version of the Riemann–Roch of Gillet–Soulé [**GS91**]. As in the hermitian case, some adelic version of Minkowski's theorem gives $\mathrm{vol}(\overline{L}) \geq \mathrm{vol}_\chi(\overline{L})$.

It is easy to generalize Corollary 9.4 for vol_χ to the adelic case, though we do not know if it is true for vol. We rewrite the statement here, since it will be used for the equidistribution.

Corollary 9.4. *Let X be a projective variety of dimension n over a number field K, and $\overline{L}, \overline{M}$ be two integrable adelic line bundles on X. Assume that $\overline{L}$ is relatively semipositive and that the generic fiber L is ample. Then*

$$\mathrm{vol}_{\chi}(\overline{L}+t\overline{M}) = (\overline{L}+t\overline{M})^{n+1} + O(t^2), \quad t \to 0.$$

Remark 9.5. One may naturally define ampleness of adelic line bundles to generalize Theorem 8.3, 8.4, 8.7, 8.8 and Lemma 8.6.

9.4. Heights defined by adelic line bundles.

9.4.1. *Weil height.* Let $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ be an adelic line bundle on a projective variety X over a number field K. Define *the height function* $h_{\overline{L}} : X(\overline{K}) \to \mathbb{R}$ *associated to* $\overline{L}$ by

$$h_{\overline{L}}(x) = \frac{1}{\deg(x)} \overline{L} \cdot x_{\mathrm{gal}}, \quad x \in X(\overline{K}).$$

Here x_{gal} is the closed point of X corresponding to the algebraic point x.

If $\overline{L}$ is integrable, *the height of any closed subvariety Y of $X_{\overline{K}}$ associated to $\overline{L}$* is defined by

$$h_{\overline{L}}(Y) = \frac{\overline{L}^{\dim Y+1} \cdot Y_{\mathrm{gal}}}{(\dim Y + 1)\deg_L(Y_{\mathrm{gal}})}.$$

Here Y_{gal} is the closed K-subvariety of X corresponding to Y, i.e. the image of the composition $Y \to X_{\overline{K}} \to X$. In particular, the height of the ambient variety X is

$$h_{\overline{L}}(X) = \frac{\overline{L}^{\dim X+1}}{(\dim X + 1)\deg_L(X)}.$$

We see that heights are just normalized arithmetic degrees.

Lemma 9.6. *For any $x \in X(\overline{K})$, one has*

$$h_{\overline{L}}(x) = -\frac{1}{\deg(x)} \sum_{v \in M_K} \sum_{z \in O(x)} \log \|s(z)\|_v.$$

Here s is any rational section of L regular and non-vanishing at x, and $O(x) = \mathrm{Gal}(\overline{K}/K)x$ is the Galois orbit of x naturally viewed as a subset of $X(\overline{K}_v)$ for every v.

Proof. It follows from the definitions. See also Example 9.3. □

Theorem 9.7. *For any adelic line bundle $\overline{L}$, the height function $h_{\overline{L}} : X(\overline{K}) \to \mathbb{R}$ is a Weil height corresponding to the line bundle L on X.*

Proof. We prove the result by three steps.

(1) Let $(X, L) = (\mathbb{P}^n_K, O(1))$, and endow L with the standard adelic metric described in Example 3.2. By Lemma 9.6, it is easy to check that the height function $h_{\overline{L}} : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$ is exactly the standard height function $h : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$ in §5.1.
(2) Go back to general X. We claim that, for any two adelic line bundles $\overline{L}_1 = (L_1, \{\|\cdot\|_{1,v}\}_v)$ and $\overline{L}_2 = (L_2, \{\|\cdot\|_{2,v}\}_v)$ with generic fiber $L_1 = L_2$, the difference $h_{\overline{L}_1} - h_{\overline{L}_1}$ is bounded. In fact, we have $\|\cdot\|_{1,v} = \|\cdot\|_{2,v}$ for almost all v, and $\|\cdot\|_{1,v}/\|\cdot\|_{2,v}$ is bounded for the remaining places. The claim follows from Lemma 9.6.

(3) By linearity, it suffices to prove the theorem for the case that L is very ample. Let $i : X \hookrightarrow \mathbb{P}^m$ be an embedding with $i^*O(1) = L$. By (2), it suffices to consider the case $\overline{L} = i^*\overline{O(1)}$ where $\overline{O(1)}$ is the line bundle $O(1)$ endowed with the standard metric. Recall that $h_{L,i} = h \circ i$ is the Weil height induced by the embedding. By (1), it is easy to see that $h_{L,i} = h_{\overline{L}}$. It proves the result.

□

9.4.2. *Canonical height.* Now we consider the dynamical case. Let (X, f, L) be a dynamical system over a number field K. By §5.2, we have the canonical height function $h_f = h_{L,f} : X(\overline{K}) \to \mathbb{R}$. Recall that $\overline{L}_f$ is the f-invariant adelic line bundle.

THEOREM 9.8. $h_{L,f} = h_{\overline{L}_f}$.

PROOF. By Theorem 9.7, $h_{\overline{L}_f}$ is a Weil height corresponding to L. Furthermore, $h_{\overline{L}_f}(f(x)) = qh_{\overline{L}_f}(x)$ by $f^*\overline{L}_f = \overline{L}_f^{\otimes q}$. Then $h_{\overline{L}_f} = h_{L,f}$ by the uniqueness of $h_{L,f}$ in Theorem 5.4. □

REMARK 9.9. Following Lemma 9.6, we can decompose canonical heights of points in terms of local heights. See Theorem 5.7.

By the theorem, it is meaningful to denote by h_f the height $h_{\overline{L}_f}$ of subvarieties of $X_{\overline{K}}$. The following result is a generalization of Theorem 5.4.

PROPOSITION 9.10. *Let Y be a closed subvariety of $X_{\overline{K}}$. Then the following are true:*

(1) $h_f(f(Y)) = qh_f(Y)$.
(2) $h_f(Y) \geq 0$, *and the equality holds if Y is preperiodic in the sense that $\{Y, f(Y), f^2(Y), \cdots\}$ is a finite set.*

PROOF. The arithmetic intersection satisfies the projection formula, which gives (1) by $f^*\overline{L}_f = \overline{L}_f^{\otimes q}$. The inequality $h_f(Y) \geq 0$ follows from the construction that $\overline{L}_f$ is a uniform limit of adelic metrics induced by ample hermitian models. The last property follows from (1). □

REMARK 9.11. The adelic line bundle $\overline{L}_f$ is "nef" in the sense that it is relatively semipositive, and it has non-negative intersection with any closed subvariety of X.

REMARK 9.12. The converse that $h_f(Y) = 0$ *implies Y is preperiodic* is not true. It was an old version of the dynamical Manin–Mumford conjecture. See Ghioca–Tucker–Zhang [**GTZ10**] for a counter-example by Ghioca and Tucker, and some recent formulations of the dynamical Manin–Mumford conjecture.

10. Proof of the equidistribution

10.1. Equidistribution in terms of adelic line bundles. We introduce an equidistribution theorem for relatively semipositive adelic line bundles, which includes the dynamical case by taking the invariant adelic line bundle.

Let X be a projective variety of dimension n over a number field K and let $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ be an adelic line bundle. Similar to 6.1, we make a definition of smallness.

DEFINITION 10.1. An infinite sequence $\{x_m\}_{m\geq 1}$ of $X(\overline{K})$ is called *$h_{\overline{L}}$-small* if $h_{\overline{L}}(x_m) \to h_{\overline{L}}(X)$ as $m \to \infty$.

Note that we require the limit of the heights to be $h_{\overline{L}}(X)$. It is compatible with the dynamical case since $h_f(X) = 0$ in the dynamical case by Proposition 9.10.

Recall that an infinite sequence of $X(\overline{K})$ is called *generic* if any infinite subsequence is Zariski dense in X. We further recall that

$$\mu_{x,v} = \frac{1}{\deg(x)} \sum_{z \in O_v(x)} \delta_z$$

is the probability measure on the analytic space $X^{\mathrm{an}}_{\mathbb{C}_v}$ at a place v of K associated to the Galois orbit of a point $x \in X(\overline{K})$. Denote by

$$\mu_{\overline{L},v} = \frac{1}{\deg_L(X)} c_1(L_{\mathbb{C}_v}, \|\cdot\|_v)^n$$

the normalized Monge–Ampère/Chambert-Loir measure on $X^{\mathrm{an}}_{\mathbb{C}_v}$.

THEOREM 10.2. *Let X be a projective variety over a number field K and let $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ be a relatively semipositive adelic line bundle on X with L ample. Let $\{x_m\}$ be a generic and $h_{\overline{L}}$-small sequence of $X(\overline{K})$. Then for any place v of K, the probability measure $\mu_{x_m,v}$ converges weakly to the measure $\mu_{\overline{L},v}$ on $X^{\mathrm{an}}_{\mathbb{C}_v}$.*

The theorem is proved by Yuan [**Yu08**], and it implies Theorem 6.2 immediately. The case that v is archimedean and $\|\cdot\|_v$ is strictly positive was due to Szpiro–Ullmo–Zhang [**SUZ97**], which implies their equidistribution on abelian varieties. The result is further generalized to the case that L is big and v is archimedean by Berman–Boucksom [**BB10**]. If L is big and v is non-archimedean, a similar result is true by Chen [**Che08c**].

10.2. Fundamental inequality. Let X be a projective variety of dimension n over a number field K, and $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ be an adelic line bundle on X. Following Zhang [**Zh95a, Zh95b**], the *essential minimum* of the height function $h_{\overline{L}}$ is

$$e_{\overline{L}}(X) = \sup_{Y \subset X} \inf_{x \in X(\overline{K}) - Y(\overline{K})} h_{\overline{L}}(x).$$

Here the sup is taken over all closed subvarieties Y of X.

Let $\{x_m\}_m$ be a generic sequence in $X(\overline{K})$. By definition,

$$\liminf_{m\to\infty} h_{\overline{L}}(x_m) \geq e_{\overline{L}}(X).$$

Furthermore, the equality can be attained by some sequences. Thus we have an alternative definition

$$e_{\overline{L}}(X) = \inf_{\{x_m\}_m \text{ generic}} \liminf_{m\to\infty} h_{\overline{L}}(x_m).$$

THEOREM 10.3 (fundamental inequality, Zhang). *Assume that L is big. Then*

$$e_{\overline{L}}(X) \geq \frac{\mathrm{vol}_\chi(\overline{L})}{(n+1)\mathrm{vol}(L)}.$$

In particular, if $\overline{L}$ is relatively semipositive and L is ample, then

$$e_{\overline{L}}(X) \geq h_{\overline{L}}(X).$$

Recall that, in the classical sense, L is big if

$$\mathrm{vol}(L) = \lim_{N\to\infty} \frac{h^0(NL)}{N^n/n!} > 0.$$

The existence of the limit follows from Fujita's approximation theorem. See [**La04**] for example.

It is worth noting that, for each $d = 1, \cdots, n$, Zhang [**Zh95a, Zh95b**] acutally introduced a number $e_{\overline{L},d}(X)$ by restricting $\mathrm{codim}(Y) \geq d$ in the definition of the essential minimum. Then the essential minimum is just $e_{\overline{L}}(X) = e_{\overline{L},1}(X)$. The fundamental inequality is an easy part of his theorem on successive minima.

REMARK 10.4. In the setting of Theorem 10.2, the existence of a generic and $h_{\overline{L}}$-small sequence is equivalent to the equality $e_{\overline{L}}(X) = h_{\overline{L}}(X)$. The equality is true in the dynamical case since both sides are zero, but it is very hard to check in general.

To prove the theorem, we need the following Minkowski type result:

LEMMA 10.5. *Let $K, X, \overline{L}$ be as above. Fix a place v_0 of K. Then for any $\epsilon > 0$, there exists a positive integer N and a non-zero section $s \in H^0(X, NL)$ satisfying*

$$\frac{1}{N} \log \|s\|_{v_0,\sup} \leq -\frac{\mathrm{vol}_\chi(\overline{L})}{(n+1)\mathrm{vol}(L)} + \epsilon$$

and

$$\log \|s\|_{v,\sup} \leq 0, \quad \forall\, v \neq v_0.$$

PROOF. It is a consequence of the adelic version of Minkowski's theorem. See [**BG06**, Appendix C] for example. □

Now we prove Theorem 10.3. Let $s \in H^0(X, NL)$ be as in the lemma. By Lemma 9.6, for any $x \in X(\overline{K}) - |\mathrm{div}(s)|(\overline{K})$,

$$h_{\overline{L}}(x) = -\frac{1}{N} \cdot \frac{1}{\deg(x)} \sum_{v\in M_K} \sum_{z\in O(x)} \log \|s(z)\|_v.$$

Then we immediately have

$$h_{\overline{L}}(x) \geq \frac{\mathrm{vol}_\chi(\overline{L})}{(n+1)\mathrm{vol}(L)} - \epsilon, \quad \forall\, x \in X(\overline{K}) - |\mathrm{div}(s)|(\overline{K}).$$

Let $\epsilon \to 0$. We obtain the theorem.

10.3. Variational principle. Now we are ready to prove Theorem 10.2. To illustrate the idea, we first consider the archimedean case. We use the variational principle of [**SUZ97**].

Fix an archimedean place v. Let ϕ be a real-valued continuous function on $X^{\mathrm{an}}_{\mathbb{C}_v}$. The goal is to prove

$$\lim_{m\to\infty} \int_{X^{\mathrm{an}}_{\mathbb{C}_v}} \phi \mu_{x_m,v} = \int_{X^{\mathrm{an}}_{\mathbb{C}_v}} \phi \mu_{\overline{L},v}.$$

By density, it suffices to assume ϕ is *smooth* in the sense that, there exists an embedding of $X^{\mathrm{an}}_{\mathbb{C}_v}$ into a projective manifold M, such that ϕ can be extended to a smooth function on M.

Denote by $\overline{O}(\phi)$ the trivial line bundle with the trivial metric at all places $w \neq v$, and with metric given by $\|1\|_v = e^{-\phi}$ at v. Denote $\overline{L}(\phi) = \overline{L} + \overline{O}(\phi)$. The smoothness of ϕ implies that $\overline{O}(\phi)$ is an integrable adelic line bundle.

Let t be a small positive rational number. By the fundamental inequality,

$$\liminf_{m\to\infty} h_{\overline{L}(t\phi)}(x_m) \geq \frac{\mathrm{vol}_\chi(\overline{L}(t\phi))}{(n+1)\deg_L(X)}.$$

Here we have used the assumption that $\{x_m\}_m$ is generic.

Note that $\overline{L}(t\phi) = \overline{L} + t\overline{O}(\phi)$. By the differentiability in Corollary 9.4,

$$\mathrm{vol}_\chi(\overline{L}(t\phi)) = (\overline{L}(t\phi))^{n+1} + O(t^2).$$

It follows that

$$\liminf_{m\to\infty} h_{\overline{L}(t\phi)}(x_m) \geq h_{\overline{L}(t\phi)}(X) + O(t^2). \tag{10.1}$$

On the other hand, it is easy to verify that

$$h_{\overline{L}(t\phi)}(x_m) = h_{\overline{L}}(x_m) + t\int_{X^{\mathrm{an}}_{\mathbb{C}_v}} \phi\mu_{x_m,v}, \tag{10.2}$$

$$h_{\overline{L}(t\phi)}(X) = h_{\overline{L}}(X) + t\int_{X^{\mathrm{an}}_{\mathbb{C}_v}} \phi\mu_{\overline{L},v} + O(t^2). \tag{10.3}$$

In fact, the first equality follows from Lemma 9.6, and the second equality follows from

$$\overline{L}^n \cdot \overline{O}(\phi) = \int_{X^{\mathrm{an}}_{\mathbb{C}_v}} \phi\ c_1(L_{\mathbb{C}_v}, \|\cdot\|_v)^n.$$

Note that we have assumed

$$\lim_{m\to\infty} h_{\overline{L}}(x_m) = h_{\overline{L}}(X).$$

Then (10.1), (10.2) and (10.3) imply

$$\liminf_{m\to\infty} \int_{X^{\mathrm{an}}_{\mathbb{C}_v}} \phi\mu_{x_m,v} \geq \int_{X^{\mathrm{an}}_{\mathbb{C}_v}} \phi\mu_{\overline{L},v}.$$

Replacing ϕ by $-\phi$ in the inequality, we get the other direction. Hence,

$$\lim_{m\to\infty} \int_{X^{\mathrm{an}}_{\mathbb{C}_v}} \phi\mu_{x_m,v} = \int_{X^{\mathrm{an}}_{\mathbb{C}_v}} \phi\mu_{\overline{L},v}.$$

It finishes the proof.

10.4. Non-archimedean case. Now we consider Theorem 10.2 for any non-archimedean place v. The proof is parallel to the archimedean case, except that we use "model functions" to replace "smooth functions". The key is that model functions are dense in the space of continuous functions by a result of Gubler.

To introduce model functions, we consider a slightly general setting. Fix a projective variety Y over a non-archimedean field k, and consider integral models of Y over finite extensions of k. Let E be a finite extension of k, and let $(\mathcal{Y}, \mathcal{M})$ be a (projective and flat) integral model of (Y_E, O_{Y_E}) over O_E. It induces a metric $\|\cdot\|_{\mathcal{M}}$ on O_{Y_E}. Then $-\log\|1\|_{\mathcal{M}}^{1/e}$, for any positive integer e, is a function on $|Y_E|$, and extends to a unique continuous function on Y_E^{an}. It pull-backs to a function on $Y_{\mathbb{C}_k}^{\mathrm{an}}$ via the natural projection $Y_{\mathbb{C}_k}^{\mathrm{an}} \to Y_E^{\mathrm{an}}$. Here $\mathbb{C}_k$ denotes the completion of the algebraic closure $\overline{k}$ of k. The resulting function on $Y_{\mathbb{C}_k}^{\mathrm{an}}$ is called a *model function* on $Y_{\mathbb{C}_k}^{\rm an}$. Alternatively, it is also induced by the integral model $(\mathcal{Y}_{O_{\mathbb{C}_k}}, \mathcal{M}_{O_{\mathbb{C}_k}})$ of $(Y_{\mathbb{C}_k}, O_{Y_{\mathbb{C}_k}})$.

THEOREM 10.6 (Gubler). *The vector space over $\mathbb{Q}$ of model functions is uniformly dense in the space of real-valued continuous functions on $Y_{\mathbb{C}_k}^{\rm an}$.*

The result is stated in [**Gu08**, Proposition 3.4] in this form. It is essentially a combination of Gubler [**Gu98**, Theorem 7.12] and [**Yu08**, Lemma 3.5].

Now we are ready to prove Theorem 10.2 when v is non-archimedean. By the density theorem proved above, it suffices to prove

$$\lim_{m\to\infty}\int_{X_{\mathbb{C}_v}^{\rm an}} \phi\mu_{x_m,v} = \int_{X_{\mathbb{C}_v}^{\rm an}} \phi\mu_{\overline{L},v}.$$

for any model function $\phi = -\log\|1\|_{\mathcal{M}}$ induced by an O_E-model $(\mathcal{X},\mathcal{M})$ of (X_E, O_{X_E}) for any finite extension E of K_v. By passing to finite extensions of K, it is not hard to reduce it to the case that $E = K_v$.

Denote by $\overline{O}(\phi)$ the trivial line bundle with the trivial metric at all places $w \neq v$, and with metric given by $\|1\|_v = e^{-\phi}$ at v. Denote $\overline{L}(\phi) = \overline{L} + \overline{O}(\phi)$. It is easy to extend $(\mathcal{X},\mathcal{M})$ to an integral model over O_K which induces the adelic line bundle $\overline{O}(\phi)$. It follows that $\overline{O}(\phi)$ is an *integrable* adelic line bundle.

Now the proof is the same as the archimedean case. Still consider the variation $\overline{L}(t\phi) = \overline{L} + t\overline{O}(\phi)$ for a small rational number t. Note that (10.1) only requires $\overline{O}(\phi)$ to be integrable, and (10.2) and (10.3) are true by definition.

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