On Faltings heights of abelian varieties with complex multiplication

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Abstract. This expository article introduces some conjectures and theorems related to the Faltings heights of abelian varieties with complex multiplication. The topics include the Colmez conjecture, the averaged Colmez conjecture, and the André–Oort conjecture.

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1. Introduction

The celebrated Gross–Zagier formula is an equality between the central derivative of certain Rankin-Selberg L-function of modular form of weight two and the Néron–Tate height of certain Heegner cycle on an modular curve. It is an astonishing identity of type "*derivative* = *height*."

In this survey, we introduce Colmez's conjectural equality between the Faltings height of a CM abelian variety and the logarithmic derivative of the relevant Artin L-function, which is another identity of

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type "*derivative = height*." We will also introduce some known results related to this conjecture. The topics of this survey include the following:

(1) the definition of Faltings height by Faltings [**Fa**],
(2) the Colmez conjecture by Colmez [**Co**],
(3) the averaged Colmez conjecture proved by Yuan–Zhang [**YZ**] and Andreatta–Goren–Howard–Madapusi-Pera [**AGHM**] independently.
(4) the proof of the André–Oort conjecture assuming the averaged Colmez conjecture by Tsimerman [**Ts**].

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2. Faltings heights

To define the Faltings height of an abelian variety over a number field, we start with the arithmetic degree of a hermitian line bundle in the setting of Arakelov geometry.

2.1. Arithmetic degrees. Let F be a number field, and denote by O_F its ring of integers. Recall that a line bundle $\mathcal{L}$ (i.e. invertible sheaf) over $\mathrm{Spec}(O_F)$ corresponds to an O_F-module, locally free of rank one. By abuse of notations, we also write $\mathcal{L}$ for this O_F-module.

By a *hermitian line bundle* over $\mathrm{Spec}(O_F)$, we mean a pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$, consisting of a line bundle $\mathcal{L}$ over $\mathrm{Spec}(O_F)$ and a collection $\|\cdot\| = \{\|\cdot\|_\sigma\}_\sigma$, where $\|\cdot\|_\sigma$ is a metric of the complex line $\mathcal{L}_\sigma = \mathcal{L} \otimes_\sigma \mathbb{C}$ for each embedding $\sigma : F \to \mathbb{C}$. The metrics are required to be invariant under the action of the complex conjugate.

The *arithmetic degree* of the hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ is defined to be

$$\widehat{\deg}(\overline{\mathcal{L}}) = \log \#(\mathcal{L}/sO_F) - \sum_{\sigma:F\to\mathbb{C}} \log \|s\|_\sigma.$$

Here $s \in \mathcal{L}$ is a nonzero element, and we will see that the definition is independent of the choice of s. In fact, for any (nonzero) prime ideal $\wp \subset O_F$, define a $F_\wp$-metric on $\mathcal{L} \otimes_{O_F} O_{F_\wp}$ by setting $\|s\|_\wp = N(\wp)^{-\mathrm{ord}_\wp(s)}$. Then the definition is equivalent to the more symmetric formula

$$\widehat{\deg}(\overline{\mathcal{L}}) = -\sum_{\wp\subset O_F} \log \|s\|_\wp - \sum_{\sigma:F\to\mathbb{C}} \log \|s\|_\sigma.$$

Then we see that the definition is independent of the choice of $s \in \mathcal{L}$ since a different choice is of the form as for some $a \in F^\times$, which does not change the arithmetic degree by the product formula of absolute values of F.

One can take inverse and tensor products of hermitian line bundles, and the arithmetic degree is additive under these operations.

2.2. Faltings heights. Let A be an abelian variety of dimension g over a number field F. Denote by O_F the integer ring of F as before. Let $\mathcal{A}$ be the Néron model of A over O_F. The *Hodge bundle* of A is defined to be

$$\omega_{\mathcal{A}} = \epsilon^* \Omega^g_{\mathcal{A}/O_F} = \pi_* \Omega^g_{\mathcal{A}/O_F},$$

where $\pi : \mathcal{A} \to \mathrm{Spec}(O_F)$ denotes the structure morphism and $\epsilon : \mathrm{Spec}(O_F) \to \mathcal{A}$ denotes the identity section. Then $\omega_{\mathcal{A}}$ is a line bundle over $\mathrm{Spec}(O_F)$.

There is a *canonical hermitian metric* $\|\cdot\| = \{\|\cdot\|_\sigma\}_\sigma$ on $\omega_{\mathcal{A}}$ given by

$$\|\alpha\|^2_\sigma = \frac{1}{(2\pi)^g} \left| \int_{A_\sigma(\mathbb{C})} \alpha \wedge \overline{\alpha} \right|,$$

where $\sigma : F \to \mathbb{C}$ is any embedding and

$$\alpha \in \omega_{\mathcal{A}} \otimes_\sigma \mathbb{C} = \Gamma(A_\sigma(\mathbb{C}), \Omega^g_{A_\sigma(\mathbb{C})/\mathbb{C}})$$

is any global holomorphic g-form on the complex torus $A_\sigma(\mathbb{C})$. Note that we have used the normalizing factor $1/(2\pi)^g$, while the normalizing factor $1/2^g$ was used by Faltings [**Fa**, §3].

The pair $\overline{\omega}_{\mathcal{A}} = (\omega_{\mathcal{A}}, \|\cdot\|)$ forms a hermitian line bundle over $\mathrm{Spec}(O_F)$. Define the *unstable Faltings height* of A to be

$$h'(A) = \frac{1}{[F:\mathbb{Q}]} \widehat{\deg}(\overline{\omega}_{\mathcal{A}}).$$

We put the word "unstable" in the name because the height may decrease under base change unless A has a semistable reduction over F. Therefore, define the *stable Faltings height* of A to be

$$h(A) = h'(A_{F'})$$

where F' is a finite extension of F such that $A_{F'}$ has everywhere semistable reduction over $O_{F'}$. The definition does not depend on the choice of F', and is invariant under base change.

The Colmez conjecture is about the stable Faltings height. For simplicity, when we say "Faltings height" in the following, we always mean "stable Faltings height."

REMARK 2.1. Let A be an abelian variety of dimension g over a number field F. By replacing F by a finite extension if necessary, we can assume that A has semistable reduction over F and that the class of $\omega_{\mathcal{A}}$ in $\mathrm{Cl}(O_F)$ is trivial. Thus we can write $\omega_{\mathcal{A}} = O_F\alpha$ for some $\alpha \in \Gamma(\mathrm{Spec}(O_F), \omega_{\mathcal{A}}) = \Gamma(\mathcal{A}, \Omega^g_{\mathcal{A}/O_F})$. If $g = 1$, α is the usual Néron differential over the elliptic curve. In terms of α, the Faltings height is just

$$h(A) = -\frac{1}{2[F:\mathbb{Q}]} \sum_{\sigma : F \to \mathbb{C}} \log \left(\frac{1}{(2\pi)^g} \left| \int_{A_\sigma(\mathbb{C})} \alpha \wedge \overline{\alpha} \right| \right).$$

It is a rational linear combination of the logarithms of periods in the sense of Kontsevich–Zagier [**KZ**].

3. The Colmez conjecture

The goal of this section is to introduce the Colmez conjecture. We have already defined Faltings height, so the major work here is to describe the Artin L-function appeared in the conjecture. We will also recall the classical Chowla–Selberg formula, which can be considered as a special case of the Colmez conjecture.

3.1. Abelian varieties with complex multiplication. We first recall the basics of abelian varieties with complex multiplications. Most results below can be found in [**Mi**, §10]. For a complete treatment, we refer to Shimura [**Sh**].

Let E/F be an CM extension; i.e., F is a totally real number field, and E is a totally imaginary quadratic field extension of F. Denote $g = [F : \mathbb{Q}]$ and $c : E \to E$ the nontrivial element of $\mathrm{Gal}(E/F)$.

By a *CM type* of E, we mean a subset Φ of $\mathrm{Hom}(E, \mathbb{C})$ such that

$$\Phi \cap \Phi^c = \emptyset, \quad \Phi \cup \Phi^c = \mathrm{Hom}(E, \mathbb{C}).$$

In other words, Φ is a subset of $\mathrm{Hom}(E, \mathbb{C})$ which picks exactly one element in each conjugate pair of elements of $\mathrm{Hom}(E, \mathbb{C})$. We also say that (E, Φ) is a *CM type*.

By *an abelian variety with complex multiplication by* E, we mean an abelian variety A of dimension g over $\mathbb{C}$, together with an injection $i : E \to \mathrm{End}_{\mathbb{C}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ of rings. One can verify that there is a unique CM type Φ of E such that

$$\mathrm{tr}(i(\alpha)|_{\mathrm{Lie}(A)}) = \sum_{\sigma \in \Phi} \sigma(\alpha), \quad \forall \alpha \in E.$$

In this case, we also say that A is *of CM type* (E, Φ). If moreover the maximal order O_E acts on A, i.e., $i(O_E) \subset \mathrm{End}_{\mathbb{C}}(A)$, then we say that A is *of CM type* (O_E, Φ).

For convenience, for any subfield $k \subset \mathbb{C}$, we say that an abelian variety A over k has complex multiplication by E (resp. is of CM type (E,Φ), of CM type (O_E,Φ)) if the base change $A_{\mathbb{C}}$ over $\mathbb{C}$ satisfies this property.

The following are some basic results of the theory of abelian varieties with complex multiplication:

(1) Let (E,Φ) be a CM type with $[E:\mathbb{Q}]=2g$. Let $O_E \to \mathbb{C}^g$ be the injection given by the (distinct) elements of Φ. Then O_E is a lattice of $\mathbb{C}^g$ under this injection, and the quotient $\mathbb{C}^g/O_E$ is an abelian variety with CM type (O_E,Φ).
(2) Any two abelian varieties of the same CM type (E,Φ) are isogenous over $\mathbb{C}$.
(3) Any abelian variety with complex multiplication is defined over a number field.
(4) Any abelian variety with complex multiplication over a number field has potentially good reduction everywhere.

We refer to [Mi, §10] for the proofs.

Let A be an abelian variety of CM type (O_E,Φ) over $\mathbb{C}$. Then A is defined over a number field H, so it makes sense to consider the Faltings height $h(A)\in\mathbb{R}$, which is independent of the choice of H. By [Co, Theorem 0.3(ii)], the height $h(A)$ depends only on the CM type Φ. We denote it by $h(A_\Phi)$ to emphasize the dependence on Φ.

3.2. Colmez conjecture. Colmez [Co] associates an Artin character to the CM type (E,Φ) as follows. Denote by $\mathbb{Q}^{\mathrm{CM}}$ the compositum of all CM fields over $\mathbb{Q}$. By [Sh, Lem. 18.2], the compositum of two CM fields is also a CM field, and the Galois closure of a CM field is also a CM field. Therefore, $\mathbb{Q}^{\mathrm{CM}}$ is Galois over $\mathbb{Q}$. Denote $G=\mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/\mathbb{Q})$. The complex conjugate c is an element in the center of G. Fix inclusions $\mathbb{Q}^{\mathrm{CM}}\subset\overline{\mathbb{Q}}\subset\mathbb{C}$. Then G acts on $\mathrm{Hom}(E,\mathbb{C})=\mathrm{Hom}(E,\mathbb{Q}^{\mathrm{CM}})$ by its action on $\mathbb{Q}^{\mathrm{CM}}$. Define a function $A^0_{E,\Phi}:G\to\mathbb{C}$ by

$$A^0_{E,\Phi}(\sigma)=\int_G |\Phi\cap(\tau\sigma\tau^{-1}\Phi)|d\tau.$$

Here the integral uses the Haar measure on G with total volume 1, and the integrand is the order of the intersection of the two subsets of $\mathrm{Hom}(E,\mathbb{Q}^{\mathrm{CM}})$. The integral is essentially a finite sum by compactness, and $A^0_{E,\Phi}$ takes only finitely many values. See [Co, §0.4] for an equivalent definition of $A^0_{E,\Phi}$.

By definition, we have

$$A^0_{E,\Phi}(\sigma)+A^0_{E,\Phi}(c\sigma)=g,\quad \sigma\in G.$$

Define a function $f_\Phi : G \to \mathbb{C}$ by

$$f_\Phi(\sigma) = A^0_{E,\Phi}(\sigma) - \frac{1}{2}g.$$

Then we have

$$f_\Phi(\sigma) + f_\Phi(c\sigma) = 0, \quad \sigma \in G.$$

By definition, f_Φ is a class function on G, i.e., it is a function on the conjugacy classes of G. One can write f_Φ uniquely as a linear combination of characters associated to finite-dimensional irreducible representations π of G over $\mathbb{C}$, and then use these irreducible representations to define L-functions. Alternatively, for the sake of analytic continuation, by Brauer's induction theorem, we can write

$$f_\Phi = \sum_{(M,\chi)} a_{(M,\chi)} \mathrm{Ind}^G_{\mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/M)}(\chi), \quad a_{(M,\chi)} \in \mathbb{C}.$$

Here the pair (M,χ) consists of a finite extension M of $\mathbb{Q}$ in $\mathbb{Q}^{\mathrm{CM}}$ and a continuous homomorphism $\chi : \mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/M) \to \mathbb{C}^\times$. By abuse of notation, $\mathrm{Ind}^G_{\mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/M)}(\chi)$ means the character of the induced representation in the summation. The summation has only finitely many nonzero terms. Define

$$Z(f_\Phi, s) = \sum_{(M,\chi)} a_{(M,\chi)} \frac{L'(\chi,s)}{L(\chi,s)}, \qquad \log\mu(f_\Phi) = \sum_{(M,\chi)} a_{(M,\chi)} \log\mu_\chi.$$

Here μ_χ is the conductor of χ for the functional equation. The definition is independent of the choice of the decomposition of f_Φ.

Finally, the Colmez conjecture ([Co, Thm. 0.3, Conj. 0.4]) is as follows:

Conjecture 3.1 (Colmez conjecture). *For any CM type (E,Φ),*

$$h(A_\Phi) = -Z(f_\Phi, 0) - \frac{1}{2}\log\mu(f_\Phi).$$

The above equation is equivalent to the conjectural formula of [Co, Thm. 0.3, Conj. 0.4] on the Faltings height. In fact, to convert our equation to the one in the loc. cit., use

$$Z(f_\Phi, 0) = Z(A^0_{E,\Phi}, 0) - \frac{1}{2}g \cdot \frac{\zeta'(0)}{\zeta(0)} = Z(A^0_{E,\Phi}, 0) - \frac{1}{2}g\log(2\pi).$$

This gives an extra term $\frac{1}{2}g\log(2\pi)$ on the right-hand side, which is also equal to the difference of the Faltings heights due to the different normalizing factors of Hermitian metrics.

Now we check that the function $Z(f_\Phi, s)$ is holomorphic at $s = 0$, so that $Z(f_\Phi, 0)$ is well-defined. Go back to the decomposition

$$f_\Phi = \sum_{(M,\chi)} a_{(M,\chi)} \mathrm{Ind}^G_{\mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/M)}(\chi).$$

Note that M is either a CM field or a totally real field by [Sh, Lem. 18.2]. We first claim that we can choose a decomposition such that all M are totally real. In fact, if M is a CM field, denote by M_+ its maximal totally real subfield. Then we have an isomorphism

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/M_+) = \mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/M) \times \{1, c\}.$$

As a consequence, the 2-dimensional representation $\mathrm{Ind}^{\mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/M_+)}_{\mathrm{Gal}(\mathbb{Q}^{\mathrm{CM}}/M)}(\chi)$ is a direct sum of two 1-dimensional characters. Thus we can replace M by M_+ by using these two characters. Now we assume that every M in the decomposition is totally real. By $f_\Phi(c\sigma) = -f_\Phi(\sigma)$ and $\chi(c\sigma) = \chi(c)\chi(\sigma)$, we can assume that $\chi(c) = -1$ for every χ by removing the ones with $\chi(c) = 1$. It is reduced to the classical statement that *if N is a totally imaginary abelian extension of a totally real field M and $\chi : \mathrm{Gal}(N/M) \to \mathbb{C}^\times$ is a character with $\chi(c) = -1$, then $L(\chi, s)$ is holomorphic and nonzero at $s = 0$.* Note that $L(\chi, s)$ is holomorphic and nonzero at $s = 1$ by decomposing $\zeta_N(s)$ into the product of $L(\omega, s)$ for all characters ω of $\mathrm{Gal}(N/M)$. This gives the result at $s = 0$ by the functional equation, since the gamma factor is holomorphic and nonzero at $s = 0, 1$.

The conjecture is known in the the following cases. If $F = \mathbb{Q}$, the conjecture is equivalent to the classical Chowla–Selberg formula [CS] proved in 1949, which will be discussed in the next subsection. If $E/\mathbb{Q}$ is abelian, the conjecture was proved by Colmez [Co] up to rational multiples of $\log 2$, and the rational multiples were eliminated by Obus [Ob] later. If $[E : \mathbb{Q}] = 4$, the conjecture was proved by Yang [Ya1, Ya2].

In history, there are a lot of works to prove, reformulate or generalize the Chowla–Selberg formula. A very interesting geometric proof of the Chowla–Selberg formula was discovered by Gross [Gr]. He also made a conjecture with Deligne for the periods of motives with complex multiplication by an abelian field. Anderson [An] reformulated the conjecture of Deligne and Gross in terms of the logarithmic derivatives of odd Dirichlet L-functions at $s = 0$. All these treatments were only up to algebraic numbers. Finally, motivated by a far-reaching conjectural product formula from the p-adic Hodge theory, Colmez [Co] used the Faltings height, instead of just the archimedean periods, to formulate the precise conjecture.

3.3. Chowla–Selberg formula. Here we review the classical Chowla–Selberg formula following the treatment of Weil [We]. Let E be an imaginary quadratic field of discriminant $-d < 0$, class number h, and $w = |O_E^\times|$. Fix an embedding $E \subset \mathbb{C}$. For any fractional ideal $\mathfrak{a}$ of O_E, denote

$$F(\mathfrak{a}) = \Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1}) = N(\mathfrak{a})^{12}|\Delta(\mathfrak{a})|^2,$$

where

$$\Delta(\mathfrak{a}) = g_2(\mathfrak{a})^3 - 27g_3(\mathfrak{a})^2$$

is the modular discriminant of $\mathfrak{a}$. By definition, $F(\mathfrak{a})$ depends only on the ideal class of $\mathfrak{a}$. In the case $\mathfrak{a} = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with $\tau = \omega_1/\omega_2 \in \mathcal{H}$, one has

$$F(\mathfrak{a}) = d^{-6}(2\mathrm{Im}(\tau))^{12}|\Delta(\tau)|^2,$$

where

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}, \quad q = e^{2\pi i\tau}.$$

The Chowla–Selberg formula takes many classical forms. One form in [We, p. 92] is

$$\prod_{\mathfrak{a}\in \mathrm{Cl}(O_E)} F(\mathfrak{a}) = \left(\frac{2\pi}{d}\right)^{12h} \prod_{i=1}^{d-1} \Gamma\left(\frac{i}{d}\right)^{6w\eta(i)}.$$

In the last line of [We, p. 91], the formula takes the form

$$\frac{\zeta_E'(0)}{\zeta_E(0)} = \frac{1}{12h}\sum_{\mathfrak{a}\in \mathrm{cl}} \log F(\mathfrak{a}).$$

This is close to the setting of the current paper. In fact, by $\zeta'(0)/\zeta(0) = \log(2\pi)$, the left-hand side has the main term $L'(0,\eta)/L(0,\eta)$. Here $\eta : (\mathbb{Z}/d\mathbb{Z})^\times \to \{\pm 1\}$ is the quadratic character associated to E. By the Kodaira spencer map on the modular curve $X_0(1)$, the right-hand side is the sum of $-2h(A)$ and some minor terms, where A is an elliptic curve with CM by O_E. Therefore, the Chowla–Selberg formula is equivalent to

$$h(A) = -\frac{1}{2}\frac{L'(\eta,0)}{L(\eta,0)} - \frac{1}{4}\log(d).$$

Here $L(\eta, s)$ is the usual Dirichlet L-function (without the gamma factors).

Example 3.2. If $E = \mathbb{Q}(i)$, then the Chowla–Selberg formula is equivalent to

$$\Delta(i) = \pi^{-6}\,\Gamma\big(\frac{1}{4}\big)^{24}.$$

REMARK 3.3. Recall that the class number formula gives

$$L(0, \eta) = 2h/w.$$

The Chowla–Selberg formula can be viewed as an arithmetic version of this formula. The relation of these two formulas is like the relation between the Waldspurger formula and the Gross–Zagier formula stated in [**YZZ**, §1.3-1.4].

4. The averaged Colmez conjecture

The Colmez conjecture is far-reaching due to the generality of the CM type. For example, the Artin L-function in the conjecture is very mysterious. However, after we average the equality in the conjecture over all CM types Φ of a fixed CM extension E/F, the right-hand side is essentially given by the logarithmic derivative of the L-function of the quadratic character associated to the quadratic extension E/F. More precisely, we have the following result.

THEOREM 4.1 (averaged Colmez conjecture). *Let E/F be a CM extension, η be the corresponding quadratic character of $\mathbb{A}_F^\times/F^\times$ associated to the extension E/F, and d_F (resp. $d_{E/F}$) be the absolute discriminant of F (resp. the norm of the relative discriminant of E/F). Then*

$$\frac{1}{2^g}\sum_{\Phi} h(A_\Phi) = -\frac{1}{2}\frac{L'(\eta,0)}{L(\eta,0)} - \frac{1}{4}\log(d_{E/F}d_F),$$

where Φ runs through the set of all CM types of E/F. Here $L(\eta, s)$ is the L-function without the Gamma factors.

The averaged formula was explicitly conjectured in [**Co**, p. 634]. It is proved independently by Yuan–Zhang [**YZ**] and Andreatta–Goren–Howard–Madapusi-Pera in [**AGHM**]. These two proofs are very different. The proof of [**AGHM**] is based on the idea of Yang [**Ya1, Ya2**] (on the Colmez conjecture for $g = 2$) and computes arithmetic intersection numbers over high-dimensional Shimura varieties of orthogonal type. The proof of [**YZ**] is inspired by the work of Yuan–Zhang–Zhang [**YZZ**] (on the Gross–Zagier formula) and computes arithmetic intersection numbers over Shimura curves.

In the following, we will sketch the main ideas of the proof of [**YZ**]. The proof will take two separate steps. The first step expresses the averaged Faltings height in terms of the height of a single CM point on a quaternionic Shimura curve. The second step proves that the height of the single CM point is given by the expected logarithmic derivative.

4.1. Step 1: averaged Faltings height vs. quaternionic height. Let E/F be the CM extension as before. Take a quaternion algebra B over F with an embedding $E \to B$ over F. Assume that B is indefinite at exactly one archimedean place of F. Let $U \subset B^\times(\mathbb{A}_f)$ be a maximal compact subgroup containing the image of $\widehat{O}_E^\times$ in $B^\times(\mathbb{A}_f)$. Consider the *Shimura curve*

$$\mathrm{Sh}(B^\times, U) = B^\times \backslash \mathcal{H}^\pm \times B^\times(\mathbb{A}_f)/U.$$

By the theory of canonical models (cf. [De]), $\mathrm{Sh}(B^\times, U)$ descends to a smooth curve X over F. Assume that B is not a matrix algebra over F so that X is projective over F. Based on [BC, Ca], we have a canonical integral model $\mathcal{X}$ of X over O_F. Note that X is the the coarse moduli space of the corresponding Deligne–Mumford stack, and our treatment can be also written in terms of the stack.

Let $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ be the arithmetic Hodge bundle of $\mathcal{X}$, which is a hermitian $\mathbb{Q}$-line bundle in the setting of Arakelov geometry. Here $\mathcal{L}$ is the relative dualizing sheaf of $\mathcal{X}$ over O_F modified by some "ramification divisor," and the hermitian metric at an archimedean place $\sigma : F \to \mathbb{C}$ is the Petersson metric

$$\|d\tau\|_\sigma = 2\,\mathrm{Im}(\tau)$$

in terms of the complex uniformization of the Shimura curve.

Let $P \in X(E^{\mathrm{ab}})$ be the algebraic point of X represented by the pair $[\tau_0, 1]$ in terms of the complex uniformization, where $\tau_0 \in \mathcal{H}$ is the unique fixed point of $E^\times$. The *height* of P is defined by

$$h_{\overline{\mathcal{L}}}(P) = \frac{1}{[F(P):F]}\widehat{\deg}(\overline{\mathcal{L}}|_{\overline{P}}),$$

where $F(P)$ denotes a field of definition of P and $\overline{P} : \mathrm{Spec}(O_{F(P)}) \to \mathcal{X}$ denotes the extension of P to $\mathcal{X}$. The arithmetic degree is defined as in §2.1. The first step is to prove the following result.

THEOREM 4.2. *Let d_B be the norm of the product of finite primes of O_F over which B is ramified. Assume that there is no finite place of F ramified simultaneously in B and E. Then*

$$\frac{1}{2^g}\sum_{\Phi} h(A_\Phi) = \frac{1}{2}h_{\overline{\mathcal{L}}}(P) - \frac{1}{4}\log(d_B d_F).$$

This theorem is proved by several manipulations of heights in Part I of [YZ]. Note that X is not of PEL type, so it does not parametrize abelian variety naturally.

Let A/H be an abelian variety of CM type (O_E, Φ) over a number field H. Assume that A has semistable reduction over H, which can be achieved by enlarging H if necessary. Then it actually has everywhere good reduction by the CM theory. As a consequence, the Néron model $\mathcal{A}$ of A is an abelian scheme over O_H. Recall that the Faltings height $h(A)$ is defined to be the arithmetic degree of the Hodge bundle

$$\omega_{\mathcal{A}} = \det \Omega_{\mathcal{A}}.$$

Here

$$\Omega_{\mathcal{A}} = \epsilon^* \Omega^1_{\mathcal{A}/O_H} = \pi_* \Omega^1_{\mathcal{A}/O_H}$$

is a vector bundle on $\mathrm{Spec}(O_H)$ of rank $g = \dim A$, where $\pi : \mathcal{A} \to \mathrm{Spec}(O_H)$ denotes the structure morphism and $\epsilon : \mathrm{Spec}(O_H) \to \mathcal{A}$ denotes the identity section.

We further assume that H contains the Galois closure of E. Considering the action of O_E on $\Omega_{\mathcal{A}}$, we can decompose it into a direct sum of line bundles on $\mathrm{Spec}(O_H)$. Accordingly, we have a decomposition

$$h(A_\Phi) = \sum_{\tau \in \Phi} h(\Phi, \tau)$$

up to some error term coming from ramifications of the number fields involved.

Let (Φ_1, Φ_2) be a nearby pair of CM types in the sense that $|\Phi_1 \cap \Phi_2| = g - 1$. Let τ_i be the complement of $\Phi_1 \cap \Phi_2$ in Φ_i for $i = 1, 2$. Define

$$h(\Phi_1, \Phi_2) = \frac{1}{2}(h(\Phi_1, \tau_1) + h(\Phi_2, \tau_2)).$$

It suffices to compute the average of $h(\Phi_1, \Phi_2)$ over all nearby pairs (Φ_1, Φ_2). In [YZ], we have actually proved

$$g \cdot h(\Phi_1, \Phi_2) = \frac{1}{2} h_{\overline{\mathcal{L}}}(P) - \frac{1}{4} \log(d_B).$$

This is surprising since the right-hand side is independent of the choice of (Φ_1, Φ_2). To prove the last formula, $h(\Phi_1, \Phi_2)$ is further reduced to the height of a CM point on the Shimura curve of certain unitary group $U(1,1)$. By some delicate local comparison of the unitary Shimura curve with the quaternionic Shimura curve, we eventually relate the heights on these two Shimura curves.

4.2. Step 2: Compute the quaternionic height. The second step is to prove the following result.

THEOREM 4.3. *In the setting of Theorem 4.1 and Theorem 4.2,*

$$h_{\overline{\mathcal{L}}}(P) = -\frac{L'(\eta, 0)}{L(\eta, 0)} + \frac{1}{2} \log \frac{d_B}{d_{E/F}}.$$

This theorem is proved by extending the method of the proof of the Gross–Zagier formula in [YZZ]. Recall that the Gross–Zagier formula is an identity between the derivative of L-series of a Hilbert modular form of parallel weight two and the height of a CM point on a modular abelian variety of GL(2)-type. This formula is proved by a comparison of the analytic kernel $\mathrm{Pr}\, I'(0,g,\phi)$ and the geometric kernel $2Z(g,(1,1),\phi)$ parametrized by certain modified Schwartz functions $\phi \in \overline{\mathcal{S}}(\mathbb{B} \times \mathbb{A}^\times)$. Here $\mathbb{B}$ is obtained from $B(\mathbb{A})$ by changing every archimedean component to the Hamiltonian algebra. More precisely, we have proved that the difference

$$\mathcal{D}(g,\phi) = \mathrm{Pr}\, I'(0,g,\phi) - 2Z(g,(1,1),\phi), \qquad g \in \mathrm{GL}_2(\mathbb{A}_F)$$

is perpendicular to cusp forms. The matching of the "main terms" of $\mathcal{D}(g,\phi)$ eventually implies the Gross–Zagier formula; On the other hand, the matching of the "degenerate terms" implies Theorem 4.3.

On the analytic side, the function $I(s,g,\phi)$ is a mixed Eisenstein–theta series, i.e. a finite linear combination of automorphic forms of the form $\theta(g)E(s,g)$. Here the theta series $\theta(g)$ and the Eisenstein series $E(s,g)$ each has weight one. The main contribution of $I'(0,g,\phi)$ to Theorem 4.3 is given by the intertwining part of the constant term of $E'(0,g)$. Via analytic continuation, the intertwining part of $E(s,g)$ is equal to a simple multiple of $L(s,\eta)/L(s+1,\eta)$. Hence, its contribution in $I'(0,g,\phi)$ gives the logarithmic derivative $L'(0,\eta)/L(0,\eta)$. Here the functional equation can be used to convert $L'(1,\eta)/L(1,\eta)$ to $L'(0,\eta)/L(0,\eta)$.

On the geometric side, the series

$$Z(g,(1,1),\phi) = \langle Z(g,\phi)P^\circ, P^\circ\rangle_{\mathrm{NT}}$$

is a Néron-Tate height pairing. Here $P^\circ = P - \xi$, where ξ is the normalized Hodge bundle, i.e., a rational multiple of the Hodge bundle $\mathcal{L}_F$ which has degree one on the geometrically connected component of X containing P. Here $Z(g,\phi)$ is Kudla's generating function, which is an automorphic form with coefficients in $\mathrm{Pic}(X^2) \otimes_{\mathbb{Z}} \mathbb{C}$. A primitive version of the generating function is the classical series $\sum_{N\geq 0} T_N q^N$ of Hecke correspondences on the square of the usual modular curve. In terms of Arakelov geometry, we can compute $\langle Z(g,\phi)P^\circ, P^\circ\rangle_{\mathrm{NT}}$ in terms of arithmetic intersection numbers. The divisor $Z(g,\phi)P$ is literally a sum of CM points of X, and it contains a multiple of P, which is our "degenerate term." The contribution of this degenerate term in $Z(g,(1,1),\phi)$ is a multiple of

$$\langle P, P\rangle = -\widehat{\deg}(\overline{\mathcal{L}}|_{\overline{P}}) = -[F(P):P]\cdot h_{\overline{\mathcal{L}}}(P).$$

Here the left-hand side is certain arithmetic intersection number on $\mathcal{X}$, and the first equality is a version of the arithmetic adjunction formula.

Finally, the matching of these two "degenerate terms" in $\mathcal{D}(g,\phi)$ gives the desired height formula in Theorem 4.3. To get this matching, we have the matching of the "main terms" of $\mathcal{D}(g,\phi)$ by explicit local computations, which is mostly done in [**YZZ**]. By the modularity of $\mathcal{D}(g,\phi)$ and the theory of pseudo-theta functions in [**YZ**], the matching of the "main terms" implies the matching of the "degenerate terms."

5. The André–Oort conjecture

By the recent work of Tsimerman [**Ts**], which is based on the previous works of Pila et al, the averaged Colmez conjecture implies the André–Oort conjecture for the Siegel modular variety. In this section, we give a sketch of this implication.

Conjecture 5.1 (André–Oort conjecture, [**And, Oo**]). *Let X be a Shimura variety over $\mathbb{C}$. Let $Y \subset X$ be a closed subvariety which contains a Zariski dense subset of special points of X. Then Y is a special subvariety.*

There are enormous progresses on the conjecture made by André, Edixhoven, Klingler, Ullmo, Yafaev, Pila, Zannier, Tsimerman et al. We describe some of them as follows, and refer to [**Sc, PT**] for more detailed descriptions. Klingler–Ullmo–Yafaev [**UY, KY**] proved the conjecture assuming the generalized Riemann hypothesis for CM fields. Pila [**Pi**] proved the conjecture when X is a power of modular curves. Tsimerman [**Ts**] proved the conjecture when X is a Siegel modular variety (and thus all Shimura varieties of abelian type) assuming the averaged Colmez conjecture (cf. Theorem 4.1). Consequently, the André–Oort conjecture is proved for all Shimura varieties of abelian type.

In the following, we sketch the proof of Tsimerman [**Ts**]. From averaged Colmez conjecture to the André–Oort conjecture (for the Siegel modular variety), there is an intermediate conjecture proposed by Edixhoven, which we will start with.

5.1. Edixhoven's conjecture implies André–Oort conjecture. Let $\mathcal{S}_g$ be the Siegel modular variety over $\mathbb{Q}$, i.e. the coarse moduli space of principally polarized abelian varieties of dimension g. Let $x \in \mathcal{S}_g(\overline{\mathbb{Q}})$ be a special point. Then x corresponds to a principally polarized abelian variety A_x over $\overline{\mathbb{Q}}$, isogenous to a product of abelian varieties with complex multiplication as defined in §3.1. Denote by R_x the center of the endomorphism algebra $\mathrm{End}_{\overline{\mathbb{Q}}}(A_x)$, which is an order in a direct sum of CM fields. Then it makes sense to talk about the discriminant $\mathrm{disc}(R_x) \in \mathbb{Z}$ of R_x. In [**EMO**, Problem 14], the following statement is proposed as a question.

CONJECTURE 5.2 (Edixhoven's conjecture). *There is a positive real number c_g depending only on g such that*

$$[\mathbb{Q}(x) : \mathbb{Q}] \geq |\mathrm{disc}(R_x)|^{c_g}$$

for any special point $x \in \mathcal{S}_g(\overline{\mathbb{Q}})$.

If $g = 1$, then A_x is an elliptic curve with complex multiplication by R_x. In this case, $\mathbb{Q}(x)$ is almost the ring class field of R_x, whose degree is given by the class number of R_x. Then the conjecture follows from the classical Brauer–Siegel theorem (cf. [**Br**]). If $g > 1$, such an argument does not work since $\mathbb{Q}(x)$ is much more complicated than the ring class field.

The key result of our interest here is Pila–Tsimerman [**PT**, Theorem 7.1], which asserts that Edixhoven's conjecture implies the André-Oort conjecture for the case $X = \mathcal{S}_g$. The proof of this theorem is built up on the works of Bombieri, Pila, Zannier, Wilkie et al. We refer to [**PT, Sc, Ts**] for more details, while we sketch some of the ideas here. The setup is to consider the complex uniformization $\pi : \mathcal{H}_g \to \mathcal{S}_g(\mathbb{C})$. Take a "nice" fundamental domain $F_g \subset \mathcal{H}_g$ of $\mathcal{S}_g(\mathbb{C})$ and denote $G_g = F_g \cap \pi^{-1}(Y(\mathbb{C}))$. A special point $x \in Y(\overline{\mathbb{Q}})$ gives roughly $[\mathbb{Q}(x) : \mathbb{Q}]$ special points of $Y(\overline{\mathbb{Q}})$ by the Galois action, and these points lift to algebraic points of G_g of degree at most $2g$. By Conjecture 5.2, they give "many" algebraic points of G_g. This suggests that connected components of G_g "behave like" hermitian symmetric subdomains of $\mathcal{H}_g$. In fact, a fundamental theorem of Pila–Wilkie [**PW**] implies that G_g contains a semi-algebraic set of $\mathcal{H}_g$. Finally, the hyperbolic Ax-Lindemann theorem proved in [**PT**] finishes the argument.

Therefore, it remains to prove Edixhoven's conjecture, which is the main theorem of Tsimerman [**Ts**]. We sketch it in the following.

5.2. Averaged Colmez conjecture implies Edixhoven's conjecture. Tsimerman's proof is a combination of a few important theorems. We describe it in a few steps.

Step 1: reduce to simple case. Note that A_x is isogenous to a product of abelian varieties with complex multiplication by CM fields. It can actually be reduced to a (simple) abelian variety with complex multiplication by a maximal order. This is done in the proof of [**Ts**, Theorem 5.1]. Then the goal becomes to prove that if E/F is a CM extension and A is an abelian variety of primitive CM type (O_E, Φ), then

$$[\mathbb{Q}(A) : \mathbb{Q}] \geq |d_E|^{c_g}$$

Here c_g depends only on $g = [F : \mathbb{Q}]$. Here $\mathbb{Q}(A)$ is the intersection of all subfields of $\mathbb{C}$ to which A can be descended to.

Step 2: bound Faltings height. The goal here is to prove

$$h(A) \leq |d_E|^{o_g(1)}.$$

Assuming the averaged Colmez conjecture in Theorem 4.1, the bound is a consequences of the following bounds:

(1) $0 < L(1, \eta) \leq |d_E|^{o_g(1)}$,
(2) $L'(1, \eta) \leq |d_E|^{o_g(1)}$,
(3) $h(A') \geq O_g(1)$.

Note we can switch $-L'(0, \eta)/L(0, \eta)$ to $L'(1, \eta)/L(1, \eta)$ by the functional equation. Part (1) follows from the classical Brauer–Siegel theorem (cf. [**Br**]). Part (2) follows form a standard subconvexity bound in analytic number theory. Part (3) holds for any abelian variety A' over $\overline{\mathbb{Q}}$ of dimension g, which is a result of Bost [**Bo**].

Step 3: isogeny of large degree. We claim that there is an abelian variety B over $\mathbb{C}$ of CM type (O_E, Φ) such that

$$\deg(A, B) \geq |d_E|^{1/4 - o_g(1)}.$$

Here $\deg(A, B)$ denotes the smallest value among the degrees of all isogenies between A and B over $\mathbb{C}$. Note that any isogeny between A and B over $\mathbb{C}$ can be descended to a number field. To prove the claim, denote by $S(O_E, \Phi)$ the set of the isomorphism classes of abelian varieties over $\mathbb{C}$ with CM type (O_E, Φ). There is a bijection $\mathrm{Cl}(O_E) \to S(O_E, \Phi)$ sending a fractional ideal $\mathcal{I}$ of O_E to $\mathbb{C}^g/\mathcal{I}$, so

$$|S(O_E, \Phi)| = h_E \geq h_E/h_F \geq |d_E|^{1/4 - o_g(1)}.$$

Here the last equality follows from the Brauer–Siegel theorem again. With the lower bound of $|S(O_E, \Phi)|$, the claim can be obtained by an easy argument.

Step 4: isogeny theorem. The last step is of an application of the powerful isogeny theorem of Masser-Wüstholz [**MW**] from diophantine geometry. We are need the version of the theorem proved in in [**GR**, Theorem 1.4], which asserts that for any isogenous abelian varieties A and B of dimension g over a number field H, we have

$$\deg(A, B) \leq \max\{h(A), [H : \mathbb{Q}]\}^{O_g(1)}.$$

Combine it with the inequality in Step 3, and note the upper bound of $h(A)$ in Step 2. This finishes the proof.

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