Adelic line bundles on quasi-projective varieties

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1 Introduction

In Diophantine geometry, height theory of algebraic points over number fields always plays a fundamental role, which can be seen in the proof of the Mordell–Weil theorem, the proofs of the Mordell conjecture by Faltings [Fal2] and Vojta [Voj], and the proofs of the Bogomolov conjecture by Ullmo [Ull] and Zhang [Zha4]. While heights can be defined and studied in elementary terms, they are best interpreted in the arithmetic intersection theory of Arakelov [Ara] and Gillet–Soulé [GS1].

To explain the idea to interpret the height function, let X be a projective variety over a number field K with a (projective and flat) integral model $\mathcal{X}$ over O_K and let $\overline{\mathcal{L}}$ be a hermitian line bundle on $\mathcal{X}$. The height of a point $P \in X(\overline{K})$ with respect to $\overline{\mathcal{L}}$ is expressed as the normalized arithmetic degree $\widehat{\deg}(\overline{\mathcal{L}}|_{\mathcal{P}})/\deg(P)$ of $\overline{\mathcal{L}}$ at the corresponding multi-section $\mathcal{P} \subset \mathcal{X}$.

However, hermitian line bundles are too restrictive to give most of the interesting heights including the Néron–Tate height of abelian varieties over number fields. This problem was resolved by Zhang [Zha2] by introducing adelic line bundles on number fields, which are limits of hermitian line bundles in a suitable sense. Adelic line bundles are crucial in the proofs of Ullmo [Ull] and Zhang [Zha4] mentioned above.

The theory of [Zha2] is only valid for projective varieties over number fields (or function fields of one variable). The goal of this paper is to extend the theory of [Zha2] to *quasi-projective* varieties over *finitely generated fields*. More precisely, let F be a finitely generated field over $\mathbb{Q}$ (or a constant field), and let X be a quasi-projective variety over F. We introduce a notion of *adelic line bundles* on X, consider their intersection theory, study their volumes of effective sections, and introduce heights associated to them.

An immediate application of our framework is a theory of canonical heights on polarized algebraic dynamical systems over quasi-projective varieties over finitely generated fields. In particular, we introduce Néron–Tate heights of abelian varieties over finitely generated fields, and extend the arithmetic Hodge index theorem of Faltings [Fal1] and Hriljac [Hri] to this setting.

Furthermore, we prove an equidistribution theorem of small points on quasi-projective varieties over number fields, generalizing the equidistribution theorems of Szpiro–Ullmo–Zhang [SUZ], Chambert-Loir [CL] and Yuan [Yua1]. On the other hand, Kühne [Kuh] recently proved a special case of this equidistribution theorem independently, and applied it to obtain a uniformity result on the Mordell conjecture after the work of Dimitrov–Gao–Habegger [DGH].

A part of our height theory extends the previous work of Moriwaki [Mor3, Mor4]. In fact, Moriwaki [Mor3, Mor4] developed a height theory for projective varieties over finitely generated fields F over $\mathbb{Q}$, depending on the choice of an arithmetic polarization of $\operatorname{Spec} F$. His motivation was to apply Arakelov geometry to varieties over arbitrary fields (of characteristic 0), and he succeeded in formulating and proving the Bogomolov conjecture in that setting. His treatment was more on the numerical theory of heights, but ours is more on the geometric theory of adelic line bundles.

The exposition of this paper uses a combination of algebraic geometry, complex algebraic geometry, Arakelov theory (cf. [Ara, GS1]) and Berkovich analytic spaces (cf. [Ber1, Ber2]). In the following, we sketch the main constructions and theorems of this paper.

1.1 Adelic line bundles

To quickly illustrate the concept, we will take an approach different from the major parts of this paper, but it will give equivalent constructions.

Let K be a number field, a function field of one variable over a field, or a complete valuation field. In the last case, if the valuation is trivial, then we call K a *constant field*. For any field K, denote by K_{triv} the constant field K with the trivial valuation.

For convenience, set $B = \operatorname{Spec} K$ if K is archimedean or constant; set $B = \operatorname{Spec} O_K$ if K is a number field or a non-archimedean field; set B to be the unique projective and regular curve over k with function field K if K is a function field of one variable over a field k.

Let X be a quasi-projective variety over K, i.e., a quasi-projective integral scheme over K. There is a natural *Berkovich analytic space* X^{an} associated to the datum (X, K). In fact, if X has an open affine cover $\{\operatorname{Spec} A_i\}_i$, then $X^{\mathrm{an}} = \cup_i \mathcal{M}(A_i)$, where $\mathcal{M}(A_i)$ is the set of multiplicative semi-norms on A_i whose restrictions to K lie in $\mathcal{M}(K)$. Here $\mathcal{M}(K)$ is defined as follows:

(1) if K is a number field, $\mathcal{M}(K)$ is the set of all valuations over K;

(2) if K is a function field of one variable over a field k, $\mathcal{M}(K)$ is the set of all valuations over K whose restriction to k is trivial;

(3) if K is a complete valuation field with valuation $|\cdot|$, $\mathcal{M}(K)$ is the set of order 1 whose only element is the valuation $|\cdot|$.

There is a natural topology on X^{an} by demanding that $\mathcal{M}(A_i)$ is open in X^{an} and that $|f| : \mathcal{M}(A_i) \to \mathbb{R}$ is continuous for all $f \in A_i$.

A *metrized line bundles on* X is a pair $\overline{L} = (L, \|\cdot\|)$ consisting of a line bundle L on X and a continuous metric $\|\cdot\|$ of L on X^{an}.

Denote by $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ the category of metrized line bundles on X. A morphism of two objects is defined to be an isometry. There is a forgetful functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}}) \longrightarrow \mathcal{P}\mathrm{ic}(X).$$

Here $\mathcal{P}\mathrm{ic}(X)$ denotes the category of line bundles on X, in which a morphism of two objects is an isomorphisms of line bundles.

Model adelic line bundles

Objects of the category $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ are too general for the purpose of intersection theory. Instead, we will define a full subcategory $\widehat{\mathcal{P}\mathrm{ic}}(X)$ of adelic line bundles in $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$, and a full subcategory $\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}$ of integrable adelic line bundles in $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ for intersection theory. For this, we will start with model adelic line bundles and consider a limit process.

An object of $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ with underlying line bundle $L \in \mathcal{P}\mathrm{ic}(X)$ is called a *model adelic line bundle* if it is induced by a projective model $(\mathcal{X}, \overline{\mathcal{L}})$ of (X, eL) over B for some positive integer e, where $\mathcal{X}$ is a flat and projective integral scheme over B with an open immersion $X \hookrightarrow \mathcal{X}_K$, and $\overline{\mathcal{L}}$ is as follows:

(1) if K is a constant field, a non-archimedean field, or a function field of one variable over a field, then $\overline{\mathcal{L}}$ is a line bundle on $\mathcal{X}$ extending eL;

(2) if K is an archimedean field or a number field, then $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ consists of a line bundle $\mathcal{L}$ on $\mathcal{X}$ extending eL and a continuous hermitian metric $\|\cdot\|$ of $\mathcal{L}(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$. The metric is required to be invariant under the complex conjugation if K is a number field.

Because of the integer e in the definition, $(\mathcal{X}, e^{-1}\overline{\mathcal{L}})$ is a projective model of (X, L) in terms of the notion of $\mathbb{Q}$-line bundles. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{mod}}$ the full subcategory of $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ consisting of model adelic line bundles on X.

To define limits, we also need to give a filtration of $\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{mod}}$ indexed by quasi-projective models $\mathcal{U}$ of X over B, i.e. a flat and quasi-projective integral scheme over B with an open immersion $X \hookrightarrow \mathcal{U}_K$. The filtration takes the form

$$\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{mod}} \simeq \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}}.$$

The category $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}}$ can also be defined similarly. In fact, define the Berkovich analytic space $\mathcal{U}^{\mathrm{an}}$ similarly. There is a canonical continuous injection $X^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}}$ with a dense image. A *metrized line bundle on* $\mathcal{U}$ is a pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ consisting of a line bundle $\mathcal{L}$ on $\mathcal{U}$ endowed with a metric $\|\cdot\|$ of $\mathcal{L}$ on $\mathcal{U}^{\mathrm{an}}$. A *model adelic line bundle on* $\mathcal{U}$ is a metrized line bundle on $\mathcal{U}$ induced by a projective model $(\mathcal{X}, \overline{\mathcal{L}})$ of $(\mathcal{U}, \mathcal{L})$ (with $\mathcal{L} \in \mathcal{P}\mathrm{ic}(\mathcal{U})$) over B. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}}$) the category of metrized line bundles (resp. model adelic line bundles) on $\mathcal{U}$.

Limit process

Let $\mathcal{U}$ be a flat and quasi-projective integral scheme over B. Choose a projective compactification $\mathcal{X}_0$ of $\mathcal{U}$ over B such that the boundary $\mathcal{X}_0 \backslash \mathcal{U}$ is exactly equal to the support of an effective Cartier divisor $\mathcal{E}_0$ on $\mathcal{X}$. If K is constant, non-archimedean, or a function field, set $\overline{\mathcal{E}}_0 = \mathcal{E}_0$. If K is archimedean or a number field, set $\overline{\mathcal{E}}_0 = (\mathcal{E}_0, g_0)$, where $g_0 > 0$ is a Green's function of $\mathcal{E}_0(\mathbb{C})$ on $\mathcal{X}_0(\mathbb{C})$. Then $\overline{\mathcal{E}}_0$ induces a Green's function $\widetilde{g}_0$ of $\mathcal{E}_0$ on $\mathcal{X}_0^{\mathrm{an}}$, which restricts to a continuous function $\widetilde{g}_0 : \mathcal{U}^{\mathrm{an}} \to \mathbb{R}_{\geq 0}$.

Consider the space $C(\mathcal{U}^{\mathrm{an}})$ of real-valued continuous functions on $\mathcal{U}^{\mathrm{an}}$. It is endowed with a *boundary topology* induced by the extended norm

$$\|f\|_{\widetilde{g}_0} := \sup_{x \in \mathcal{U}^{\mathrm{an}},\ \widetilde{g}_0(x) > 0} \frac{|f(x)|}{\widetilde{g}_0(x)}.$$

We refer to [Bee] for basics of extended norms, which are allowed to take values in $[0, \infty]$ but still required to satisfy the triangle inequality. The boundary topology is independent of the choice of $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$. Moreover, $C(\mathcal{U}^{\mathrm{an}})$ is complete with respect to the boundary topology.

We say that a sequence $\overline{\mathcal{L}}_i = (\mathcal{L}_i, \|\cdot\|_i)$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})$ *converges* to an object $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})$ if there are isomorphisms $\tau_i : \mathcal{L} \to \mathcal{L}_i$ such that the sequence $-\log(\tau_i^* \|\cdot\|_i / \|\cdot\|)$ converges to 0 in $C(\mathcal{U}^{\mathrm{an}})$ under the boundary topology.

Adelic line bundles

An object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})$ is called an *adelic line bundle on* $\mathcal{U}$ if it is isomorphic to the limit of a sequence in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}}$. An adelic line bundle on $\mathcal{U}$ is called *strongly nef* if it is the limit of a sequence in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}}$ induced by projective models $(\mathcal{X}_i, \overline{\mathcal{L}}_i)$ over B such that $\overline{\mathcal{L}}_i$ is nef on $\mathcal{X}_i$. An adelic line bundle $\overline{\mathcal{L}}$ on $\mathcal{U}$ is called *nef* if there exists a strongly nef adelic line bundle $\overline{\mathcal{M}}$ on $\mathcal{U}$ such that $\overline{\mathcal{L}}^{\otimes a} \otimes \overline{\mathcal{M}}$ is strongly nef for all positive integers a. An adelic line bundle on $\mathcal{U}$ is called *integrable* if it is isometric to $\overline{\mathcal{L}}_1 \otimes \overline{\mathcal{L}}_2^\vee$ for two strongly nef adelic line bundle $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$ on $\mathcal{U}$.

Denote by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})$ the full subcategory of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})$ consisting of adelic line bundles on $\mathcal{U}$. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{nef}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{int}}$) the full subcategory of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})$ consisting of nef (resp. integrable) adelic line bundles on $\mathcal{U}$.

Return to the quasi-projective variety X over K. Define

$$\widehat{\mathcal{P}\mathrm{ic}}(X) := \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{nef}} := \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{nef}},$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}} := \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{int}}.$$

Objects of $\widehat{\mathcal{P}\mathrm{ic}}(X)$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{nef}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}$) are called *adelic line bundles* (resp. *nef adelic line bundles*, *integrable adelic line bundles*) on X.

Denote by $\widehat{\mathrm{Pic}}(X)$ (resp. $\widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$, $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$, $\widehat{\mathrm{Pic}}(X^{\mathrm{an}})$) the *group* of isomorphism classes of objects of $\widehat{\mathcal{P}\mathrm{ic}}(X)$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{nef}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$).

We can further extend the definition to quasi-projective varieties over finitely generated fields. Namely, let F be a finitely generated field over K, and let V be a quasi-projective variety over F. Then we define

$$\widehat{\mathcal{P}\mathrm{ic}}(V) := \varinjlim_{U \to W} \widehat{\mathcal{P}\mathrm{ic}}(U),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(V)_{\mathrm{nef}} := \varinjlim_{U \to W} \widehat{\mathcal{P}\mathrm{ic}}(U)_{\mathrm{nef}},$$

$$\widehat{\mathcal{P}\mathrm{ic}}(V)_{\mathrm{int}} := \varinjlim_{U \to W} \widehat{\mathcal{P}\mathrm{ic}}(U)_{\mathrm{int}}.$$

Here the limit is over all flat morphisms $U \to W$ of quasi-projective varieties over K whose generic fibers are isomorphic to $V \to \operatorname{Spec} F$.

As mentioned at the beginning, the terminology here is different from the major parts of this paper. To clarify, assume that K is either a number field or a constant field, which the other cases can be interpreted similarly. Then the term $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ is the same as that in the major parts, and the current terms

$$\widehat{\mathcal{P}\mathrm{ic}}(X), \quad \widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{mod}}, \quad \widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{nef}}, \quad \widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}$$

actually means, in terms of the terminology in the major parts, the essential images of

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k), \quad \widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{mod}}, \quad \widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{nef}}, \quad \widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{int}}$$

in $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ under the fully faithful analytification functor in Proposition 3.4.1. Here $k = \mathbb{Z}$ if K is a number field; and $k = K$ if K is a constant field.

Functoriality

Let $f : X \to Y$ be a morphism of quasi-projective varieties over K. Then we have a pull-back functor

$$f^* : \widehat{\mathcal{P}\mathrm{ic}}(Y) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X).$$

If K is a number field or a function field of one variable, then for any valuation v of K, we have a canonical localization functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X_{K_v}).$$

This applies particularly to the trivial valuation of K, in which case we denote the functor as

$$\widehat{\mathcal{P}\mathrm{ic}}(X) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X_{\mathrm{triv}}), \quad \overline{L} \longmapsto \overline{L}|_{X_{\mathrm{triv}}}.$$

Here X_{triv} denotes (X, K_{triv}), i.e., the datum (X, K) with K endowed with the trivial valuation.

All these functors preserve the subcategories of model (resp. nef, integrable) adelic line bundles.

1.2 Intersection theory and heights

Our intersection theory includes an absolute intersection pairing, and a relative intersection pairing extending the Deligne pairing.

Intersections and heights

Let K be a number field, a function field of one variable, or a constant field. Set $\dim K = 1$ if K is a number field or a function field of one variable; set $\dim K = 0$ if K is a constant field.

Let X be a quasi-projective variety over K. Then the absolute intersection pairing is a symmetric and multi-linear pairing

$$\widehat{\mathrm{Pic}}(X)^d_{\mathrm{int}} \longrightarrow \mathbb{R},$$

where $d = \dim X + \dim K$. This is the limit version of the intersection theory in algebraic geometry and the arithmetic intersection theory of Gillet–Soulé. See Proposition 4.1.1.

Now assume that K is a number field or a function field of one variable. Let X be a quasi-projective variety over K of dimension n. Let $\overline{L}$ be an integrable adelic line bundle on X. Define a *height function*

$$h_{\overline{L}} : X(\overline{K}) \longrightarrow \mathbb{R}$$

by

$$h_{\overline{L}}(x) := \frac{\widehat{\deg}(\overline{L}|_{x'})}{\deg(x')}.$$

Here x' denotes the closed point of X containing x, $\deg(x')$ denotes the degree of the residue field of x' over K, $\overline{L}|_{x'}$ denotes the *pull-back* of $\overline{L}$ to $\widehat{\mathrm{Pic}}(x')_{\mathrm{int}}$, and $\widehat{\deg} : \widehat{\mathrm{Pic}}(x')_{\mathrm{int}} \to \mathbb{R}$ is by the intersection theory.

More generally, for any closed $\overline{K}$-subvariety Z of X, define *the height of Z with respect to $\overline{L}$* as

$$h_{\overline{L}}(Z) := \frac{(\overline{L}|_{Z'})^{\dim Z+1}}{(\dim Z+1)(\overline{L}|_{Z'_{\mathrm{triv}}})^{\dim Z}} \ \in \mathbb{R}.$$

Here Z' denotes the image of $Z \to X$ (which is a closed subvariety of X over K), and

$$\overline{L} \longmapsto \overline{L}|_{Z'} \longmapsto \overline{L}|_{Z'_{\mathrm{triv}}}$$

denotes the image of $\overline{L}$ via the functorial maps

$$\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(Z')_{\mathrm{int}} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(Z'_{\mathrm{triv}})_{\mathrm{int}},$$

and the self-intersections are as in the above intersection theory.

Deligne pairing and relative heights

Let K be a number field or a constant field. Let $f : X \to Y$ be a projective and flat morphism of of relative dimension n between quasi-projective varieties over K. Assume that Y is normal.

Theorem 1.2.1 (Theorem 4.1.3)**.** *The Deligne pairing on model adelic line bundles induces a symmetric and multilinear functor*

$$\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(Y)_{\mathrm{int}},$$

When restricted to nef adelic line bundles, the functor induces a functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{nef}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(Y)_{\mathrm{nef}}.$$

Moreover, the functors are compatible with base changes of the form $Y' \to Y$, where Y' is a quasi-projective normal variety over K such that $X' = X \times_Y Y'$ is integral.

In the setting of the theorem, let $F = K(Y)$ be the function field of Y, and $X_F \to \operatorname{Spec} F$ the generic fiber of $X \to Y$. Let $\overline{L}$ be an object of $\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}}$. By this, we can define a *vector-valued* height function

$$\mathfrak{h}_{\overline{L}} : X(\overline{F}) \longrightarrow \widehat{\mathrm{Pic}}(F)_{\mathrm{int},\mathbb{Q}}.$$

Here the group

$$\widehat{\mathrm{Pic}}(F)_{\mathrm{int}} := \varinjlim_{U} \widehat{\mathrm{Pic}}(U)_{\mathrm{int}},$$

where U runs through all open subsets of Y.

More generally, for any closed $\overline{F}$-subvariety Z of X_F, define *the vector-valued height of Z with respect to $\overline{L}$* as

$$\mathfrak{h}_{\overline{L}}(Z) := \frac{\langle \overline{L}|_{Z'} \rangle^{\dim Z+1}}{(\dim Z + 1)(\overline{L}|_{Z'_F})^{\dim Z}} \ \in \widehat{\mathrm{Pic}}(F)_{\mathrm{int},\mathbb{Q}}.$$

Here Z' denotes the image of $Z \to X$, Z'_F is the generic fiber of $Z' \to Y$, and

$$\overline{L} \longmapsto \overline{L}|_{Z'} \longmapsto \overline{L}|_{Z'_F}$$

denotes the image of $\overline{L}$ via the functorial maps

$$\widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathrm{int}} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(Z')_{\mathrm{int}} \longrightarrow \mathcal{P}\mathrm{ic}(Z'_F).$$

Note that the first self-intersection is the Deligne pairing, and the second self-intersection is just the degree on the projective variety Z'_F in the classical sense. The height is well-defined only if the denominator is nonzero.

When F is polarized in sense of Moriwaki [Mor3], then we can also define the Moriwaki heights. If K is a number field or a finite field, and if X is projective over F, we obtain a Northcott property of the Moriwaki heights from that of [Mor3]. In general, we obtain the fundamental inequality for the Moriwaki height following the strategy of [Mor3].

1.3 Volumes and equidistribution theorems

As in the projective case, we can define effective sections of adelic line bundles, study their volumes, and prove equidistribution theorems on quasi-projective varieties.

Volumes

Let K be a number field or a constant field. Let X be a quasi-projective variety over K. Let $\overline{L} = (L, \|\cdot\|)$ be an adelic line bundle on X. Define

$$\widehat{H}^0(X, \overline{L}) := \{s \in H^0(X, L) : \|s(x)\| \leq 1, \ \forall\, x \in X^{\mathrm{an}}\}.$$

Elements of $\widehat{H}^0(X, \overline{L})$ are called *effective sections* of $\overline{L}$ on X. If K is a number field, denote

$$\widehat{h}^0(X, \overline{L}) := \log \# \widehat{H}^0(X, \overline{L});$$

if K is a constant field, denote

$$\widehat{h}^0(X, \overline{L}) := \dim_K \widehat{H}^0(X, \overline{L}).$$

We check that $\widehat{h}^0(X, \overline{L})$ is always a finite real number. In this setting, we have the following fundamental results.

Theorem 1.3.1 (Theorem 5.2.1, Theorem 5.2.2)**.** *Let K be a number field or a constant field. Let X be a quasi-projective variety over K. Let $\overline{L}, \overline{M}$ be adelic line bundles on X. Denote $d = \dim X + \dim K$. Then*

(1) The limit

$$\widehat{\mathrm{vol}}(X, \overline{L}) = \lim_{m\to\infty} \frac{d!}{m^d} \widehat{h}^0(X, m\overline{L})$$

exists.

(2) If $\overline{L}$ is the limit of a sequence of model adelic line bundles induced by a sequence $\{(\mathcal{X}_i, \overline{\mathcal{L}}_i)\}_{i\geq 1}$ of projective models of (X, L) over O_K, then

$$\widehat{\mathrm{vol}}(X, \overline{L}) = \lim_{i\to\infty} \widehat{\mathrm{vol}}(\mathcal{X}_i, \overline{\mathcal{L}}_i).$$

(3) If $\overline{L}$ is nef, then

$$\widehat{\mathrm{vol}}(X, \overline{L}) = \overline{L}^d.$$

(4) If $\overline{L}, \overline{M}$ are nef, then

$$\widehat{\mathrm{vol}}(X, \overline{L} - \overline{M}) \geq \overline{L}^d - d\,\overline{L}^{d-1}\overline{M}.$$

Part (1) generalizes the classical result of Fujita (cf. [Laz2, 11.4.7]) for line bundles on projective varieties and the result of Yuan [Yua2] and Chen [Che1] for hermitian line bundles on projective arithmetic varieties. Part (2) allows us to transfer many previous results in the projective case to the quasi-projective case. Part (3) generalizes the classical Hilbert–Samuel formula in algebraic geometry, and the arithmetic Hilbert–Samuel formula proved by Gillet–Soulé [GS2], Bismut–Vasserot [BV], and Zhang [Zha1]. Part (4) generalizes the classical theorem of Siu [Siu] and the arithmetic bigness theorem of Yuan [Yua1].

In the setting of the theorem, we say that $\overline{L}$ is *big* if $\widehat{\mathrm{vol}}(X, \overline{L}) > 0$. We will see that in this case, we will have nice lower bounds of the height function associated to $\overline{L}$.

Height inequality

Let K be a number field or a function field of one variable. Let X be a quasi-projective variety over K. Let $\overline{L}$ be an integrable adelic line bundle on X. Denote by

$$\deg_{\overline{L}}(X_{\mathrm{triv}}) = (\overline{L}|_{X_{\mathrm{triv}}})^{\dim X}$$

the self-intersection number of $\overline{L}|_{X_{\mathrm{triv}}}$ on X_{triv}.

As a quick consequence of the above fundamental results on volumes, we have the following height inequality. For its application to dynamical systems, we refer to Theorem 6.2.2.

Theorem 1.3.2 (Theorem 5.3.5)**.** *Let K be a number field or a function field of one variable. Let $\pi : X \to S$ be a morphism of quasi-projective varieties over K. Let $\overline{L} \in \widehat{\mathrm{Pic}}(X)$ and $\overline{M} \in \widehat{\mathrm{Pic}}(S)$ be adelic line bundles.*

(1) If $\overline{L}$ is big on X, then there exist $\epsilon > 0$ and a Zariski open and dense subvariety U of X such that

$$h_{\overline{L}}(x) \geq \epsilon\, h_{\overline{M}}(\pi(x)), \qquad \forall\, x \in U(\overline{K}).$$

(2) If $\overline{L}$ is nef on X and $\deg_{\overline{L}}(X_{\mathrm{triv}}) > 0$, then for any $c > 0$, there exist $\epsilon > 0$ and a Zariski open and dense subvariety U of X such that

$$h_{\overline{L}}(x) \geq \epsilon\, h_{\overline{M}}(\pi(x)) - c, \qquad \forall\, x \in U(\overline{K}).$$

Equidistribution

One of the most important theorems of this paper is an equidistribution theorem of small points of a quasi-projective variety over a number field or a function field of one variable.

Theorem 1.3.3 (Theorem 5.4.3)**.** *Let K be a number field or a function field of one variable. Let X be a quasi-projective variety over K. Let $\overline{L}$ be a nef adelic line bundle on X such that $\deg_{\overline{L}}(X_{\mathrm{triv}}) > 0$. Let $\{x_m\}_m$ be a generic sequence in $X(\overline{K})$ such that $\{h_{\overline{L}}(x_m)\}_m$ converges to $h_{\overline{L}}(X)$. Then the Galois orbit of $\{x_m\}_m$ is equidistributed in $X_{K_v}^{\mathrm{an}}$ with respect to $d\mu_{\overline{L},v}$ for any place v of K.*

Here $d\mu_{\overline{L},v}$ is a canonical probability measure on $X_{K_v}^{\mathrm{an}}$, defined using the recent theory of Chambert-Loir and Ducros in [CD] if v is non-archimedean. This generalizes the Monge–Ampère measure and the Chambert-Loir measure from the projective case to the quasi-projective case.

If X is projective over K, the equidistribution theorem is proved by the works of Szpiro–Ullmo–Zhang [SUZ], Chambert-Loir [CL], and Yuan [Yua1]. Our current theorem still follows the variational principle of the pioneering work [SUZ], applying our adelic Hilbert–Samuel formula and adelic bigness theorem.

We can further generalize our equidistribution theorem in two different aspects, which gives us an equidistribution theorem (Theorem 5.4.6) and an equidistribution conjecture (Conjecture 5.4.1). The equidistribution theorem considers a projective and flat morphism of quasi-projective varieties over a number field or a function field of one variable, and its proof follows a strategy of Moriwaki [Mor3]. The equidistribution conjecture considers quasi-projective varieties over finitely generated fields, which is stated as follows.

Conjecture 1.3.4 (Conjecture 5.4.1)**.** *Let K be a number field or a constant field. Let F be a finitely generated field over K. Let v be a non-trivial valuation of F. Assume that the restriction of v to K is trivial if K is a constant field. Let X be a quasi-projective variety over F. Let $\overline{L}$ be a nef adelic line bundle on X such that $\deg_{\overline{L}}(X_{\mathrm{triv}}) > 0$. Let $\{x_m\}_m$ be a generic sequence of small points in $X(\overline{F})$. Then the Galois orbit of $\{x_m\}_m$ is equidistributed in $X_{F_v}^{\mathrm{an}}$ with respect to $d\mu_{\overline{L},v}$.*

1.4 Algebraic dynamics

Here we apply the theory of adelic line bundles to algebraic dynamics.

Algebraic dynamics

Let K be a number field or a constant field. Let S be a quasi-projective variety over K with function field $F = K(S)$. Let (X, f, L) be a *polarized algebraic dynamical system* over S, i.e., X is a flat and projective integral scheme over S, $f : X \to X$ is an endomorphism over S, and L is an f-ample $\mathbb{Q}$-line bundle and satisfying $f^*L \simeq qL$ for some rational number $q > 1$.

By Tate's limiting argument, we can construct a canonical adelic $\mathbb{Q}$-line bundle $\overline{L}_f \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q},\mathrm{nef}}$ extending L which is f-invariant in that $f^*\overline{L}_f \simeq q\overline{L}_f$. Here $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q},\mathrm{nef}}$ denotes the sub-semigroup of $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ consisting of positive rational multiples of elements of $\widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$.

For any closed $\overline{F}$-subvariety Z of X_F, we have the canonical height

$$\mathfrak{h}_f(Z) = \mathfrak{h}_{\overline{L}_f}(Z) := \frac{\left\langle \overline{L}_f|_{Z'}^{\dim Z+1} \right\rangle}{(\dim Z + 1)\deg_L(Z'_F)} \in \widehat{\mathrm{Pic}}(F)_{\mathbb{Q},\mathrm{nef}}.$$

In particular, we have a height function

$$\mathfrak{h}_f : X(\overline{F}) \longrightarrow \widehat{\mathrm{Pic}}(F)_{\mathbb{Q},\mathrm{nef}}.$$

These heights can also be obtained by Tate's limiting argument.

The height function $\mathfrak{h}_f$ is f-invariant. As a consequence, $\mathfrak{h}_f(x) = 0$ for a preperiodic point $x \in X(\overline{F})$. In the minimal case that K is a number field or a finite constant field, $\mathfrak{h}_f$ satisfies the Northcott property. In this case, $\mathfrak{h}_f(x) = 0$ for a point $x \in X(\overline{F})$ implies that x is preperiodic under f.

Equidistribution of small points

Our equidistribution conjecture naturally implies an equidistribution conjecture of preperiodic points.

Conjecture 1.4.1 (Conjecture 6.1.5). *Let v be a non-trivial valuation of F. Assume that the restriction of v to K is trivial if K is a constant field. Let $\{x_m\}_m$ be a generic sequence of preperiodic points in $X(\overline{F})$. Then the Galois orbit of $\{x_m\}_m$ is equidistributed in $X_{F_v}^{\mathrm{an}}$ with respect to the canonical measure $d\mu_{L,f,v}$.*

As an example of our equidistribution theorem (cf. Theorem 5.4.3), we deduce the following equidistribution theorem of small points on non-degenerated subvarieties in a family of polarized algebraic dynamical systems.

Theorem 1.4.2 (Theorem 6.2.3)**.** *Let K be a number field or a function field of one variable. Let S be a quasi-projective variety over K. Let (X, f, L) be a polarized dynamical system over S. Let Y be a non-degenerate closed subvariety of X over K. Let $\{y_m\}_{m\geq 1}$ be a generic sequence of $Y(\overline{K})$ such that $h_{\overline{L}_f}(y_m) \to 0$. Then for any place v of K, the Galois orbit of $\{y_m\}_{m\geq 1}$ is equidistributed on the analytic space Y_v^{an} with respect to the canonical measure $d\mu_{\overline{L}_f|_Y, v}$.*

The theorem generalizes the equidistribution theorem of DeMarco–Mavrak [DM] for families of elliptic curves, and confirms the conjecture (REC) of Kühne [Kuh] for abelian schemes. A weaker version of the theorem for abelian schemes is proved by [Kuh, Thm. 1] independently. The proof of [Kuh] is a limit version of the original proof in [SUZ] and uses a result of Dimitrov–Gao–Habegger [DGH] for uniformity in the limit process.

In the theorem, a closed subvariety Y of X is called *non-degenerate* if $\deg_{\overline{L}}(Y_{\mathrm{triv}}) > 0$. This is equivalent to the property that $\overline{L}|_{Y_{\mathrm{triv}}}$ is *big*. If X is an abelian scheme and K is a number field, our definition of "non-degenerate" agrees with that of [DGH], which uses Betti maps in the complex analytic setting.

Heights of points of a non-degenerate subvariety

Let K be a number field or a function field of one variable. Let S be a quasi-projective variety over K. Let (X, f, L) be a polarized dynamical system over S. Let Y be a closed subvariety of X over K.

If Y is a section of $X \to S$, our vector-valued height of adelic line bundles generalizes and re-interprets the Tate–Silverman specialization theorem of [Tat, Sil2, Sil3, Sil4] and the work [DM] from families of elliptic curves to families of algebraic dynamical systems. See Lemma 6.2.1 for more details.

If X is an abelian scheme and Y is non-degenerate in X, there is a height inequality of points of Y by [GH, DGH], which plays a fundamental role in the treatment of the uniform Mordell–Lang problem in [DGH, Kuh]. In terms of our theory, we have a simple interpretation of the height inequality, which is also valid on families of algebraic dynamical systems. As the non-degeneracy is interpreted as the bigness of $\overline{L}|_{Y_{\mathrm{triv}}}$, the height inequality is also interpreted by the bigness of some adelic line bundle. In fact, applying Theorem 1.3.2(2) to the morphism $Y \to S$ and the adelic line bundle $\overline{L}_f|_Y$ on Y, we can have a lower bound of the canonical height of points on Y by Weil heights on S. See Theorem 6.2.2 for more details.

Equidistribution of PCF maps

Let S be a smooth and quasi-projective variety over a number field K. Let $X = \mathbb{P}^1_S$ be the projective line over S, and let $f : X \to X$ be a finite morphism over S of degree $d > 1$. A point $y \in S(\overline{K})$ is called *post-critically finite* (PCF) if all the ramification points (i.e. critical points) of $f_y : X_y \to X_y$ are preperiodic under f_y.

Denote by $\mathcal{M}_d$ the moduli space over K of endomorphisms of $\mathbb{P}^1$ of degree d. Inside $\mathcal{M}_d$, there is closed subvariety corresponding to flexible Lattés maps. By the moduli property, there is a morphism $S \to \mathcal{M}_d$.

The main result here is the following equidistribution theorem of Galois orbits of PCF points.

Theorem 1.4.3 (Theorem 6.3.6)**.** *Assume that the morphism $S \to \mathcal{M}_d$ is generically finite and its image is not contained in the flexible Lattès locus. Let $\{y_m\}_m$ be a generic sequence of PCF points of $S(\overline{K})$. Then the Galois orbit of $\{y_m\}_m$ is equidistributed in $S^{\mathrm{an}}_{K_v}$ with respect to $d\mu_{\overline{M},v}$ for any place v of K.*

If S is a family of *polynomial* maps on $\mathbb{P}^1$, the theorem was previously proved by Favre–Gauthier [FG]. Their strategy was to reduce the problem to the equidistribution of Yuan [Yua1].

Now we explain our proof of the theorem, which will also introduce the key term $\overline{M}$ in the statement. Denote by R the ramification divisor of the finite morphism $f : X \to X$, viewed as a (possibly non-reduced) closed subscheme in X. Then R is finite and flat of degree $2d-2$ over S, and the fiber R_y of R above any point $y \in S$ is the ramification divisor of $f_y : X_y \to X_y$.

Let L be a $\mathbb{Q}$-line bundle on X, of degree one on fibers of $X \to S$, such that $f^*L \simeq dL$. Denote by $\overline{L} = \overline{L}_f$ the f-invariant extension of L in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q},\mathrm{nef}}$ such that $f^*\overline{L} \simeq d\overline{L}$. Denote

$$\overline{M} := N_{R/S}(\overline{L}|_R) \in \widehat{\mathrm{Pic}}(S)_{\mathbb{Q},\mathrm{nef}}.$$

Here the norm map is the Deligne pairing of relative dimension 0.

Consider the height function

$$h_{\overline{M}} : S(\overline{K}) \longrightarrow \mathbb{R}_{\geq 0}.$$

For any $y \in S(\overline{K})$, the height $h_{\overline{M}}(y) = 0$ if and only if y is PCF in S. Then we are in the situation to apply the previous equidistribution theorem (Theorem 5.4.3) to $(S, \overline{M})$, except that we need to check the condition $\deg_{\overline{M}}(S_{\text{triv}}) > 0$.

This requires the bifurcation measure introduced by DeMarco [DeM1, DeM2] and further studied by Bassanelli–Berteloot [BB]. In fact, $\deg_{\overline{M}}(S_{\text{triv}})$ is exactly equal to the total volume of the bifurcation measure on $S_\sigma(\mathbb{C})$ for any embedding $\sigma : K \to \mathbb{C}$. Then $\deg_{\overline{M}}(S_{\text{triv}}) > 0$ is eventually equivalent to the condition on $S \to \mathcal{M}_g$ by the works of [BB, GOV]. This proves the theorem, and moreover confirms that the equilibrium measure $d\mu_{\overline{M},\sigma}$ is a constant multiple of the bifurcation measure for any embedding $\sigma : K \to \mathbb{C}$.

In the end, we note that the theorem also holds for a family of morphisms on $\mathbb{P}^n$ with a slightly weaker statement. In particular, the construction of the adelic line bundle $\overline{M}$ works in the same way. We refer to Theorem 6.3.5 for more details.

Hodge index theorem on curves

In the end, we present our generalization of the arithmetic Hodge index theorem of Faltings [Fal1] and Hriljac [Hri] to finitely generated fields. We refer to Theorem 6.5.1 for a detailed account.

Let K be a number field or a constant field. Let F be a finitely generated field over K, and let $\pi : X \to \operatorname{Spec} F$ be a smooth, projective, and geometrically connected curve of genus $g > 0$. Denote by $J = \underline{\mathrm{Pic}}^0_{X/F}$ the Jacobian variety of X and by Θ the symmetric theta divisor on J. By the dynamical system $(J, [2], \Theta)$, we have a Néron–Tate height function

$$\hat{\mathfrak{h}} : \mathrm{Pic}^0(X_{\overline{F}}) \longrightarrow \widehat{\mathrm{Pic}}(F)_{\mathbb{Q},\mathrm{nef}}.$$

The height function is quadratic as in the classical case.

Theorem 1.4.4 (Theorem 6.5.1)**.** *Let K be a number field or a constant field. Let F be a finitely generated field over K, and let $\pi : X \to \operatorname{Spec} F$ be a smooth, projective, and geometrically connected curve. Let M be a line bundle* *on X with* $\deg M = 0$*. Then there is an adelic line bundle* $\overline{M}_0 \in \widehat{\mathrm{Pic}}(X)_{\mathrm{int},\mathbb{Q}}$ *with underlying line bundle M such that*

$$\pi_*\langle \overline{M}_0, \overline{V}\rangle = 0, \quad \forall\, \overline{V} \in \widehat{\mathrm{Pic}}(X)_{\mathrm{vert},\mathbb{Q}}.$$

Moreover, for such an adelic line bundle,

$$\pi_*\langle \overline{M}_0, \overline{M}_0\rangle = -2\,\widehat{\mathfrak{h}}(M).$$

In the theorem, $\widehat{\mathrm{Pic}}(X)_{\mathrm{vert},\mathbb{Q}}$ denotes the kernel of the forgetful map $\widehat{\mathrm{Pic}}(X)_{\mathrm{int},\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$.

1.5 Connection to other recent works

As a generalization of the theory of adelic line bundles of Zhang [Zha2], we expect our current theory to be a language to treat the arithmetic of quasi-projective varieties over finitely generated fields, which includes particularly varieties descended from an arbitrary field by the Lefschetz principle, and moduli spaces of geometric objects and their universal objects.

In terms of precise works, the following recent works are related to this paper:

(1) In our another paper [YZ2], based on the theory of this paper, we are generalizing the Hodge index theorem for adelic line bundles in [YZ1] from number fields to finitely generated fields over $\mathbb{Q}$ and consider its application to algebraic dynamics.

(2) Carney [Car2] has further treated the geometric version of [YZ2], extending the results to finitely generated fields over a base field.

(3) Based on the work of Dimitrov-Gao-Habegger [DGH], Kühne [Kuh] has proved a special case of our equidistribution theorem independently, and applied it to obtain a uniform upper bound on the number of rational points in the Mordell conjecture.

(4) Influenced by our formulation, Gauthier [Gau] has extended the equidistribution theorem of [Kuh] to more general settings, which has a large overlap with our equidistribution theorems.

(5) In a forthcoming work, Yuan [Yua4] will use positivity of adelic line bundles to treat the uniform Mordell–Lang problem. This gives a new proof of the results of [GH, DGH, Kuh], which works for both number fields and function fields of any characteristic.

1.6 Terminology and notation

The terminology of the introduction is slightly different from that of the remaining part of the paper. In the following, we introduce the terminology for the remaining part.

We will introduce a uniform system of terminology and notations for both the arithmetic case and the geometric case. To achieve this, we need frequent abuse of terminology.

Our base ring k be either $\mathbb{Z}$ or an arbitrary field. This is divided into two cases:

(1) (arithmetic case) $k = \mathbb{Z}$. In this case, the adelic line bundles will be limits of hermitian line bundles on projective integral schemes over $\mathbb{Z}$. The intersection theory is obtained by this limit process.

(2) (geometric case) k is an arbitrary field. In this case, the adelic line bundles will be limits of usual line bundles on projective varieties over k. The intersection theory is obtained by this limit process.

By a *finitely generated field F over k*, we mean a field F which is finitely generated over the fraction field of k. For any integral scheme X over k, denote by $k(X)$ the *function field* of X.

By a *quasi-projective variety* (resp. *projective variety*) over k, we mean an integral scheme which is flat and quasi-projective (resp. projective) over k. For a quasi-projective variety $\mathcal{U}$ over k, a *projective model* means a projective variety $\mathcal{X}$ over k endowed with an open immersion $\mathcal{U} \to \mathcal{X}$ over k. In the arithmetic case (that $k = \mathbb{Z}$), we may also use the terms *quasi-projective arithmetic variety* and *projective arithmetic variety* to emphasize the situation.

In the arithmetic case, for a projective arithmetic variety $\mathcal{X}$ over $\mathbb{Z}$, we have the group $\widehat{\mathrm{Div}}(\mathcal{X})$ of arithmetic divisors on $\mathcal{X}$, and the group $\widehat{\mathrm{Pic}}(\mathcal{X})$ and the category $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ of hermitian line bundles on $\mathcal{X}$.

In the geometric case, for a projective variety $\mathcal{X}$ over a field k, an arithmetic divisor means a Cartier divisor, a hermitian line bundle means a line bundle, and we write $\widehat{\mathrm{Div}}(\mathcal{X})$, $\widehat{\mathrm{Pic}}(\mathcal{X})$, $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ for $\mathrm{Div}(\mathcal{X})$, $\mathrm{Pic}(\mathcal{X})$, $\mathcal{P}\mathrm{ic}(\mathcal{X})$. We take this convention in other similar situations.

This abuse of notation is only one-way. For example, by Div, Pic or $\mathcal{P}$ic in the arithmetic case, we still mean the ones without the archimedean components.

Below are a few conventions which are not directly related to the base k but taken throughout this paper.

(1) Denote $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ for any abelian group M.

(2) Except in §2.7 and §3.6, all schemes are assumed to be noetherian.

(3) By a *variety* over a field, we mean an integral scheme, separated and of finite type over the field. We do not require it to be geometrically integral.

(4) By a *curve* over a field, we mean a variety over the field of dimension 1.

(5) All *divisors* in this paper are *Cartier divisors*, unless otherwise instructed.

(6) By a *line bundle* on a scheme, we mean an invertible sheaf on the scheme. We often write or mention tensor products of line bundles additively, so $a\mathcal{L} - b\mathcal{M}$ means $\mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes(-b)}$.

(7) All the categories of (adelic, metrized) line bundles are groupoids, so the morphisms are isomorphisms.

(8) A *functor* between two categories may also be called a map or a homomorphism sometimes.

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2 Adelic divisors and adelic line bundles

In this section, we develop a theory of adelic divisors and adelic line bundles on essentially quasi-projective schemes. The contents of this section are explained in the introduction.

2.1 Preliminaries on arithmetic varieties

In this subsection, we review some basic notions of arithmetic divisors and hermitian line bundles on projective arithmetic varieties. These are standard terminology, and most of them are reviewed in [YZ1, Appendix 1]. Since the content here is purely arithmetic, we do not take the uniform terminology in §1.6.

2.1.1 Metrics on analytic spaces

We refer to [Dem2, Chapter II] or [Rem] for detailed introductions to complex analytic varieties. For convenience, all complex analytic varieties in this section are assumed to be reduced and irreducible.

Let X be a (reduced and irreducible) complex analytic variety. The default topology on X is the complex topology unless otherwise instructed. The *regular locus* (or equivalently *smooth locus*) X^{reg} is a complex manifold, which is Zariski open and dense in X. In the following, we introduce metrics of line bundles on X with different type of regularities.

We take the notion of smooth differential forms following [Kin, §1.1]. For integers $p, q \geq 0$, a *smooth* (p, q)*-form on* X is a smooth (p, q)-form α on X^{reg} such that for any point $x \in X$, there is an open neighborhood U of x in X and an analytic map $i : U \to M$ to a complex manifold M under which U is a closed analytic subvariety of M, such that $\alpha|_{U^{\mathrm{reg}}}$ can be extended to a smooth (p, q)-form on M.

The case $p = q = 0$ gives the notion of smooth functions. Namely, a *smooth function* on X is a continuous function $f : X \to \mathbb{C}$ such that for any point $x \in X$, there is an open neighborhood U of x in X and an analytic map $i : U \to M$ to a complex manifold M under which U is a closed analytic subvariety of M, such that $f|_U$ can be extended to an infinitely differentiable function $\tilde{f} : M \to \mathbb{C}$. Note that the smoothness here is stronger than that in [Zha1].

With the definition of smooth (p,q)-forms, most notions and operations on differential forms and currents on complex manifolds can be extended to complex analytic varieties.

Let L be a line bundle on X. By a *continuous metric* (resp. *smooth metric*) of L on X, we mean the assignment of a metric $\|\cdot\|$ to the fiber $L(x)$ above every point $x \in X$, which varies continuously (resp. smoothly) in that for any local section s of L defined on an open subset U of X, the function $\|s(x)\|^2$ is continuous (resp. smooth) in $x \in U$.

For any continuous metric $\|\cdot\|$ of L on X, the *Chern current*

$$c_1(L, \|\cdot\|) := \frac{1}{\pi i} \partial\overline{\partial} \log \|s\| + \delta_{\mathrm{div}(s)}$$

is a $(1,1)$-current on X. Here s is any meromorphic section of L on X, and the definition is independent of the choice of s.

A continuous metric $\|\cdot\|$ of L on X is called *semipositive* if the *Chern current* is a positive current; equivalently, $\|\cdot\|$ is semipositive if for any local section s of L analytic and everywhere non-vanishing on an open subset U of X, the function $-\log \|s(x)\|$ is pluri-subharmonic (psh) on U. We refer to [Kli] for basic theory of psh functions, and to [Mor7, §1] for a brief extension of psh functions to singular analytic varieties.

A continuous metric $\|\cdot\|$ of L on X is called *integrable* if it is the quotient of two semipositive metrics; i.e., there are line bundles $(L_1, \|\cdot\|_1)$ and $(L_2, \|\cdot\|_2)$ endowed with semipositive metrics on X such that $(L, \|\cdot\|)$ is isometric to $(L_1, \|\cdot\|_1) \otimes (L_2, \|\cdot\|_2)^\vee$.

As the well-known special case, if X is smooth and the metric $\|\cdot\|$ of L is smooth, the Chern current is represented by the *Chern form* $c_1(L, \|\cdot\|)$ (by abuse of notation). In this case, $\|\cdot\|$ is semipositive on X if and only if $c_1(L, \|\cdot\|)$ is positive semi-definite as a smooth $(1,1)$-form on X. If X is general and the metric $\|\cdot\|$ is smooth, then $\|\cdot\|$ is semipositive on X if and only if $c_1(L|_{X^{\mathrm{reg}}}, \|\cdot\|)$ is positive semi-definite as a smooth $(1,1)$-form on X^{reg}.

Note that most terminologies were introduced by Moriwaki [Mor6] under different names. For example, a semipositive continuous metric is called a metric of $(C^0 \cap \mathrm{PSH})$-*type* in the loc. cit..

Let X be a complex projective variety. Let L be a line bundle on X. Then any smooth metric $\|\cdot\|$ of L on X is the quotient of two semipositive metrics of line bundles on X. In fact, if X is smooth, take an ample line bundle A with a positive metric $\|\cdot\|_A$, then $(L, \|\cdot\|) \otimes (A, \|\cdot\|_A)^{\otimes m}$ also have a positive metric for sufficiently large m. If X is singular, take a closed embedding $i : X \to \mathbb{P}^N$ and set $(A, \|\cdot\|_A) = i^*(\mathcal{O}(1), \|\cdot\|_{\mathrm{FS}})$, where $\|\cdot\|_{\mathrm{FS}}$ is the Fubini-Study metric. By some extra argument, $(L, \|\cdot\|) \otimes (A, \|\cdot\|_A)^{\otimes m}$ still has a semipositive metric for sufficiently large m. As a consequence, smooth metrics are integrable.

Note that [Zha2] also has a notion of "semipositive metrics" and "integrable metrics". We will see that our notion is essentially equivalent to those of the loc. cit. on complex projective varieties. Let L be a line bundle on a complex projective variety X with a continuous metric $\|\cdot\|$. Then we have the following comparisons:

(1) If L is ample, then $\|\cdot\|$ is semipositive (in our sense) if and only if it is semipositive in the sense of [Zha2]. In fact, by [CGZ, Cor. C], any semipositive continuous metric on L is an increasing limit of semipositive smooth metrics on L. This limit process is uniform since all the metrics are continuous. Then it is semipositive in the sense of [Zha2]. The inverse direction follows from the fact that the decreasing limit of psh functions is again psh.

(2) Any semipositive metric $\|\cdot\|$ (without assuming that L is ample) is the quotient of two semipositive continuous metrics on ample line bundles. This is trivial by tensoring $(L, \|\cdot\|)$ by an ample line bundle with a semipositive continuous metric.

(3) A continuous metric $\|\cdot\|$ is integrable if and only if it is integrable in the sense of [Zha2]. This follows from (1) and (2).

Let X be a complex projective variety of dimension n. Let $(L_1, \|\cdot\|_1), \cdots, (L_n, \|\cdot\|_n)$ be line bundles with integrable metrics on X. Then there is a (signed) Monge–Ampère measure $c_1(L_1, \|\cdot\|_1) \cdots c_1(L_n, \|\cdot\|_n)$ on X. It is reduced to semipositive metrics by linearity, and then apply the approximation method of [BT, Thm. 2.1] (or [Dem1, Cor. 1.6]).

2.1.2 Green's functions on analytic spaces

Let X be a complex analytic variety. We introduce Green's functions of divisors following the above treatment of metrics of line bundles.

Let D be an (analytic) Cartier divisor on X with support $|D|$. A *continuous Green's function* (resp. *smooth Green's function*) g of D on X is a function $g : X \setminus |D| \to \mathbb{R}$ such that for any meromorphic function f on an open subset U of X satisfying $\mathrm{div}(f) = D|_U$, the function $g + \log|f|$ can be extended to a continuous (resp. smooth) function on U.

Note that the pair (D, g) defines a pair $(\mathcal{O}(D), \|\cdot\|_g)$ with the metric defined by $\|s_D\|_g = e^{-g}$. Here s_D is the section of $\mathcal{O}(D)$ corresponding to the meromorphic function 1 on X.

By this correspondence, g is continuous (resp. smooth) if and only if $\|\cdot\|_g$ is continuous (resp. smooth). Moreover, we say that g is *semipositive* (resp. *integrable*) if $\|\cdot\|_g$ is semipositive (resp. integrable).

All the definition and results for metrics and Green's functions extend to finite disjoint unions of analytic variety (resp. complex projective variety) easily.

2.1.3 Hermitian line bundles on arithmetic varieties

By a *projective arithmetic variety (resp. quasi-projective arithmetic variety)* $\mathcal{X}$, we mean an integral scheme, projective (resp. quasi-projective) and flat over $\mathbb{Z}$. We usually denote by $\mathbb{Q}(\mathcal{X})$ the function field of $\mathcal{X}$.

Let $\mathcal{X}$ be a projective arithmetic variety. A *hermitian line bundle on* $\mathcal{X}$ is a pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$, where $\mathcal{L}$ is a line bundle (equivalently an invertible sheaf) on $\mathcal{X}$, and $\|\cdot\|$ is a *continuous metric* of $\mathcal{L}(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$, invariant under the action of the complex conjugation.

An *isometry* from a hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ to another hermitian line bundle $\overline{\mathcal{L}}' = (\mathcal{L}', \|\cdot\|')$ is an isomorphism $\mathcal{L} \to \mathcal{L}'$ of coherent sheaves compatible with the metrics.

Denote by $\widehat{\mathrm{Pic}}(\mathcal{X})$ the *group of isometry classes* of hermitian line bundles on $\mathcal{X}$. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ the *category* of hermitian line bundles on $\mathcal{X}$, in which the morphisms are isometries of hermitian line bundles. This is a groupoid by definition.

Note that for a hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$, we only require the metric to be continuous (instead of smooth). For convenience, define $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathrm{smth}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathrm{int}}$) to be the full subcategory of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ of hermitian line bundles with smooth metric (resp. integrable metrics). Define $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathrm{smth}}$ (resp. $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathrm{int}}$) to be the subgroup of $\widehat{\mathrm{Pic}}(\mathcal{X})$ similarly.

2.1.4 Arithmetic divisors

Let $\mathcal{X}$ be a projective arithmetic variety. An *arithmetic divisor on* $\mathcal{X}$ is a pair $\overline{\mathcal{D}} = (\mathcal{D}, g_{\mathcal{D}})$, where $\mathcal{D}$ is a Cartier divisor on $\mathcal{X}$, and $g_{\mathcal{D}}$ is a *continuous Green's function* of $\mathcal{D}(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$, invariant under the action of the complex conjugation. A *principal arithmetic divisor on* $\mathcal{X}$ is an arithmetic divisor of the form

$$\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f|)$$

for any rational function $f \in \mathbb{Q}(\mathcal{X})^{\times}$ on $\mathcal{X}$.

Denote by $\widehat{\mathrm{Div}}(\mathcal{X})$ the group of arithmetic divisors on $\mathcal{X}$, and by $\widehat{\mathrm{Pr}}(\mathcal{X})$ the group of principal arithmetic divisors on $\mathcal{X}$. Then we have the *arithmetic divisor class group*

$$\widehat{\mathrm{CaCl}}(\mathcal{X}) = \widehat{\mathrm{Div}}(\mathcal{X})/\widehat{\mathrm{Pr}}(\mathcal{X}).$$

An arithmetic divisor $\overline{\mathcal{D}} = (\mathcal{D}, g_{\mathcal{D}}) \in \widehat{\mathrm{Div}}(\mathcal{X})$ is *effective* (resp. *strictly effective*) if $\mathcal{D}$ is an effective Cartier divisor on $\mathcal{X}$ and the Green's function $g_{\mathcal{D}} \geq 0$ (resp. $g_{\mathcal{D}} > 0$) on $\mathcal{X}(\mathbb{C}) - |\mathcal{D}(\mathbb{C})|$.

There is a canonical map

$$\widehat{\mathrm{Div}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{X}), \qquad \overline{\mathcal{D}} \longmapsto \mathcal{O}(\overline{\mathcal{D}}),$$

which induces an isomorphism

$$\widehat{\mathrm{CaCl}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{X}).$$

The inverse image of a hermitian line bundle $\overline{\mathcal{L}}$ is represented by the divisor

$$\widehat{\mathrm{div}}(s) = \widehat{\mathrm{div}}_{(\mathcal{X},\overline{\mathcal{L}})}(s) := (\mathrm{div}(s), -\log\|s\|),$$

where s is any nonzero rational section of $\mathcal{L}$ on $\mathcal{X}$.

Similar to hermitian metrics, we only require the Green's functions to be continuous (instead of smooth). Define $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathrm{smth}}$ (resp. $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathrm{int}}$) to be the subgroup of $\widehat{\mathrm{Div}}(\mathcal{X})$ of arithmetic divisors corresponding to line bundles with smooth metric (resp. integrable metrics).

2.1.5 Intersection theory and nefness

Let $\mathcal{X}$ be a projective arithmetic variety of dimension n. Let $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n$ be hermitian line bundles with integrable metrics on $\mathcal{X}$. Then there is a well-defined *intersection number* $\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_n$. This is the Gillet–Soulé intersection number if the metrics of $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n$ are smooth, extended to semipositive metrics with ample underlying line bundle by a limit process and to integrable metrics by linearity as in [Zha2]. In particular, if s_n is a rational section of $\mathcal{L}_n$ on $\mathcal{X}$ with $\mathrm{div}(s_n) = \sum_{i=1}^r m_i Z_i$ for prime divisor Z_i of $\mathcal{X}$, then we have the induction formula

$$\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_n = \sum_{i=1}^r m_i(\overline{\mathcal{L}}_1|_{Z_i} \cdots \overline{\mathcal{L}}_{n-1}|_{Z_i}) - \int_{\mathcal{X}(\mathbb{C})} \log \|s_n\| c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}).$$

For approximation, if $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n$ are hermitian line bundles with semipositive metrics on $\mathcal{X}$, and $\{\overline{\mathcal{L}}_{i,j}\}_{j\geq 1}$ is a sequence of hermitian line bundles with underlying line bundle $\mathcal{L}_i$ and with semipositive metrics converging to the metric of $\overline{\mathcal{L}}_i$ uniformly, then

$$\overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_n = \lim_{j\to\infty} \overline{\mathcal{L}}_{1,j} \cdot \overline{\mathcal{L}}_{1,j} \cdots \overline{\mathcal{L}}_{n,j}.$$

We say that a hermitian line bundle $\overline{\mathcal{L}}$ on a projective arithmetic variety $\mathcal{X}$ is *nef* if the following conditions hold:

(1) the (continuous) metric of $\overline{\mathcal{L}}$ is semipositive,

(2) $\overline{\mathcal{L}}$ has a non-negative (arithmetic) degree on any 1-dimensional integral closed subscheme of $\mathcal{X}$.

Lemma 2.1.1. *Let $\mathcal{X}$ be a projective arithmetic variety of dimension n. Let $\overline{\mathcal{L}}$ be a nef hermitian line bundle on $\mathcal{X}$. Then the intersection number $\overline{\mathcal{L}}^n \geq 0$.*

Proof. This is well-known. See [Zha1, Lem. 4.5] for the prototype and [Mor3, Prop. 2.3] for the smooth case. If $\mathcal{L}_{\mathbb{Q}}$ is ample on $\mathcal{X}_{\mathbb{Q}}$, the semipositive continuous metric on $\mathcal{L}$ is a uniform limit of semipositive smooth metrics on $\mathcal{L}$ by [CGZ, Cor. C], so the above result also imply $\overline{\mathcal{L}}^n \geq 0$ in this case. If $\mathcal{L}_{\mathbb{Q}}$ is not ample, take a nef hermitian line bundle $\overline{\mathcal{A}}$ on $\mathcal{X}$ with an ample underlying line bundle $\mathcal{A}$. For any positive integer m, $m\mathcal{L}_{\mathbb{Q}} + \mathcal{A}_{\mathbb{Q}}$ is ample, and thus $(m\overline{\mathcal{L}} + \overline{\mathcal{A}})^n \geq 0$. This implies $\overline{\mathcal{L}}^n \geq 0$. □

Let $\mathcal{X}$ be a projective arithmetic variety. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathrm{nef}}$ the full subcategory of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ of nef hermitian line bundles on $\mathcal{X}$, and by $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathrm{nef}}$ the sub-semigroup of $\widehat{\mathrm{Pic}}(\mathcal{X})$ of nef line bundles on $\mathcal{X}$.

An arithmetic divisor $\overline{D}$ on $\mathcal{X}$ is *nef* if the hermitian line bundle $\mathcal{O}(\overline{D})$ is nef on $\mathcal{X}$. Denote by $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathrm{nef}}$ the sub-semigroup of $\widehat{\mathrm{Div}}(\mathcal{X})$ of nef line bundles on $\mathcal{X}$.

2.2 Objects of mixed coefficients

The goal of this subsection is to introduce notations for divisors and line bundles of mixed coefficients, i.e., $\mathbb{Q}$-line bundles and $\mathbb{Q}$-divisors which are integral on an open subscheme of the ambient scheme. These are less standard, but will be crucial to define effective sections of adelic line bundles in our theory.

For clarity, in this subsection, we do not take the uniform terminology in §1.6, but to introduce all the terms case by case.

2.2.1 $\mathbb{Q}$-divisors and $\mathbb{Q}$-line bundles

When we say divisors, we always mean Cartier divisors, unless otherwise specified. When we want to distinguish the usual divisors (resp. line bundles) from $\mathbb{Q}$-divisors (resp. $\mathbb{Q}$-line bundles), we often say integral divisors (resp. integral line bundles).

Let $\mathcal{X}$ be a scheme. Denote by $\mathrm{Div}(\mathcal{X}) = H^0(\mathcal{X}, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ the group of *Cartier divisors* on $\mathcal{X}$. Here $\mathcal{K}_X$ is the sheaf of rational functions on X. The image of $H^0(\mathcal{X}, \mathcal{K}_X^\times)$ in $\mathrm{Div}(\mathcal{X})$ is the subgroup of *principle Cartier divisors* on $\mathcal{X}$, denoted by $\mathrm{Pr}(\mathcal{X})$.

The *support* $|\mathcal{D}|$ of a Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ is the complement of the maximal open subscheme of $\mathcal{X}$ on which $\mathcal{D}$ is trivial. A Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ is called *effective* if it lies in the image of the semi-group $H^0(\mathcal{X}, \mathcal{O}_X/\mathcal{O}_X^\times)$ in $\mathrm{Div}(\mathcal{X})$.

An element of $\mathrm{Div}(\mathcal{X})_\mathbb{Q} = \mathrm{Div}(\mathcal{X}) \otimes_\mathbb{Z} \mathbb{Q}$ is called a (Cartier) $\mathbb{Q}$-*divisor* of $\mathcal{X}$. A $\mathbb{Q}$-divisor $\mathcal{D} \in \mathrm{Div}(\mathcal{X})_\mathbb{Q}$ is called *effective* if for some positive integer m, the multiple $m\mathcal{D}$ is an effective (integral) divisor in $\mathrm{Div}(\mathcal{X})$.

Denote by $\mathcal{P}\mathrm{ic}(\mathcal{X})$ the *category* of line bundles on $\mathcal{X}$, in which the objects are line bundles (or equivalently invertible sheaves) on $\mathcal{X}$, and the morphisms are isomorphisms of line bundles. Denote by $\mathcal{P}\mathrm{ic}(\mathcal{X})_\mathbb{Q}$ the *category* of $\mathbb{Q}$-line bundles on $\mathcal{X}$, in which the objects are pairs $(a, \mathcal{L})$ (or just written as $a\mathcal{L}$) with $a \in \mathbb{Q}$ and $\mathcal{L} \in \mathcal{P}\mathrm{ic}(\mathcal{X})$, and a morphism of two such objects is defined to be

$$\mathrm{Hom}(a\mathcal{L}, a'\mathcal{L}') := \varinjlim_m \mathrm{Hom}(am\mathcal{L}, a'm\mathcal{L}'),$$

where m runs through positive integers such that am and $a'm$ are both integers, so that $am\mathcal{L}$ and $a'm\mathcal{L}'$ are viewed as integral line bundles, and "Hom" on the right-hand side represents isomorphisms of integral line bundles. For the direct system, for any $m|n$, there is a transition map

$$\mathrm{Hom}(am\mathcal{L}, a'm\mathcal{L}') \to \mathrm{Hom}(an\mathcal{L}, a'n\mathcal{L}')$$

locally given by taking (n/m)-th power of an isomorphism. The group of isomorphism classes of objects of $\mathcal{P}\mathrm{ic}(\mathcal{X})_{\mathbb{Q}}$ is isomorphic to $\mathrm{Pic}(\mathcal{X})_{\mathbb{Q}} = \mathrm{Pic}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $a\mathcal{L}$ be a $\mathbb{Q}$-line bundle on $\mathcal{X}$ with $a \in \mathbb{Q}$ and $\mathcal{L} \in \mathcal{P}\mathrm{ic}(\mathcal{X})$. A *section* of $a\mathcal{L}$ on $\mathcal{X}$ is an element of $\mathrm{Hom}(\mathcal{O}_X, a\mathcal{L}) = \varinjlim_m \Gamma(\mathcal{X}, am\mathcal{L})$, where m runs through positive integers with $am \in \mathbb{Z}$. If $\mathcal{X}$ is an integral scheme, a *rational section* of $a\mathcal{L}$ on $\mathcal{X}$ is an element of $\mathrm{Hom}(\mathcal{O}_\eta, a\mathcal{L}_\eta) = \varinjlim_m \Gamma(\eta, am\mathcal{L})$, where η is the generic point of $\mathcal{X}$, and m runs through positive integers with $am \in \mathbb{Z}$. If s is a section represented by $s_m \in \Gamma(\mathcal{X}, am\mathcal{L})$ or a rational section represented by $s_m \in \Gamma(\eta, am\mathcal{L})$, then define

$$\mathrm{div}(s) := \frac{1}{m}\mathrm{div}(s_m).$$

This is a $\mathbb{Q}$-divisor on $\mathcal{X}$.

If $\mathcal{X}$ is a projective variety over a field, a $\mathbb{Q}$-divisor $\mathcal{D} \in \mathrm{Div}(\mathcal{X})_{\mathbb{Q}}$ (resp. $\mathbb{Q}$-line bundle $\mathcal{L} \in \mathcal{P}\mathrm{ic}(\mathcal{X})_{\mathbb{Q}}$) is called *nef* if for some positive integer m, the multiple $m\mathcal{D}$ (resp. $m\mathcal{L}$) is a nef divisor on $\mathcal{X}$ (resp. nef line bundle on $\mathcal{X}$) in the usual sense.

2.2.2 Arithmetic $\mathbb{Q}$-divisors and hermitian $\mathbb{Q}$-line bundles

The above $\mathbb{Q}$-notions extend easily to the arithmetic situation. We sketch them briefly.

Let $\mathcal{X}$ be a projective arithmetic variety. An element of $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}} = \widehat{\mathrm{Div}}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is called an *arithmetic* $\mathbb{Q}$-*divisor*. So an arithmetic $\mathbb{Q}$-divisor is still represented by a pair $(\mathcal{D}, g_{\mathcal{D}})$, where $\mathcal{D}$ is a Cartier $\mathbb{Q}$-divisor on $\mathcal{X}$, and $g_{\mathcal{D}}$ is a Green's function of $\mathcal{D}$ on $\mathcal{X}$ defined similarly.

An arithmetic $\mathbb{Q}$-divisor $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ is called *effective* (resp. *strictly effective*) if for some positive integer m, the multiple $m\overline{\mathcal{D}}$ is an effective (resp. strictly effective) (integral) arithmetic divisor in $\widehat{\mathrm{Div}}(\mathcal{X})$.

Denote by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathbb{Q}}$ the *category* of hermitian $\mathbb{Q}$-line bundles on $\mathcal{X}$, in which the objects are pairs $(a, \overline{\mathcal{L}})$ (or just written as $a\overline{\mathcal{L}}$) with $a \in \mathbb{Q}$ and

$\overline{\mathcal{L}} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$, and the morphism of two such objects is defined to be

$$\mathrm{Hom}(a\overline{\mathcal{L}}, a'\overline{\mathcal{L}}') := \varinjlim_{m} \mathrm{Isom}(am\overline{\mathcal{L}}, a'm\overline{\mathcal{L}}'),$$

where "Isom" represents isometries, and m runs through positive integers such that am and $a'm$ are both integers.

If s is a section of $a\mathcal{L}$ represented by $s_m \in \Gamma(\mathcal{X}, am\mathcal{L})$ or a rational section of $a\mathcal{L}$ represented by $s_m \in \Gamma(\eta, am\mathcal{L})$, where $\eta \in \mathcal{X}$ is the generic point as above, then define

$$\mathrm{div}(s) := \frac{1}{m}\mathrm{div}(s_m), \qquad \widehat{\mathrm{div}}(s) := \frac{1}{m}\widehat{\mathrm{div}}(s_m).$$

These are respectively $\mathbb{Q}$-divisors and arithmetic $\mathbb{Q}$-divisors on $\mathcal{X}$.

An arithmetic $\mathbb{Q}$-divisor $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ (resp. hermitian $\mathbb{Q}$-line bundle $\overline{\mathcal{L}} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathbb{Q}}$) is called *nef* if for some positive integer m, the multiple $m\overline{\mathcal{D}}$ (resp. $m\overline{\mathcal{L}}$) is a nef arithmetic divisor in $\widehat{\mathrm{Div}}(\mathcal{X})$ (resp. nef hermitian line bundle in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$) in the usual sense.

2.2.3 $(\mathbb{Q}, \mathbb{Z})$-divisors

Let $\mathcal{U}$ be an open subscheme of an integral scheme $\mathcal{X}$. Define $\mathrm{Div}(\mathcal{X},\mathcal{U})$ to be the fiber product of the natural map $\phi : \mathrm{Div}(\mathcal{X})_{\mathbb{Q}} \to \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}$ with the natural map $\psi : \mathrm{Div}(\mathcal{U}) \to \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}$; i.e.,

$$\mathrm{Div}(\mathcal{X},\mathcal{U}) = \ker(\phi - \psi : \mathrm{Div}(\mathcal{X})_{\mathbb{Q}} \oplus \mathrm{Div}(\mathcal{U}) \to \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}).$$

In other words, $\mathrm{Div}(\mathcal{X},\mathcal{U})$ is the group of pairs $(\mathcal{D}, \mathcal{D}')$, where $\mathcal{D} \in \mathrm{Div}(\mathcal{X})_{\mathbb{Q}}$ and $\mathcal{D}' \in \mathrm{Div}(\mathcal{U})$ have equal images in $\mathrm{Div}(\mathcal{U})_{\mathbb{Q}}$.

An element $(\mathcal{D}, \mathcal{D}')$ of $\mathrm{Div}(\mathcal{X},\mathcal{U})$ is called a $(\mathbb{Q}, \mathbb{Z})$*-divisor on* $(\mathcal{X},\mathcal{U})$ or a $\mathbb{Q}$*-divisor of* $\mathcal{X}$ *integral on* $\mathcal{U}$. We usually call $\mathcal{D}$ *the rational part* of $(\mathcal{D}, \mathcal{D}')$, and call $\mathcal{D}'$ *the integral part* of $(\mathcal{D}, \mathcal{D}')$.

By definition, there are projection maps

$$\mathrm{Div}(\mathcal{X},\mathcal{U}) \longrightarrow \mathrm{Div}(\mathcal{X})_{\mathbb{Q}}, \qquad \mathrm{Div}(\mathcal{X},\mathcal{U}) \longrightarrow \mathrm{Div}(\mathcal{U}).$$

By abuse of notations, we may abbreviate an element $(\mathcal{D}, \mathcal{D}')$ of $\mathrm{Div}(\mathcal{X},\mathcal{U})$ as $\mathcal{D}$, and then write $\mathcal{D}|_{\mathcal{U}}$ for $\mathcal{D}'$, viewed as an integral divisor on $\mathcal{U}$.

There are canonical maps

$$\mathrm{Div}(\mathcal{X}) \longrightarrow \mathrm{Div}(\mathcal{X},\mathcal{U}), \quad \mathcal{E} \longmapsto (\mathcal{E}, \mathcal{E}_{\mathcal{U}})$$

and

$$\mathrm{Div}(\mathcal{U})_{\mathrm{tor}} \longrightarrow \mathrm{Div}(\mathcal{X},\mathcal{U}), \quad \mathcal{T} \longmapsto (0,\mathcal{T}).$$

Then we have a canonical exact sequence

$$0 \longrightarrow \mathrm{Div}(\mathcal{U})_{\mathrm{tor}} \longrightarrow \mathrm{Div}(\mathcal{X},\mathcal{U}) \longrightarrow \mathrm{Div}(\mathcal{X})_{\mathbb{Q}} \longrightarrow \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}/\mathrm{Div}(\mathcal{U}).$$

Then there is a canonical isomorphism

$$\mathrm{Div}(\mathcal{X},\mathcal{U})_{\mathbb{Q}} \longrightarrow \mathrm{Div}(\mathcal{X})_{\mathbb{Q}}.$$

Take quotient

$$\mathrm{CaCl}(\mathcal{X},\mathcal{U}) := \mathrm{Div}(\mathcal{X},\mathcal{U})/\,\mathrm{Pr}(\mathcal{X}).$$

Here $\mathrm{Pr}(\mathcal{X})$ is mapped to $\mathrm{Div}(\mathcal{X},\mathcal{U})$ via $\mathrm{Div}(\mathcal{X}) \to \mathrm{Div}(\mathcal{X},\mathcal{U})$. Note that $\mathrm{Pr}(\mathcal{X}) \to \mathrm{Div}(\mathcal{X},\mathcal{U})$ is not necessarily injective, but the quotient makes sense by group action. There are canonical maps

$$\mathrm{CaCl}(\mathcal{X},\mathcal{U}) \longrightarrow \mathrm{CaCl}(\mathcal{X})_{\mathbb{Q}}, \quad \mathrm{CaCl}(\mathcal{X},\mathcal{U}) \longrightarrow \mathrm{CaCl}(\mathcal{U}).$$

An element of $\mathrm{Div}(\mathcal{X},\mathcal{U})$ is called *effective* if its images in $\mathrm{Div}(\mathcal{X})_{\mathbb{Q}}$ and $\mathrm{Div}(\mathcal{U})$ are both effective.

If $\mathcal{X}$ is a projective variety over a field, an element of $\mathrm{Div}(\mathcal{X},\mathcal{U})$ is called *nef* if its image in $\mathrm{Div}(\mathcal{X})_{\mathbb{Q}}$ is nef.

2.2.4 Arithmetic $(\mathbb{Q},\mathbb{Z})$-divisors

The above mixed notions extend easily to the arithmetic situation.

Let $\mathcal{U}$ be an open subscheme of a projective arithmetic variety $\mathcal{X}$. Define $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ to be the fiber product of the natural map $\widehat{\phi}:\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}} \to \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}$ with the natural map $\widehat{\psi}:\mathrm{Div}(\mathcal{U}) \to \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}$; i.e.,

$$\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) = \ker(\widehat{\phi}-\widehat{\psi}:\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}} \oplus \mathrm{Div}(\mathcal{U}) \to \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}).$$

In other words, $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ is the group of pairs $(\overline{\mathcal{D}},\mathcal{D}')$, where $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ and $\mathcal{D}' \in \mathrm{Div}(\mathcal{U})$ have equal images in $\mathrm{Div}(\mathcal{U})_{\mathbb{Q}}$. Note that the second component uses $\mathrm{Div}(\mathcal{U})$ (instead of $\widehat{\mathrm{Div}}(\mathcal{U})$), so it puts no condition on the Green's function.

An element of $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ is called an *arithmetic* $(\mathbb{Q},\mathbb{Z})$*-divisor on* $(\mathcal{X},\mathcal{U})$ or an *arithmetic* $\mathbb{Q}$*-divisor of* $\mathcal{X}$ *integral on* $\mathcal{U}$. We usually call $\overline{\mathcal{D}}$ *the rational part* of $(\overline{\mathcal{D}},\mathcal{D}')$, and call $\mathcal{D}'$ *the integral part* of $(\overline{\mathcal{D}},\mathcal{D}')$.

There are projection maps

$$\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) \to \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}, \qquad \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) \to \mathrm{Div}(\mathcal{U})$$

By abuse of notations, we may abbreviate an element $(\overline{\mathcal{D}},\mathcal{D}')$ of $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ as $\overline{\mathcal{D}}$, and then write $\mathcal{D}|_{\mathcal{U}}$ for $\mathcal{D}'$, viewed as an integral divisor on $\mathcal{U}$.

There are canonical maps

$$\widehat{\mathrm{Div}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}), \quad \mathrm{Div}(\mathcal{U})_{\mathrm{tor}} \longrightarrow \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}).$$

We have a canonical exact sequence

$$0 \longrightarrow \mathrm{Div}(\mathcal{U})_{\mathrm{tor}} \longrightarrow \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}} \longrightarrow \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}/\mathrm{Div}(\mathcal{U}).$$

Then there is a canonical isomorphism

$$\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}.$$

There is also a canonical injection

$$\ker(\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}} \to \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}), \qquad \overline{\mathcal{D}} \longmapsto (\overline{\mathcal{D}},0).$$

Take quotient

$$\widehat{\mathrm{CaCl}}(\mathcal{X},\mathcal{U}) := \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})/\widehat{\mathrm{Pr}}(\mathcal{X}).$$

Here the quotient is via the composition $\widehat{\mathrm{Pr}}(\mathcal{X}) \to \widehat{\mathrm{Div}}(\mathcal{X}) \to \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$.

An element of $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ is called *effective* if its images in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ and $\mathrm{Div}(\mathcal{U})$ are both effective.

An element of $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ is called *nef* if its image in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ is nef.

2.3 Essentially quasi-projective schemes

The goal of this subsection is to define some basic terms about arithmetic models, and introduce essentially quasi-projective schemes, a special class of schemes on which we can define adelic divisors and adelic line bundles naturally.

2.3.1 Pro-open immersions

A morphism $i : X \to Y$ of schemes is called a *pro-open immersion* if it satisfies the following conditions:

(i) i is injective as a map between the underlying topological spaces;

(ii) i induces isomorphisms between the local rings; i.e., for any point $x \in X$, the induced map $\mathcal{O}_{Y,i(x)} \to \mathcal{O}_{X,x}$ is an isomorphism.

By Raynaud [Ray1, Prop. 1.1], pro-open immersions are exactly flat monomorphisms, and they are systematically studied in the loc. cit. In fact, we have the following equivalent definitions.

Proposition 2.3.1. *Let $i : X \to Y$ be a morphism of quasi-compact schemes. Then the following are equivalent:*

(1) The morphism i is a flat monomorphism; i.e, i is flat, and $\mathrm{Hom}(S, X) \to \mathrm{Hom}(S, Y)$ *is injective for any scheme S.*

(2) The morphism i is a pro-open immersion; i.e., i induces an injection between the underlying spaces and isomorphisms between the local rings.

(3) The map $i : X \to Y$ is a homeomorphism of X to its image $i(X)$ endowed with topology induced from Y; the image $i(X)$ is equal to the intersection of its open neighborhoods in Y; the natural morphism $(X, \mathcal{O}_X) \to (X, i^{-1}\mathcal{O}_Y)$ is an isomorphism of ringed spaces. Here $i^{-1}\mathcal{O}_Y$ denotes the pull-back as abelian sheaves.

Proof. See [Ray1, Prop. 1.1, Prop. 1.2]. □

Another property of [Ray1, Prop. 1.2] is as follows.

Lemma 2.3.2. *Let $i : X \to Y$ be a pro-open morphism of quasi-compact schemes. If Y is noetherian, then X is also noetherian.*

To justify the term "pro-open", note that a pro-open immersion to a scheme Y is given by the projective limit of some system of open subschemes of Y; see [Ray1, Prop. 2.3]. We refer to the loc. cit. for more properties.

2.3.2 Essentially quasi-projective schemes

Let k be either $\mathbb{Z}$ or a field. We take the convention in §1.6 for objects over k.

Recall that, by a *projective variety (resp. quasi-projective variety) over* k, we mean an integral scheme, projective (resp. quasi-projective) and flat over k. We make the following further definitions.

(1) **Essentially quasi-projective schemes.** A flat integral noetherian scheme X over k is called *essentially quasi-projective over* k if there is a pro-open immersion $i : X \to \mathcal{X}$ over k for some projective variety $\mathcal{X}$ over k.

(2) **Quasi-projective models and projective models.** Let X be a flat and essentially quasi-projective integral scheme over k. By a *quasi-projective model* (resp. *projective model*) of X over k, we mean a pro-open immersion $X \to \mathcal{U}$ (resp. $X \to \mathcal{X}$) for a quasi-projective variety $\mathcal{U}$ (resp. projective variety) over k.

The following are three important and natural classes of essentially quasi-projective schemes over k:

(a) a quasi-projective variety over k,

(b) a quasi-projective variety X over a finitely generated field F over k (including the case $X = \operatorname{Spec} F$),

(c) the spectrum of the local ring of a quasi-projective variety over k at a point.

In this paper, we are mainly concerned with case (a) and (b). If X is in case (a), any pro-open immersion $X \to \mathcal{X}$ to a projective variety $\mathcal{X}$ over k is necessarily an open immersion, so the notions of projective models regarding X as a quasi-projective variety and as an essentially quasi-projective variety coincide. If X is in case (b), its quasi-projective model is actually not as arbitrary as it seems. In fact, Lemma 2.3.3 asserts that it essentially comes from the generic fiber of a morphism $\mathcal{U} \to \mathcal{V}$ of quasi-projective varieties over k.

2.3.3 More properties

The following result describes the pro-open immersion in case (b).

Lemma 2.3.3. *Let F be a finitely generated field over k, X be a quasi-projective variety over F, and $i : X \to \mathcal{U}$ be a quasi-projective model of X over k. Then there is an open subscheme $\mathcal{U}'$ of $\mathcal{U}$ containing the image of X together with a flat morphism $\mathcal{U}' \to \mathcal{V}$ of quasi-projective varieties over k, such that the generic fiber of $\mathcal{U}' \to \mathcal{V}$ is isomorphic to $X \to \operatorname{Spec} F$.*

Furthermore, if X is projective over F, then we can assume that the flat morphism $\mathcal{U}' \to \mathcal{V}$ is projective.

Proof. The last statement follows from the quasi-projective case by choosing an open subscheme of $\mathcal{V}$, since projectivity is an open condition.

For the quasi-projective case, let $\mathcal{V}$ be a quasi-projective model of $\operatorname{Spec} F$ over k. Then the rational map $\mathcal{U} \dashrightarrow \mathcal{V}$ is defined on an open neighborhood of X in $\mathcal{U}$. Replacing $\mathcal{U}$ by an open subscheme if necessary, we can assume that the rational map extends to a morphism $\mathcal{U} \to \mathcal{V}$. Denote by $\eta \in \mathcal{V}$ the generic point and by $\mathcal{U}_\eta \to \eta$ the generic fiber of $\mathcal{U} \to \mathcal{V}$. By the universal property of the fiber product of $\mathcal{U} \to \mathcal{V}$ and $\eta \to \mathcal{V}$, we have a morphism $j : X \to \mathcal{U}_\eta$ over F, whose composition with $\mathcal{U}_\eta \to \mathcal{U}$ is exactly $i : X \to \mathcal{U}$.

The morphism $j : X \to \mathcal{U}_\eta$ is flat and of finite type, so it is an open map. In particular, the image $j(X)$ is open in $\mathcal{U}_\eta$. By Proposition 2.3.1(3), $j : X \to \mathcal{U}_\eta$ is an open immersion. Then the result follows. $\square$

In the general case, we have the following result about the inverse systems of quasi-projective models.

Lemma 2.3.4. *Let X be a flat and essentially quasi-projective integral scheme over k, and let $\mathcal{U}$ be a fixed quasi-projective model of X over k. Then the inverse system of open neighborhoods of X in $\mathcal{U}$ is cofinal to the inverse system of quasi-projective models of X over k.*

Proof. Let $\mathcal{U}'$ be a quasi-projective model of X over k. Then the rational map $\mathcal{U} \dashrightarrow \mathcal{U}'$ is defined on an open neighborhood $\mathcal{U}''$ of X in $\mathcal{U}$. Then the system $\{\mathcal{U}''\}$ is cofinal to the system $\{\mathcal{U}'\}$. $\square$

2.3.4 Effectivity of Cartier divisors

Over a normal scheme, effectivity of a Cartier divisor is easy, as it can be checked in terms of effectivity of the corresponding Weil divisor. Then we have the following result.

Lemma 2.3.5. *Let $\mathcal{X}$ be a normal integral scheme, and let $\psi : \mathcal{X}' \to \mathcal{X}$ be a birational proper morphism of integral schemes. Let $\mathcal{D}$ be a Cartier divisor on $\mathcal{X}$. Then the following are equivalent:*

(1) $\mathcal{D}$ is effective on $\mathcal{X}$;

(2) $\psi^*\mathcal{D}$ is effective on $\mathcal{X}'$;

(3) $\mathcal{D}$ is effective as a $\mathbb{Q}$-divisor on $\mathcal{X}$; i.e., $m\mathcal{D}$ is effective for some positive integer m.

Proof. The proof is straight-forward using Weil divisors, except that to prove (2) implies (1), we need to replace $\mathcal{X}'$ by its normalization. □

Without normality, the situation is very delicate. The following result solves the problem for our purpose. Recall that for a dominant morphism of integral schemes $V \to W$, we say that W *is integrally closed in* V if the normalization of W in V is isomorphic to W.

Lemma 2.3.6. *Let $i : X \to \mathcal{X}$ be a pro-open immersion of integral noetherian schemes. Assume that $\mathcal{X}$ is integrally closed in X. Then a Cartier divisor $\mathcal{D}$ on $\mathcal{X}$ is effective if and only if the following two conditions hold simultaneously:*

(1) the pull-back $\mathcal{D}|_X$ is effective on X;

(2) for any $v \in \mathcal{X} \setminus X$ of codimension one in $\mathcal{X}$, the valuation $\mathrm{ord}_v(\mathcal{D})$ in the local ring $\mathcal{O}_{\mathcal{X},v}$ is non-negative.

Proof. We first claim that the local ring $\mathcal{O}_{\mathcal{X},v}$ in (2) is a discrete valuation ring. In fact, the base change of $X \to \mathcal{X}$ by $\operatorname{Spec} \mathcal{O}_{\mathcal{X},v} \to \mathcal{X}$ is exactly $\operatorname{Spec} k(X) \to \operatorname{Spec} \mathcal{O}_{\mathcal{X},v}$. Here $k(X)$ denotes the function field of X, which is also the fraction field of $\mathcal{O}_{\mathcal{X},v}$. As a consequence, $\operatorname{Spec} \mathcal{O}_{\mathcal{X},v}$ is integrally closed in $\operatorname{Spec} k(X)$. Then $\mathcal{O}_{\mathcal{X},v}$ is a discrete valuation ring since it has dimension 1.

To prove the lemma, we only need to prove the "if" part. If $\mathcal{X}$ is normal, then we can write Cartier divisors in terms of Weil divisors, and the effectivity of them are equivalent. Then the result is easy.

In the general case, let f be a local equation of $\mathcal{D}$ in an affine open subscheme $\mathcal{W}$ of $\mathcal{X}$. Then $f \in k(\mathcal{W})^\times$, and we need to prove $f \in \mathcal{O}(\mathcal{W})$. By the normal case, f is regular on the normalization $\mathcal{W}'$ of $\mathcal{W}$. As a consequence,

f is integral over $\mathcal{O}(\mathcal{W})$. Note that $f \in \mathcal{O}(\mathcal{W} \cap X)$ by the assumption that $\mathcal{D}|_X$ is effective. By assumption, $\mathcal{O}(\mathcal{W})$ is integrally closed in $\mathcal{O}(\mathcal{W} \cap X)$. Therefore, $f \in \mathcal{O}(\mathcal{W})$. This finishes the proof. $\square$

Now we have the following variant of Lemma 2.3.5 for non-normal schemes.

Lemma 2.3.7. *Let $i' : X \to \mathcal{X}'$ and $i : X \to \mathcal{X}$ be pro-open immersions of integral noetherian schemes, and let $\psi : \mathcal{X}' \to \mathcal{X}$ be a birational and proper morphism such that $i \circ \psi = i'$. Assume that $\mathcal{X}$ is integrally closed in X. Let $\mathcal{D}$ be a Cartier divisor on $\mathcal{X}$. Then the following are equivalent:*

(1) $\mathcal{D}$ is effective on $\mathcal{X}$;

(2) $\psi^*\mathcal{D}$ is effective on $\mathcal{X}'$;

Proof. It is trivial that (1) implies (2). For the opposite direction, replace $\mathcal{X}'$ by its normalization in X, and apply Lemma 2.3.6. $\square$

2.4 Adelic divisors

Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. The goal of this section is to define adelic divisors on X. We will start the definition for a quasi-projective variety over k, and the general case is obtained as a direct limit over all quasi-projective models.

2.4.1 Adelic divisors on a quasi-projective variety

Let k be either $\mathbb{Z}$ or a field. Take the uniform terminology in §1.6. Let $\mathcal{U}$ be a quasi-projective variety over k.

Let $\mathcal{X}$ be a projective model of $\mathcal{U}$ over k. In the spirit of §1.6, denote by $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ the group of arithmetic $(\mathbb{Q},\mathbb{Z})$-divisors on $(\mathcal{X},\mathcal{U})$. Hence, in the geometric case (that k is a field), we take the convention $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) = \mathrm{Div}(\mathcal{X},\mathcal{U})$, and an arithmetic $(\mathbb{Q},\mathbb{Z})$-divisor in this case just means a $(\mathbb{Q},\mathbb{Z})$-divisor. Both cases are introduced in §2.2.

Projective models $\mathcal{X}$ of $\mathcal{U}$ over k form an inverse system. Using pull-back morphisms, we can form the direct limits:

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}),$$

$$\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Pr}}(\mathcal{X}).$$

Here the subscript "mod" represents "model divisors", as these divisors are defined on single projective models. Now we are going to introduce a topology on $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$.

For any $\overline{\mathcal{D}}, \overline{\mathcal{E}} \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$, write $\overline{\mathcal{D}} \geq \overline{\mathcal{E}}$ or $\overline{\mathcal{E}} \leq \overline{\mathcal{D}}$ if $\overline{\mathcal{D}} - \overline{\mathcal{E}}$ is effective. It is a partial order in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$. This induces a partial order in $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ by the law that $\overline{\mathcal{D}} \geq \overline{\mathcal{E}}$ or $\overline{\mathcal{E}} \leq \overline{\mathcal{D}}$ in $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ if the image of $\overline{\mathcal{D}} - \overline{\mathcal{E}}$ in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ and the image of $\overline{\mathcal{D}} - \overline{\mathcal{E}}$ in $\mathrm{Div}(\mathcal{U})$ are both effective. By direct limit, we have an induced partial order in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$, and we will use the same symbols for it.

In the geometric case (that k is a field), by a *boundary divisor of* $\mathcal{U}/k$, we mean a pair $(\mathcal{X}_0, \mathcal{E}_0)$ consisting of a projective model $\mathcal{X}_0$ of $\mathcal{U}$ over k and an effective Cartier divisor $\mathcal{E}_0$ on $\mathcal{X}_0$ with support equal to $\mathcal{X}_0 \setminus \mathcal{U}$. To see the existence of $(\mathcal{X}_0, \mathcal{E}_0)$, take any projective model $\mathcal{X}_0'$ of $\mathcal{U}$ over k, and blow-up $\mathcal{X}_0'$ along the reduced center $\mathcal{X}_0' \setminus \mathcal{U}$. We get a projective model $\mathcal{X}_0$ of $\mathcal{U}$, and the exceptional divisor of $\mathcal{X}_0 \to \mathcal{X}_0'$ is a Cartier divisor with support $\mathcal{X}_0 \setminus \mathcal{U}$.

In the arithmetic case ($k = \mathbb{Z}$), by a *boundary divisor of* $\mathcal{U}/k$, we mean a pair $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ consisting of a projective model $\mathcal{X}_0$ of $\mathcal{U}$ over k and a *strictly effective* Cartier divisor $\overline{\mathcal{E}}_0$ on $\mathcal{X}_0$ such that the support of the finite part $\mathcal{E}_0$ is equal to $\mathcal{X}_0 \setminus \mathcal{U}$.

To unify the terminology, in the geometric case, write $\overline{\mathcal{E}}_0 = \mathcal{E}_0$ in $\widehat{\mathrm{Div}}(\mathcal{X}_0) = \mathrm{Div}(\mathcal{X}_0)$. Then in both cases, a boundary divisor is written in the form $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$, and the following are for the both cases.

For any $r \in \mathbb{Q}$, view $r\overline{\mathcal{E}}_0$ as an element of $\widehat{\mathrm{Div}}(\mathcal{X}_0, \mathcal{U})$ by setting its integral part in $\mathrm{Div}(\mathcal{U})$ to be 0. Then $r\overline{\mathcal{E}}_0$ is also viewed as an element of $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathrm{mod}}$.

Let $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ be a boundary divisor of $\mathcal{U}/k$. We have a *boundary norm*

$$\|\cdot\|_{\overline{\mathcal{E}}_0} : \widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}} \longrightarrow [0, \infty]$$

defined by

$$\|\overline{\mathcal{D}}\|_{\overline{\mathcal{E}}_0} := \inf\{\epsilon \in \mathbb{Q}_{>0} : \ -\epsilon\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}} \leq \epsilon\overline{\mathcal{E}}_0\}.$$

Here we take the convention that $\inf(\emptyset) = \infty$. Note that $\|\cdot\|_{\overline{\mathcal{E}}_0}$ can take value infinity, but it is an *extended norm* in the sense of [Bee, Def. 1.1]. We refer to [Bee] for more theory on extended norms. In our situation, we have the following basic properties.

Lemma 2.4.1. *The boundary norm* $\|\cdot\|_{\overline{\mathcal{E}}_0}$ *on* $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ *satisfies the following properties:*

(1) $\|\overline{\mathcal{D}}\|_{\overline{\mathcal{E}}_0} = 0$ *if and only if* $\overline{\mathcal{D}} = 0$;

(2) $\|\overline{\mathcal{D}}_1 + \overline{\mathcal{D}}_2\|_{\overline{\mathcal{E}}_0} \leq \|\overline{\mathcal{D}}_1\|_{\overline{\mathcal{E}}_0} + \|\overline{\mathcal{D}}_2\|_{\overline{\mathcal{E}}_0}$;

(3) $\|a\overline{\mathcal{D}}\|_{\overline{\mathcal{E}}_0} \leq |a| \cdot \|\overline{\mathcal{D}}\|_{\overline{\mathcal{E}}_0}$ *for any nonzero* $a \in \mathbb{Z}$. *The inequality is strict if and only if both* $\mathcal{D} \neq 0$ *and* $a\mathcal{D} = 0$ *hold in* $\mathrm{Div}(\mathcal{U})$, *where* $\mathcal{D}$ *denotes the image of* $\overline{\mathcal{D}}$ *in* $\mathrm{Div}(\mathcal{U})$.

Moreover, if $(\mathcal{X}_0', \overline{\mathcal{E}}_0')$ *is another boundary divisor, then* $\|\cdot\|_{\overline{\mathcal{E}}_0'}$ *is equivalent to* $\|\cdot\|_{\overline{\mathcal{E}}_0}$ *in the sense that there is a real number* $r > 1$ *such that*

$$r^{-1}\|\cdot\|_{\overline{\mathcal{E}}_0} \leq \|\cdot\|_{\overline{\mathcal{E}}_0'} \leq r\|\cdot\|_{\overline{\mathcal{E}}_0}.$$

Proof. Note that (2) and (3) are automatic by definition. For (1), assume that $\|\overline{\mathcal{D}}\|_{\overline{\mathcal{E}}_0} = 0$ for some $\overline{\mathcal{D}}$; i.e., $-\epsilon\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}} \leq \epsilon\overline{\mathcal{E}}_0$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ for all positive rational numbers ϵ. Assume that $\overline{\mathcal{D}}$ comes from $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ for a projective model $\mathcal{X}$ of $\mathcal{U}$, and assume that $\mathcal{X}$ dominates $\mathcal{X}_0$ and is integrally closed in $\mathcal{U}$. By Lemma 2.3.7, $-\epsilon\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}} \leq \epsilon\overline{\mathcal{E}}_0$ holds in $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ for all positive rational numbers ϵ. Then we can conclude $\overline{\mathcal{D}} = 0$ by Lemma 2.3.6.

For the equivalence of the two norms, it suffices to find a rational number $r > 1$ such that $r^{-1}\overline{\mathcal{E}}_0 \leq \overline{\mathcal{E}}_0' \leq r\overline{\mathcal{E}}_0$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$. In fact, we can find a third projective model $\mathcal{Y}$ of $\mathcal{U}$ dominating both $\mathcal{X}_0$ and $\mathcal{X}_0'$, and we can further assume that $\mathcal{Y}$ is integrally closed in $\mathcal{U}$. Then we only need to treat the inequalities over $\mathcal{Y}$, which is an easy consequence of Lemma 2.3.6. □

The *boundary topology on* $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ is the topology induced by the boundary norm $\|\cdot\|_{\overline{\mathcal{E}}_0}$. Thus a neighborhood basis at 0 of the topology is given by

$$B(\epsilon, \widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}) := \{\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}} : \ -\epsilon\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}} \leq \epsilon\overline{\mathcal{E}}_0\}, \quad \epsilon \in \mathbb{Q}_{>0}.$$

By translation, it gives a neighborhood basis at any point. The topology does not depend on the choice of the boundary divisor $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ by the lemma.

Let $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ be the *completion* of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ with respect to the boundary topology. An element of $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ is called an *adelic divisor* (or a *compactified divisor*) of $\mathcal{U}/k$. By definition, an adelic divisor is represented by a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$, i.e., a sequence $\{\overline{\mathcal{D}}_i\}_{i\geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ satisfying the property that there is a sequence $\{\epsilon_i\}_{i\geq 1}$ of positive rational numbers converging to 0 such that

$$-\epsilon_i\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}}_{i'} - \overline{\mathcal{D}}_i \leq \epsilon_i\overline{\mathcal{E}}_0, \quad i' \geq i \geq 1.$$

The sequence $\{\overline{\mathcal{D}}_i\}_{i\geq 1}$ represents 0 in $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ if and only if there is a sequence $\{\delta_i\}_{i\geq 1}$ of positive rational numbers converging to 0 such that

$$-\delta_i\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}}_i \leq \delta_i\overline{\mathcal{E}}_0, \quad i \geq 1.$$

Define the *class group of adelic divisors* of $\mathcal{U}$ to be

$$\widehat{\mathrm{CaCl}}(\mathcal{U}/k) := \widehat{\mathrm{Div}}(\mathcal{U}/k)/\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{\mathrm{mod}}.$$

The map $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) \to \mathrm{Div}(\mathcal{U})$ induces maps

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}} \longrightarrow \mathrm{Div}(\mathcal{U}), \quad \widehat{\mathrm{CaCl}}(\mathcal{U}/k)_{\mathrm{mod}} \longrightarrow \mathrm{CaCl}(\mathcal{U}).$$

We call these maps *restriction maps* or *forgetful maps*.

Remark 2.4.2. In the arithmetic case $k = \mathbb{Z}$, for the definition

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}} = \lim_{\overrightarrow{\mathcal{X}}} \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}),$$

we allow elements of $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ to have continuous Green's functions, instead of smooth Green's functions. See §2.1 for the definitions of these terms. However, both choices give the same completion $\widehat{\mathrm{Div}}(\mathcal{U}/k)$, since continuous functions on $\mathcal{X}(\mathbb{C})$ can be approximated by smooth functions uniformly.

2.4.2 Completion of the divisor class group

Let $\mathcal{U}$ be a quasi-projective variety over k. Consider the class group of model divisors:

$$\widehat{\mathrm{CaCl}}(\mathcal{U}/k)_{\mathrm{mod}} = \lim_{\overrightarrow{\mathcal{X}}} \widehat{\mathrm{CaCl}}(\mathcal{X},\mathcal{U}) \simeq \widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}/\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{\mathrm{mod}}.$$

It is endowed with the quotient topology induced by the boundary topology of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$. On the other hand,

$$\widehat{\mathrm{CaCl}}(\mathcal{U}/k) = \widehat{\mathrm{Div}}(\mathcal{U}/k)/\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{\mathrm{mod}}$$

is not defined to be the completion of $\widehat{\mathrm{CaCl}}(\mathcal{U}/k)_{\mathrm{mod}}$. However, the following result asserts that these two are actually isomorphic.

Lemma 2.4.3. *The group $\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{\mathrm{mod}}$ is discrete in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ under the boundary topology. Therefore, $\widehat{\mathrm{CaCl}}(\mathcal{U}/k)$ is canonically isomorphic to the completion of $\widehat{\mathrm{CaCl}}(\mathcal{U}/k)_{\mathrm{mod}}$.*

Proof. It suffices to prove the first statement. In the following, we assume the arithmetic case $k = \mathbb{Z}$, since the geometric case is similar and easier.

Assume that there is a sequence $\overline{\mathcal{D}}_i$ in $\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{\mathrm{mod}}$ converging to 0. Then there is a sequence $\{\epsilon_i\}_{i\geq 1}$ of positive rational numbers converging to 0 such that $\epsilon_i\overline{\mathcal{E}}_0 \pm \overline{\mathcal{D}}_i \geq 0$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ for any $i \geq 1$. Assume that $\overline{\mathcal{D}}_i = \widehat{\mathrm{div}}_{\mathcal{X}_i}(f_i)$ for a projective model $\mathcal{X}_i$ of $\mathcal{U}$ and a rational function $f_i \in \mathbb{Q}(\mathcal{X}_i)^\times = \mathbb{Q}(\mathcal{U})^\times$.

We first consider the case that $\mathcal{U}$ and $\mathcal{X}_i$ are normal for $i \geq 0$. For $i = 0$, recall that the projective model $\mathcal{X}_0$ is the one chosen to define $\overline{\mathcal{E}}_0$. By Lemma 2.3.5, the relation $\epsilon_i\overline{\mathcal{E}}_0 \pm \overline{\mathcal{D}}_i \geq 0$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ is the same as the relation $\epsilon_i\overline{\mathcal{E}}_0 \pm \widehat{\mathrm{div}}_{\mathcal{X}_0}(f_i) \geq 0$ in $\widehat{\mathrm{Div}}(\mathcal{X}_0)_{\mathbb{Q}}$. When ϵ_i is small enough, we must have $\mathrm{div}_{\mathcal{X}_0}(f_i) = 0$. This implies that $f_i \in \mathcal{O}(\mathcal{X}_0)^\times = O_K^\times$. Here K is the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}(\mathcal{X}_0)$, and O_K is the ring of algebraic integers in K. In the setting of Dirichlet's unit theorem, the image of $O_K^\times$ in $\mathbb{R}^r$ under the logarithms of archimedean absolute values is discrete. Then the relation $\epsilon_i g \pm \log|f_i| \geq 0$ from the Green's function implies that $|f_i|_\sigma = 1$ for any archimedean place σ of K and for sufficiently large i. Therefore, f_i is a root of unity and thus $\overline{\mathcal{D}}_i = 0$ for such i. This proves the normal case.

For the general case that $\mathcal{X}_i$ is not normal, denote by $\mathcal{X}_i'$ (resp. $\mathcal{U}'$) the normalization of $\mathcal{X}_i$ (resp. $\mathcal{U}$) for all $i \geq 0$. Consider the pull-back of the relation $\epsilon_i\overline{\mathcal{E}}_0 \pm \overline{\mathcal{D}}_i \geq 0$ to the normalizations. Then the previous case implies that f_i is a root of unity for sufficiently large i. This implies the image of $\overline{\mathcal{D}}_i = \widehat{\mathrm{div}}_{\mathcal{X}_i}(f_i)$ in $\widehat{\mathrm{Div}}(\mathcal{X}_i)_{\mathbb{Q}}$ is 0. On the other hand, by definition of Cauchy sequences, the integral part $\mathcal{D}_i|_{\mathcal{U}}$ is constant in $\mathrm{Div}(\mathcal{U})$. Therefore, the sequence $\overline{\mathcal{D}}_i$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$, which is a subgroup of $\varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}} \oplus \mathrm{Div}(\mathcal{U})$, is eventually constant. The proof is complete. □

2.4.3 Adelic divisors on essentially quasi-projective schemes

Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k as in §2.3. The set of quasi-projective models $\mathcal{U}$ of X

over k form an inverse system. Define

$$\begin{aligned}\widehat{\mathrm{Div}}(X/k) : &= \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/k), \\ \widehat{\mathrm{CaCl}}(X/k) : &= \varinjlim_{\mathcal{U}} \widehat{\mathrm{CaCl}}(\mathcal{U}/k).\end{aligned}$$

We call elements of $\widehat{\mathrm{Div}}(X/k)$ *adelic divisors* of X/k.

If X is a quasi-projective variety over k, then X itself is the final object of the inverse system of quasi-projective models of X over k. In this case, the definitions in terms of direct limits are compatible with the original ones for quasi-projective varieties.

There are many functorial properties of $\widehat{\mathrm{Div}}(X/k)$ and $\widehat{\mathrm{CaCl}}(X/k)$, which will be introduced together with the theory of adelic line bundles.

2.5 Adelic line bundles

Now we define adelic line bundles on essentially quasi-projective schemes to match the above definition of adelic divisors. We will use the notion of hermitian $\mathbb{Q}$-line bundles in §2.2 and arithmetic models in §2.3.

Throughout this subsection, let k be either $\mathbb{Z}$ or an arbitrary field. Take the uniform terminology in §1.6.

2.5.1 Adelic line bundles on a quasi-projective variety

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{U}$ be a quasi-projective variety over k.

Let us first introduce a notation for *model adelic divisors of rational maps.* Let $\mathcal{X}_1, \mathcal{X}_2$ be projective models of $\mathcal{U}$ over k. Let $\overline{\mathcal{L}}_i$ be a hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}_i$ for $i = 1, 2$. By a *rational map* $\ell : \overline{\mathcal{L}}_1 \dashrightarrow \overline{\mathcal{L}}_2$ over $\mathcal{U}$, we mean an isomorphism $\ell : \mathcal{L}_1|_{\mathcal{U}} \to \mathcal{L}_2|_{\mathcal{U}}$ of $\mathbb{Q}$-line bundles on $\mathcal{U}$. Let $\mathcal{Y}$ be a projective model of $\mathcal{U}$ with morphisms $\tau_i : \mathcal{Y} \to \mathcal{X}_i$ of projective models of $\mathcal{U}$. View ℓ as a rational section of $\tau_1^*\mathcal{L}_1^\vee \otimes \tau_2^*\mathcal{L}_2$ on $\mathcal{Y}$, so that it defines an arithmetic $\mathbb{Q}$-divisor $\widehat{\mathrm{div}}_{\mathcal{Y}}(\ell)$ on $\mathcal{Y}$ using the metric of $\tau_1^*\overline{\mathcal{L}}_1^\vee \otimes \tau_2^*\overline{\mathcal{L}}_2$. Set $\widehat{\mathrm{div}}(\ell)$ to be the image of $\widehat{\mathrm{div}}_{\mathcal{Y}}(\ell)$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod},\mathbb{Q}}$. We also view $\widehat{\mathrm{div}}(\ell)$ as an element of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ by setting the integral part on $\mathcal{U}$ to be 0. The definition of $\widehat{\mathrm{div}}(\ell)$ is independent of the choice of $\mathcal{Y}$.

Let $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ be a boundary divisor as in §2.4. Namely, $\mathcal{X}_0$ is a projective model of $\mathcal{U}$ and $\overline{\mathcal{E}}_0 = (\mathcal{E}_0, g_0)$ is a (strictly) effective arithmetic divisor on $\mathcal{X}_0$ whose finite part $\mathcal{E}_0$ has support equal to $\mathcal{X}_0 \setminus \mathcal{U}$.

Define the *category* $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ *of adelic line bundles* on $\mathcal{U}$ as follows. An object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ is a pair $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ where:

(1) $\mathcal{L}$ is an object of $\mathcal{P}\mathrm{ic}(\mathcal{U})$, i.e., a line bundle on $\mathcal{U}$;

(2) $\mathcal{X}_i$ is a projective model of $\mathcal{U}$ over k;

(3) $\overline{\mathcal{L}}_i$ is an object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}_i)_{\mathbb{Q}}$, i.e. a hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}_i$;

(4) $\ell_i : \mathcal{L} \to \mathcal{L}_i|_{\mathcal{U}}$ is an isomorphism in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$.

The sequence is required to satisfy the *Cauchy condition* as follows. By (4), we obtain an isomorphism $\ell_i\ell_1^{-1} : \mathcal{L}_1|_{\mathcal{U}} \to \mathcal{L}_i|_{\mathcal{U}}$ of $\mathbb{Q}$-line bundles, and thus a rational map $\ell_i\ell_1^{-1} : \overline{\mathcal{L}}_1 \dashrightarrow \overline{\mathcal{L}}_i$ over $\mathcal{U}$. By the above notations, it defines a model divisor $\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$. Then the *Cauchy condition* is that the sequence $\{\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})\}_{i\geq 1}$ is a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ under the boundary topology. More precisely, there is a sequence $\{\epsilon_i\}_{i\geq 1}$ of positive rational numbers converging to 0 such that

$$-\epsilon_i\overline{\mathcal{E}}_0 \leq \widehat{\mathrm{div}}(\ell_{i'}\ell_1^{-1}) - \widehat{\mathrm{div}}(\ell_i\ell_1^{-1}) \leq \epsilon_i\overline{\mathcal{E}}_0$$

in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ for any $i' \geq i \geq 1$. The relation can also be written as

$$-\epsilon_i\overline{\mathcal{E}}_0 \leq \widehat{\mathrm{div}}(\ell_{i'}\ell_i^{-1}) \leq \epsilon_i\overline{\mathcal{E}}_0$$

in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ for any $i' \geq i \geq 1$.

For convenience, the object $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ is also called a *Cauchy sequence* in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{mod}}$. For simplicity, we may abbreviate $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ as $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i))$ or simply as $(\mathcal{L}, \mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)$.

A morphism from an object $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ to another object $(\mathcal{L}', (\mathcal{X}'_i, \overline{\mathcal{L}}'_i, \ell'_i)_{i\geq 1})$ of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ is an isomorphism $\iota : \mathcal{L} \to \mathcal{L}'$ of the integral line bundles on $\mathcal{U}$ satisfying the following properties. As above, the composition $\ell'_i\iota\ell_i^{-1} : \mathcal{L}_i|_{\mathcal{U}} \to \mathcal{L}'_i|_{\mathcal{U}}$ induces a rational map $\ell'_i\iota\ell_i^{-1} : \overline{\mathcal{L}}_i \dashrightarrow \overline{\mathcal{L}}'_i$, and thus defines a model divisor $\widehat{\mathrm{div}}(\ell'_i\iota\ell_i^{-1})$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ whose image in $\mathrm{Div}(\mathcal{U})$ is 0. Then we require the sequence $\{\widehat{\mathrm{div}}(\ell'_i\iota\ell_i^{-1})\}_{i\geq 1}$ of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ converges to 0 in $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ under the boundary topology; i.e., there is a sequence $\{\delta_i\}_{i\geq 1}$ of positive rational numbers converging to 0 such that

$$-\delta_i\overline{\mathcal{E}}_0 \leq \widehat{\mathrm{div}}(\ell'_i\iota\ell_i^{-1}) \leq \delta_i\overline{\mathcal{E}}_0, \quad i \geq 1.$$

Note that the sequence $\{\widehat{\mathrm{div}}(\ell_i'\iota\ell_i^{-1})\}_{i\geq 1}$ is already a Cauchy sequence by

$$\widehat{\mathrm{div}}(\ell_i'\iota\ell_i^{-1}) = \widehat{\mathrm{div}}(\ell_i'\ell_1'^{-1}) - \widehat{\mathrm{div}}(\ell_i\ell_1^{-1}) + \widehat{\mathrm{div}}(\iota_1).$$

By definition, any morphism in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ is an isomorphism, so $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ is a groupoid. This category is equipped with a *tensor product* given by

$$(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)) \otimes (\mathcal{L}', (\mathcal{X}_i', \overline{\mathcal{L}}_i', \ell_i')) := (\mathcal{L}\otimes\mathcal{L}', (\mathcal{W}_i, \tau_i^*\overline{\mathcal{L}}_i \otimes \tau_i'^*\overline{\mathcal{L}}_i', \ell_i\otimes\ell_i')),$$

where $\mathcal{W}_i$ is the Zariski closure of the image of the diagonal map $\mathcal{U} \to \mathcal{X}_i \times_k \mathcal{X}_i'$, and $\tau_i : \mathcal{W}_i \to \mathcal{X}_i$ and $\tau_i' : \mathcal{W}_i \to \mathcal{X}_i'$ are the two projection maps. It is also equipped with a *dual* given by

$$(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i))^\vee := (\mathcal{L}^\vee, (\mathcal{X}_i, \overline{\mathcal{L}}_i^\vee, \ell_i^\vee)).$$

Then the tensor product of an element with its dual is isomorphic to the neutral object $(\mathcal{O}_\mathcal{U}, (\mathcal{X}_0, \overline{\mathcal{O}}_{\mathcal{X}_0}, 1))$.

An object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ is called an *adelic line bundle* (or a *compactified line bundle*) on $\mathcal{U}$. Define $\widehat{\mathrm{Pic}}(\mathcal{U}/k)$ to be the *group* of isomorphism classes of objects of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$, where the group operation is the above tensor product. We usually write an adelic line bundle in the form $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$, and call $\mathcal{L}$ *the underlying line bundle* of $\overline{\mathcal{L}}$ on $\mathcal{U}$.

As in the classical case, we have the following result.

Proposition 2.5.1. *Let $\mathcal{U}$ be a quasi-projective variety over k. Then there is a canonical isomorphism*

$$\widehat{\mathrm{CaCl}}(\mathcal{U}/k) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U}/k).$$

Proof. The proof is a routine, but we write in details to familiarize the terminologies here. It suffices to define a map

$$\widehat{\mathrm{Div}}(\mathcal{U}/k) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U}/k)$$

and check that it is surjective with kernel $\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{\mathrm{mod}}$.

To define the map, take an element $\overline{\mathcal{D}}$ of the left-hand side. Then $\overline{\mathcal{D}}$ is represented by a Cauchy sequence $\{\overline{\mathcal{D}}_i\}_{i\geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$. Let $\{\mathcal{X}_i\}_{i\geq 1}$ be a system of projective models of $\mathcal{U}$ such that $\overline{\mathcal{D}}_i \in \widehat{\mathrm{Div}}(\mathcal{X}_i, \mathcal{U})$ for any $i \geq 1$. Note that $\mathcal{D}_1|_\mathcal{U} = \mathcal{D}_i|_\mathcal{U}$ is an integral divisor on $\mathcal{U}$. Set the image of $\overline{\mathcal{D}}$ in

$\widehat{\mathrm{Pic}}(\mathcal{U}/k)$ to be the isomorphism class of the sequence $(\mathcal{L},(\mathcal{X}_i,\overline{\mathcal{L}}_i,\ell_i))$, where $\mathcal{L}=\mathcal{O}(\mathcal{D}_1|_{\mathcal{U}})$, $\overline{\mathcal{L}}_i=\mathcal{O}(\overline{\mathcal{D}}_i)$, and $\ell_i:\mathcal{L}\to\mathcal{L}_i|_{\mathcal{U}}$ is the isomorphism induced by the equality $\mathcal{D}_1|_{\mathcal{U}}=\mathcal{D}_i|_{\mathcal{U}}$ in $\mathrm{Div}(\mathcal{U})$. By definition, $\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})=\overline{\mathcal{D}}_i-\overline{\mathcal{D}}_1$. Then we see that $(\mathcal{L},(\mathcal{X}_i,\overline{\mathcal{L}}_i,\ell_i))$ satisfies the Cauchy condition. This defines the map.

Now assume that the above adelic divisor $\overline{\mathcal{D}}$ lies in the kernel of the map. It follows that there is an isomorphism from $(\mathcal{O}_{\mathcal{U}},(\mathcal{X}_0,\overline{\mathcal{O}}_{\mathcal{X}_0},1))$ to $(\mathcal{L},(\mathcal{X}_i,\overline{\mathcal{L}}_i,\ell_i))$. This includes an isomorphism $\mathcal{O}_{\mathcal{U}}\to\mathcal{O}(\mathcal{D}_1|_{\mathcal{U}})$, which is given by the multiplication by some $f\in\Gamma(\mathcal{U},\mathcal{O}_{\mathcal{U}})^\times$ with $\mathrm{div}(f)=\mathcal{D}_1|_{\mathcal{U}}=0$ on $\mathcal{U}$. The further properties of the isomorphism are equivalent to that $\mathcal{D}_i$ converges to $-\widehat{\mathrm{div}}(f)$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$. This proves that the kernel is $\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{\mathrm{mod}}$.

To see the surjectivity of the map, let $\overline{\mathcal{L}}=(\mathcal{L},(\mathcal{X}_i,\overline{\mathcal{L}}_i,\ell_i))$ be an adelic line bundle on $\mathcal{U}$. For any rational section s of $\mathcal{L}$, denote

$$\widehat{\mathrm{div}}_{\overline{\mathcal{L}}}(s):=\widehat{\mathrm{div}}_{(\mathcal{X}_1,\overline{\mathcal{L}}_1)}(s)+\lim_{i\to\infty}\widehat{\mathrm{div}}(\ell_i\ell_1^{-1}),$$

which is an element of $\widehat{\mathrm{Div}}(\mathcal{U}/k)$. This gives a preimage of $\overline{\mathcal{L}}$. Then the map is surjective. □

2.5.2 Nef and integrable adelic line bundles

In §2.2, we have recalled the notion of nef hermitian line bundles on arithmetic varieties. This notion generalizes to adelic line bundles by the limit process as follows.

Definition 2.5.2. Let $\mathcal{U}$ be a quasi-projective variety over k.

(1) We say that an adelic line bundle $\overline{\mathcal{L}}\in\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ is *strongly nef* if it is isomorphic to an object $(\mathcal{L},(\mathcal{X}_i,\overline{\mathcal{L}}_i,\ell_i))$ where each $\overline{\mathcal{L}}_i$ is nef on $\mathcal{X}_i$.

(2) We say that an adelic line bundle $\overline{\mathcal{L}}\in\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ is *nef* if there exists a strongly nef adelic line bundle $\overline{\mathcal{M}}\in\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ such that $a\overline{\mathcal{L}}+\overline{\mathcal{M}}$ is strongly nef for all positive integers a.

(3) We say that an adelic line bundle in $\overline{\mathcal{L}}\in\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ is *integrable* if it is isomorphic to the difference of two strongly nef ones in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$.

It is obvious that "strongly nef" implies "nef", and "nef" implies "integrable". Denote by

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{snef}},\quad \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{nef}},\quad \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{int}}$$

respectively the *full subcategories* of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ of strongly nef objects, nef objects, and integrable objects. Denote by

$$\widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{snef}}, \quad \widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{nef}}, \quad \widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{int}}$$

respectively the *subgroups* of $\widehat{\mathrm{Pic}}(\mathcal{U}/k)$ of strongly nef elements, nef elements, and integrable elements. Then $\widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{snef}}$ and $\widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{nef}}$ are semigroups and $\widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{int}}$ is a group.

The preimages of

$$\widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{snef}}, \quad \widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{nef}}, \quad \widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{int}}$$

in $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ are denoted by

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{snef}}, \quad \widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{nef}}, \quad \widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{int}}$$

respectively. Their elements are respectively called *strongly nef* adelic divisors on $\mathcal{U}/k$, *nef* adelic divisors on $\mathcal{U}/k$, *integrable* adelic divisors on $\mathcal{U}/k$.

2.5.3 Definition on essentially quasi-projective schemes

Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. Define

$$\begin{aligned}\widehat{\mathrm{Pic}}(X/k): &= \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/k),\\ \widehat{\mathcal{P}\mathrm{ic}}(X/k): &= \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k).\end{aligned}$$

Here the limits are over all quasi-projective models $\mathcal{U}$ of X over k. The category $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ defined by the direct limit is understood as follows. An object of $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ is a pair $(\overline{\mathcal{L}},\mathcal{U})$, where $\mathcal{U}$ is a quasi-projective model of X over k and $\overline{\mathcal{L}}$ is an object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$. A morphism $(\overline{\mathcal{L}},\mathcal{U}) \to (\overline{\mathcal{L}}',\mathcal{U}')$ between two objects of $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ is an isomorphism $\iota : \mathcal{L}|_X \to \mathcal{L}'|_X$ in $\mathcal{P}\mathrm{ic}(X)$ satisfying the property that for some quasi-projective model $\mathcal{V}$ of X over k endowed with open immersions $\psi : \mathcal{V} \to \mathcal{U}$ and $\psi' : \mathcal{V} \to \mathcal{U}'$ extending the identity morphism $X \to X$, the isomorphism $\iota : \mathcal{L}|_X \to \mathcal{L}'|_X$ can be extended to an isomorphism $\mathcal{L}|_{\mathcal{V}} \to \mathcal{L}'|_{\mathcal{V}}$ in $\mathcal{P}\mathrm{ic}(\mathcal{V})$ and induces an isomorphism $\overline{\mathcal{L}}|_{\mathcal{V}} \to \overline{\mathcal{L}}'|_{\mathcal{V}}$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{V}/k)$. Here we take the convention $\mathcal{L}|_X = (\mathcal{L}|_{\mathcal{U}})|_X$, and if $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$, then $\overline{\mathcal{L}}|_{\mathcal{V}} = (\mathcal{L}|_{\mathcal{V}}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i|_{\mathcal{V}})_{i\geq 1})$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{V}/k)$.

By definition, $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ is a groupoid. We call objects of $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ *adelic line bundles* on X/k.

As an easy consequence of Lemma 2.5.1, there is a canonical isomorphism

$$\widehat{\mathrm{CaCl}}(X/k) \longrightarrow \widehat{\mathrm{Pic}}(X/k)$$

for any flat and essentially quasi-projective integral scheme X over k.

Let $\mathbf{P}$ represent one of the symbols in $\{\mathrm{snef}, \mathrm{nef}, \mathrm{int}\}$. Define

$$\begin{aligned} \widehat{\mathrm{Div}}(X/k)_{\mathbf{P}} : &= \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathbf{P}}, \\ \widehat{\mathrm{CaCl}}(X/k)_{\mathbf{P}} : &= \varinjlim_{\mathcal{U}} \widehat{\mathrm{CaCl}}(\mathcal{U}/k)_{\mathbf{P}}, \\ \widehat{\mathrm{Pic}}(X/k)_{\mathbf{P}} : &= \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathbf{P}}, \\ \widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathbf{P}} : &= \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathbf{P}}. \end{aligned}$$

Objects of $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{snef}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{nef}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{int}}$) are called *strongly nef* (resp. *nef*, *integrable*) adelic line bundles on X/k. Elements of $\widehat{\mathrm{Div}}(X/k)_{\mathrm{snef}}$ (resp. $\widehat{\mathrm{Div}}(X/k)_{\mathrm{nef}}$, $\widehat{\mathrm{Div}}(X/k)_{\mathrm{int}}$) are called *strongly nef* (resp. *nef*, *integrable*) adelic divisors on X/k.

In special situations, we take the following simplified or alternative notations:

(1) The definitions also work for $X = \mathrm{Spec}\, F$ for a finitely generated field F over k. We will write

$$\widehat{\mathrm{Pic}}(F/k) = \widehat{\mathrm{Pic}}((\mathrm{Spec}\, F)/k).$$

Apply this similarly to the other groups or categories.

(2) If k is minimal, i.e., $k = \mathbb{Z}$ or $k = \mathbb{F}_p$ for a prime p, then we may omit the dependence on k in the groups or categories, as k is determined by X as an abstract scheme. In this case, we will simply write

$$\widehat{\mathrm{Pic}}(X) = \widehat{\mathrm{Pic}}(X/k), \quad \widehat{\mathrm{Pic}}(F) = \widehat{\mathrm{Pic}}(F/k).$$

This includes particularly the arithmetic case. Apply this similarly to the other groups or categories.

(3) If k is a field, we may also write

$$\widehat{\mathrm{Div}}(X/k), \quad \widehat{\mathrm{CaCl}}(X/k), \quad \widehat{\mathcal{P}\mathrm{ic}}(X/k), \quad \widehat{\mathrm{Pic}}(X/k)$$

as

$$\widetilde{\mathrm{Div}}(X/k), \quad \widetilde{\mathrm{CaCl}}(X/k), \quad \widetilde{\mathcal{P}\mathrm{ic}}(X/k), \quad \widetilde{\mathrm{Pic}}(X/k).$$

This is to emphasize that there is no archimedean component involved in the terms.

We can compare our definition with that of Moriwaki [Mor4] in the setting of projective varieties over finitely generated fields. Let F be a finitely generated field over k, and X be a projective variety over F. Then our adelic line bundle on X comes from an adelic line bundle $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ on a quasi-projective model $\mathcal{U}$ of X over k. We will see that the sequence $\{(\mathcal{X}_i, \overline{\mathcal{L}}_i)\}_{i\geq 1}$ is close to the notion of an adelic sequence in [Mor4, §3.1]. In fact, by Lemma 2.3.3, we can shrink $\mathcal{U}$ such that there is a projective and flat morphism $\mathcal{U} \to \mathcal{V}$ extending $X \to \operatorname{Spec} F$, where $\mathcal{V}$ is a quasi-projective model of $\operatorname{Spec} F$ over k. We can further take a boundary divisor $(\mathcal{Y}, \overline{\mathcal{E}}_0)$ of $\mathcal{V}$ over k, and assume that there is a morphism $\mathcal{X}_i \to \mathcal{Y}$ extending $X \to \operatorname{Spec} F$ for every $i \geq 1$. If $\overline{\mathcal{L}}_i$ is nef, then $\{(\mathcal{X}_i, \overline{\mathcal{L}}_i)\}_{i\geq 1}$ is indeed an adelic sequence in the loc. cit.. Note that the loc. cit. defines a Cauchy condition in terms of both effectivity and intersection numbers, and we define the Cauchy condition purely in terms of effectivity.

2.5.4 Forgetful maps

Let $\mathcal{U}$ be a quasi-projective variety over k. For any projective model $\mathcal{X}$ of $\mathcal{U}$, there are forgetful maps

$$\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U}) \longrightarrow \mathrm{Div}(\mathcal{U}), \qquad \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}, \mathcal{U}) \longrightarrow \mathcal{P}\mathrm{ic}(\mathcal{U}).$$

Taking limits, they induce forgetful maps

$$\widehat{\mathrm{Div}}(\mathcal{U}/k) \longrightarrow \mathrm{Div}(\mathcal{U}), \qquad \widehat{\mathrm{Pic}}(\mathcal{U}/k) \longrightarrow \mathrm{Pic}(\mathcal{U}), \qquad \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k) \longrightarrow \mathcal{P}\mathrm{ic}(\mathcal{U}).$$

Here the last two maps send an object $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i))$ to $\mathcal{L}$.

Let X be a flat and essentially quasi-projective integral scheme over k. Then the above maps induce forgetful maps

$$\widehat{\mathrm{Div}}(X/k) \longrightarrow \mathrm{Div}(X), \qquad \widehat{\mathrm{Pic}}(X/k) \longrightarrow \mathrm{Pic}(X), \qquad \widehat{\mathcal{P}\mathrm{ic}}(X/k) \longrightarrow \mathcal{P}\mathrm{ic}(X).$$

As a convention, we usually write an object of $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ in the form $\overline{L}$, where L is understood to be the image of $\overline{L}$ in $\mathcal{P}\mathrm{ic}(X)$. We often refer L as the *underlying line bundle of* $\overline{L}$, and refer $\overline{L}$ as an *adelic extension of* L. We take similar conventions for $\widehat{\mathrm{Pic}}(X/k)$ and $\widehat{\mathrm{Div}}(X/k)$.

2.5.5 Functoriality

Here we introduce a few functorial maps between the Picard groups and the divisor groups. In the following, $\mathbf{P}$ represents one of the symbols in $\{\mathrm{void}, \mathrm{int}, \mathrm{nef}, \mathrm{snef}\}$, and take the convention that $\widehat{\mathrm{Pic}}(X/k)_{\mathbf{P}}$ for "$\mathbf{P} = \mathrm{void}$" means $\widehat{\mathrm{Pic}}(X/k)$.

Pull-back. Let k be either $\mathbb{Z}$ or a field. Let $f : X' \to X$ be a morphism of flat and essentially quasi-projective integral schemes over k. Then there are canonical maps

$$f^* : \widehat{\mathrm{Pic}}(X/k)_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Pic}}(X'/k)_{\mathbf{P}},$$
$$f^* : \widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathbf{P}} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X'/k)_{\mathbf{P}}.$$

In fact, for quasi-projective models $X' \to \mathcal{U}'$ and $X \to \mathcal{U}$ over k, the rational map $\mathcal{U}' \dashrightarrow \mathcal{U}$ is defined in an open neighborhood of X' in $\mathcal{U}'$. Replacing $\mathcal{U}'$ by that neighborhood if necessary, we obtain a morphism $f_{\mathcal{U}} : \mathcal{U}' \to \mathcal{U}$. Then it suffices to define a canonical functor $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k) \to \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}'/k)$.

Let $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i))$ be a Cauchy sequence in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{mod}}$. There is a projective model $\mathcal{X}_i'$ of $\mathcal{U}'$ with a morphism $f_i : \mathcal{X}_i' \to \mathcal{X}_i$ extending $f_{\mathcal{U}} : \mathcal{U}' \to \mathcal{U}$. This can be achieved by taking any projective model $\mathcal{X}_i'$ of $\mathcal{U}'$, and blow-up $\mathcal{X}_i'$ along a suitable center supported on $\mathcal{X}_i' \setminus \mathcal{U}'$. Set the image of $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i))$ under f^* to be $(f_{\mathcal{U}}^*\mathcal{L}, (\mathcal{X}_i', f_i^*\overline{\mathcal{L}}_i, f_{\mathcal{U}}^*\ell_i))$. To prove that the later is a Cauchy sequence in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}'/k)_{\mathrm{mod}}$, we need to compare the boundary topologies.

Let $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ (resp. $(\mathcal{X}_0', \overline{\mathcal{E}}_0')$) be a boundary divisor of $\mathcal{U}$ (resp. $\mathcal{U}'$) over k. As above, we can further assume that there is a morphism $f_0 : \mathcal{X}_0' \to \mathcal{X}_0$ extending $\mathcal{U}' \to \mathcal{U}$. Note that $f_0^*\overline{\mathcal{E}}_0$ is supported in $\mathcal{X}_0' \setminus \mathcal{U}'$. As in our proof that the boundary topology is independent of the choice of the boundary divisor in §2.4, there is a rational number $c > 0$ such that $f_0^*\overline{\mathcal{E}}_0 \leq c\overline{\mathcal{E}}_0'$. This gives the compatibility of the boundary topologies.

Hence, we have a functor $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k) \to \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}'/k)$ and a functor $\widehat{\mathcal{P}\mathrm{ic}}(X/k) \to \widehat{\mathcal{P}\mathrm{ic}}(X'/k)$. The functor keeps tensor products.

In the above construction, if $f : X' \to X$ is dominant, there is also a canonical map

$$f^* : \widehat{\mathrm{Div}}(X/k)_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Div}}(X'/k)_{\mathbf{P}}.$$

We will see in Corollary 3.4.2 that these maps are injective if X' and X are normal and f is birational.

Varying the base. Let k'/k be a finitely generated extension of fields, and X be an essentially quasi-projective integral scheme over k'. Then there are canonical maps

$$\widehat{\mathrm{Div}}(X/k)_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Div}}(X/k')_{\mathbf{P}},$$

$$\widehat{\mathrm{Pic}}(X/k)_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Pic}}(X/k')_{\mathbf{P}},$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathbf{P}} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X/k')_{\mathbf{P}}.$$

To define the maps, note that if $\mathcal{U}$ (resp. $\mathcal{V}$) is a quasi-projective model of X (resp. $\mathrm{Spec}\, k'$) over k, then by shrinking $\mathcal{U}$, we can assume that there is a flat morphism $\mathcal{U} \to \mathcal{V}$ extending $X \to \mathrm{Spec}\, k'$. This is similar to Lemma 2.3.3. Then it suffices to define a map $\widehat{\mathrm{Div}}(\mathcal{U}/k) \to \widehat{\mathrm{Div}}(X/k')$ and its analogue for the line bundles. By composition, we can further assume that X is isomorphic to the generic fiber of $\mathcal{U} \to \mathcal{V}$.

Fix a projective model $\mathcal{B}$ of $\mathcal{V}$ over k. For any projective model $\mathcal{X}$ of $\mathcal{U}$ over k, we can assume that there is a morphism $\mathcal{X} \to \mathcal{B}$ extending $\mathcal{U} \to \mathcal{V}$ by blowing-up $\mathcal{X}$. Then the generic fiber $\mathcal{X}_\eta$ of $\mathcal{X} \to \mathcal{B}$ is a projective model of X over k'. Finally, the map is induced by the natural map $\mathrm{Div}(\mathcal{X}) \to \mathrm{Div}(\mathcal{X}_\eta)$.

If k'/k is a finite extension, the above maps are actually isomorphisms.

Base change 1: geometric case. Let k'/k be a finitely generated extension of fields, and X be an essentially quasi-projective integral scheme over k. Assume that the base change $X_{k'}$ is still integral. Then there are canonical maps

$$\widehat{\mathrm{Div}}(X/k)_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Div}}(X_{k'}/k')_{\mathbf{P}},$$

$$\widehat{\mathrm{Pic}}(X/k)_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Pic}}(X_{k'}/k')_{\mathbf{P}},$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathbf{P}} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X_{k'}/k')_{\mathbf{P}}.$$

This is induced by the fact that, if $\mathcal{U}$ is a quasi-projective (resp. projective) model of X over k, then $\mathcal{U}_{k'}$ is a quasi-projective (resp. projective) model of $X_{k'}$ over k'. Then the maps are induced by the pull-back maps via the base changes.

Base change 2: from $\mathbb{Z}$ to $\mathbb{Q}$. Let X be a flat and essentially quasi-projective integral scheme over $\mathbb{Z}$. For any projective model $\mathcal{X}$ of X over $\mathbb{Z}$, the generic fiber $\mathcal{X}_{\mathbb{Q}}$ is a projective model of $X_{\mathbb{Q}}$ over $\mathbb{Q}$. There are natural maps

$$\widehat{\mathrm{Div}}(\mathcal{X}) \longrightarrow \mathrm{Div}(\mathcal{X}_{\mathbb{Q}}), \quad \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \mathrm{Pic}(\mathcal{X}_{\mathbb{Q}}), \quad \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}) \longrightarrow \mathcal{P}\mathrm{ic}(\mathcal{X}_{\mathbb{Q}}).$$

These maps induce canonical maps

$$\widehat{\mathrm{Div}}(X/\mathbb{Z})_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Div}}(X_{\mathbb{Q}}/\mathbb{Q})_{\mathbf{P}}, \quad \overline{D} \longmapsto \widetilde{D},$$

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Pic}}(X_{\mathbb{Q}}/\mathbb{Q})_{\mathbf{P}}, \quad \overline{L} \longmapsto \widetilde{L},$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})_{\mathbf{P}} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X_{\mathbb{Q}}/\mathbb{Q})_{\mathbf{P}}, \quad \overline{L} \longmapsto \widetilde{L}.$$

We call $\widetilde{D}$ (resp. $\widetilde{L}$) the *geometric part* of $\overline{D}$ (resp. $\overline{L}$) over $\mathbb{Q}$.

Base change 3: from $\mathbb{Z}$ to $\mathbb{F}_p$. Let X be a flat and essentially quasi-projective integral scheme over $\mathbb{Z}$. Let p be a prime number such that the fiber $X_{\mathbb{F}_p}$ of X over p is integral (and non-empty). For any projective model $\mathcal{X}$ of X over $\mathbb{Z}$, the Zariski closure $\mathcal{X}'_{\mathbb{F}_p}$ of $X_{\mathbb{F}_p}$ in $\mathcal{X}_{\mathbb{F}_p}$ is a projective model of $X_{\mathbb{F}_p}$ over $\mathbb{F}_p$. There are natural maps

$$\widehat{\mathrm{Div}}(\mathcal{X}) \longrightarrow \mathrm{Div}(\mathcal{X}'_{\mathbb{F}_p}), \quad \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \mathrm{Pic}(\mathcal{X}'_{\mathbb{F}_p}), \quad \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}) \longrightarrow \mathcal{P}\mathrm{ic}(\mathcal{X}'_{\mathbb{F}_p}).$$

These maps induce canonical maps

$$\widehat{\mathrm{Div}}(X/\mathbb{Z})_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Div}}(X_{\mathbb{F}_p}/\mathbb{F}_p)_{\mathbf{P}},$$

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Pic}}(X_{\mathbb{F}_p}/\mathbb{F}_p)_{\mathbf{P}},$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z})_{\mathbf{P}} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X_{\mathbb{F}_p}/\mathbb{F}_p)_{\mathbf{P}}.$$

Mixed situation. Let F be a finitely generated field over $\mathbb{Q}$, and X be a quasi-projective variety over F. Combining the above constructions, we obtain compositions

$$\widehat{\mathrm{Div}}(X/\mathbb{Z})_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Div}}(X/\mathbb{Q})_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Div}}(X/F)_{\mathbf{P}},$$

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Pic}}(X/\mathbb{Q})_{\mathbf{P}} \longrightarrow \widehat{\mathrm{Pic}}(X/F)_{\mathbf{P}}.$$

If X is projective over F, then the compositions are just the forgetful maps defined above. In general, the image of an element of $\widehat{\mathrm{Div}}(X/\mathbb{Z})_{\mathbf{P}}$ (resp. $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbf{P}}$) in $\widehat{\mathrm{Div}}(X/F)_{\mathbf{P}}$ (resp. $\widehat{\mathrm{Pic}}(X/F)_{\mathbf{P}}$) is called the *geometric part* of this element over F.

2.5.6 Extension to $\mathbb{Q}$-coefficients

Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. To work with $\mathbb{Q}$-line bundles on X, we write

$$\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q}} = \widehat{\mathrm{Pic}}(X/k) \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad \widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}} = \widehat{\mathrm{Pic}}(X/k)_{\mathrm{int}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We further write $\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q},\mathrm{snef}}$ (resp. $\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q},\mathrm{nef}}$) as the sub-semigroup of $\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q}}$ consisting of *positive* rational multiples of elements of $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{snef}}$ (resp. $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{nef}}$). Extend the notations to $\widehat{\mathrm{Div}}$ and $\widehat{\mathrm{CaCl}}$ similarly.

Let $\mathcal{U}$ be a quasi-projective variety over k. Then we can also interpret elements of the above groups directly in terms of Cauchy sequences. In fact, by the isomorphism $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})_{\mathbb{Q}} \to \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$, the group $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathbb{Q}}$ is simply isomorphic to the completion of

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod},\mathbb{Q}} = \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$$

with respect to the boundary topology defined similarly, and

$$\widehat{\mathrm{CaCl}}(\mathcal{U}/k)_{\mathbb{Q}} \simeq \widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod},\mathbb{Q}}/\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{\mathbb{Q}}.$$

On the other hand, $\widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathbb{Q}}$ is the group of isomorphism classes of objects of a *category* $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathbb{Q}}$ defined as follows. An object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathbb{Q}}$ is a sequence $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i))$, whose definition is similar to the integral case, except that $\mathcal{L}$ is an object of $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$ (instead of $\mathcal{P}\mathrm{ic}(\mathcal{U})$). A morphism from an object $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i))$ to another object $(\mathcal{L}', (\mathcal{X}_i', \overline{\mathcal{L}}_i', \ell_i'))$ is also similar to the integral case, except that it is given by an isomorphism $\mathcal{L} \to \mathcal{L}'$ of $\mathbb{Q}$-line bundles (instead of integral line bundles) on $\mathcal{U}$ which induces an isometry between the two objects in a similar sense.

Note that we do not derive $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathbb{Q}}$ from the category $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ by multiplying a rational number to the objects, but they give equivalent categories. We choose the current definition for its simplicity.

These descriptions can be used to define $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathbb{Q}}$, $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathbb{Q}}$ and $\widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathbb{Q}}$ without introducing their integral versions $\widehat{\mathrm{Div}}(\mathcal{U}/k)$, $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ and $\widehat{\mathrm{Pic}}(\mathcal{U}/k)$ first. Moreover, we can even define the integral versions from the $\mathbb{Q}$-versions, which can be served as a slightly different approach of the theory. For example, $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ can be defined by the canonical exact sequence

$$0 \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}/k) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathbb{Q}} \oplus \mathrm{Div}(\mathcal{U}) \longrightarrow \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}.$$

Here the last arrow sends a pair $(\overline{\mathcal{D}}, \mathcal{D}')$ to $\overline{\mathcal{D}}|_{\mathcal{U}} - \mathcal{D}'$. Here $\overline{\mathcal{D}} \mapsto \overline{\mathcal{D}}|_{\mathcal{U}}$ is the forgetful map as in §2.5.4.

2.6 Examples and connections

Here we present some usual examples of adelic line bundles, and relate the general theory to Zariski–Riemann spaces.

2.6.1 Families of algebraic dynamical systems

This example is our major motivation to introduce the theory of adelic line bundles. Let k be either $\mathbb{Z}$ or a field. Let the base S be either one of the following:

(a) a quasi-projective variety over k,

(b) a quasi-projective variety over a field F which is finitely generated field over k.

Let (X, f, L) be a *polarized dynamical system* over S; i.e.,

(1) X is a flat projective scheme over S;

(2) $f : X \to X$ is a morphism over S;

(3) $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ is a $\mathbb{Q}$-line bundle, relatively ample over S, such that $f^*L = qL$ from some rational number $q > 1$.

By Tate's limiting argument, there is a nef adelic line bundle $\overline{L}_f \in \widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q}}$ extending L such that $f^*\overline{L}_f = q\overline{L}_f$.

The construction is similar to the case that S is the spectrum of a number field in [Zha2]. We refer to Theorem 6.1.1 for the current case.

There are many natural polarized dynamical systems over bases S of positive dimensions. For example, this happens if S is a moduli space, which includes the moduli space of endomorphisms of $\mathbb{P}^N$ of fixed degree $d > 1$ and a fine moduli space of polarized abelian varieties over k.

2.6.2 Hodge bundle for Faltings height

In the proof of the Mordell conjecture by Faltings [Fal2], Faltings heights of abelian varieties over number fields are interpreted in terms of the height function associated to the Hodge bundle on certain moduli space of abelian varieties. It is known that the Hodge bundle and the Faltings metric do not form a hermitian line bundle due to the singularity of the metric at the boundary, but they do form an adelic line bundle in the framework of this paper. Here we describe the situation briefly. We will state the result for general families of abelian varieties instead of just moduli spaces.

The base ring here is $k = \mathbb{Z}$. Let S be either one of the following:

(a) an integral quasi-projective scheme over $\mathbb{Z}$,

(b) a quasi-projective variety over $\mathbb{Q}$.

Let $\pi : X \to S$ be a principally polarized abelian scheme of relative dimension g. Denote by $e : S \to X$ its identity section.

The *Hodge bundle* on S is the line bundle

$$\omega(S) = e^*\Omega^g_{X/S} \simeq \pi_*\Omega^g_{X/S}.$$

The *Faltings metric* of $\omega(S)$ on $S(\mathbb{C})$ is defined by

$$\|\alpha\|^2_{\mathrm{Fal}} = \frac{1}{2^g}\left|\int_{X_s(\mathbb{C})} \alpha \wedge \bar{\alpha}\right| = \frac{i^{g^2}}{2^g}\int_{X_s(\mathbb{C})} \alpha \wedge \bar{\alpha}$$

for any point $s \in S(\mathbb{C})$ and any element α of the fiber

$$\omega(S)(s) = e_s^*\Omega^g_{X_s/\mathbb{C}} \simeq \Gamma(X_s, \Omega^g_{X_s/\mathbb{C}}).$$

Then we have a metrized line bundle $(\omega(S), \|\cdot\|_{\mathrm{Fal}})$ on S. If S is not projective over $\mathbb{Z}$, then it is not a hermitian line bundle in our strict sense. In general, it does not extend to a hermitian line bundle on a projective model of S over $\mathbb{Z}$ due to the logarithmic singularity of the metric at the boundary. However, we will see that $(\omega(S), \|\cdot\|_{\mathrm{Fal}})$ extends canonically to an adelic line bundle $\overline{\omega(S)}$ on $S/\mathbb{Z}$, and the height function associated to $\overline{\omega(S)}$ exactly computes the stable Faltings heights of the abelian varieties on fibers of $X \to S$.

The precise statement and proof require the analytification and the height function of adelic line bundles, which will be introduced in the future sections, so we postpone the treatment to Theorem 5.5.1.

2.6.3 Arithmetic curves

Let K be a number field. Here we compute the group $\widehat{\mathrm{Pic}}(K)$ of adelic line bundles. Note that our definition in this case agrees with that in Zhang [Zha2].

Lemma 2.6.1. *Let $\mathcal{U}$ be an open subscheme of $\mathcal{X} = \mathrm{Spec}\, O_K$. Denote*

$$\widehat{\mathrm{Pic}}^0(\mathcal{U}) = \ker(\widehat{\deg} : \widehat{\mathrm{Pic}}(\mathcal{U}) \to \mathbb{R}),$$

$$\widehat{\mathrm{Pic}}^0(K) = \ker(\widehat{\deg} : \widehat{\mathrm{Pic}}(K) \to \mathbb{R}).$$

There is a canonical exact sequence

$$0 \longrightarrow (\mathcal{O}(\mathcal{U})^\times/\mu_K) \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}) \longrightarrow \widehat{\mathrm{Pic}}^0(\mathcal{U}) \longrightarrow \mathrm{Pic}(\mathcal{U}) \longrightarrow 0$$

and a canonical isomorphism

$$(K^\times/\mu_K) \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}) \longrightarrow \widehat{\mathrm{Pic}}^0(K).$$

Here μ_K is the group of roots of unity in K.

Proof. It suffices to prove the results for $\mathcal{U}$. Denote $\mathcal{E} = \mathcal{X} \setminus \mathcal{U}$, endowed with the reduced scheme structure. Denote by $|\mathcal{X}|, |\mathcal{U}|, |\mathcal{E}|$ the set of closed points of the corresponding schemes. Recall that

$$\widehat{\mathrm{Pic}}(\mathcal{U}) = \widehat{\mathrm{Div}}(\mathcal{U})/\widehat{\mathrm{Pr}}(\mathcal{U})_{\mathrm{mod}}.$$

Note that $\mathcal{X}$ is the only normal projective model of $\mathcal{U}$. We simply have

$$\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}} = \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}), \qquad \widehat{\mathrm{Pr}}(\mathcal{U})_{\mathrm{mod}} = \widehat{\mathrm{Pr}}(\mathcal{X}).$$

Explicitly,

$$\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) = (\oplus_{v\in|\mathcal{U}|}\mathbb{Z}) \oplus \mathbb{Q}^{|\mathcal{E}|} \oplus \mathbb{R}^{M_\infty}, \qquad \widehat{\mathrm{Pr}}(\mathcal{X}) = \widehat{\mathrm{div}}_{\mathcal{X}}(K^\times).$$

Here

$$\widehat{\mathrm{div}}_{\mathcal{X}} : K^\times \longrightarrow \widehat{\mathrm{Div}}(\mathcal{X})$$

is the map of taking principal divisors.

Take the arithmetic divisor

$$\overline{\mathcal{E}}_0 = (\mathcal{E}, 1) = \sum_{v\in|\mathcal{E}|\cup M_\infty} [v].$$

It defines the boundary topology on $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$. The completion gives

$$\widehat{\widehat{\mathrm{Div}}}(\mathcal{U}) = (\oplus_{v\in|\mathcal{U}|}\mathbb{Z}) \oplus \mathbb{R}^{|\mathcal{E}|\cup M_\infty}.$$

The restriction map $\widehat{\widehat{\mathrm{Div}}}(\mathcal{U}) \to \mathrm{Div}(\mathcal{U})$ induces a canonical exact sequence

$$0 \longrightarrow \widehat{\mathrm{Pr}}(\mathcal{X}) \cap \mathbb{R}^{|\mathcal{E}|\cup M_\infty} \longrightarrow \mathbb{R}^{|\mathcal{E}|\cup M_\infty} \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U}) \longrightarrow \mathrm{Pic}(\mathcal{U}) \longrightarrow 0.$$

It is easy to have

$$\widehat{\mathrm{Pr}}(\mathcal{X}) \cap \mathbb{R}^{|\mathcal{E}|\cup M_\infty} \simeq \widehat{\mathrm{div}}_{\mathcal{X}}(\mathcal{O}(\mathcal{U})^\times) \simeq \mathcal{O}(\mathcal{U})^\times/\mu_K.$$

By Dirichlet's unit theorem, $\widehat{\mathrm{div}}_{\mathcal{X}}(\mathcal{O}(\mathcal{U})^\times)$ is a full lattice of the hyperplane

$$(\mathbb{R}^{|\mathcal{E}|\cup M_\infty})_0 := \ker(\widehat{\deg} : \mathbb{R}^{|\mathcal{E}|\cup M_\infty} \to \mathbb{R}).$$

This gives an isomorphism

$$\widehat{\mathrm{div}}_{\mathcal{X}}(\mathcal{O}(\mathcal{U})^\times) \otimes_{\mathbb{Z}} \mathbb{R} = (\mathbb{R}^{|\mathcal{E}|\cup M_\infty})_0.$$

The result follows. □

2.6.4 Line bundles on Zariski–Riemann spaces

Our model adelic line bundles can be realized on some generalized Zariski–Riemann space as introduced by Temkin [Tem]. Here we recall the definitions and connections briefly. The treatment here will not be used in this paper elsewhere.

To illustrate the idea, we restrict to the geometric case that k is a field. Let X be an essentially quasi-projective integral scheme over k, as defined in §2.3. Define *the Zariski–Riemann space associated to* X to be the ringed space

$$\widetilde{X} = \varprojlim_{\mathcal{X}} \mathcal{X},$$

where the limit is over all projective models $\mathcal{X}$ of X over k. In the limit process, the underlying space $\widetilde{X}$ is endowed with the limit topology, i.e., the coarsest topology so that all the projections $p_{\mathcal{X}} : \widetilde{X} \to \mathcal{X}$ to projective models $\mathcal{X}$ of X are continuous. The structure sheaf $\mathcal{O}_{\widetilde{X}}$ is defined to be the direct limit of $p_{\mathcal{X}}^{-1}\mathcal{O}_{\mathcal{X}}$ over all projective models $\mathcal{X}$ of X.

If $X = \operatorname{Spec} F$ for a finitely generated field F over k, the space $\widetilde{X}$ is exactly the classical Zariski–Riemann space introduced by Zariski. In the general situation, the space $\widetilde{X}$ is the relative Zariski–Riemann space $\mathrm{RZ}_X(\mathcal{X}_0)$ introduced by [Tem, §B2], here $\mathcal{X}_0$ is a fixed projective model of X over k.

By definition, there is a canonical morphism $X \to \widetilde{X}$ induced by the morphism $X \to \mathcal{X}$. Another key property is that $\widetilde{X}$ is quasi-compact; see [Tem, Prop. B.2.3].

Since $\widetilde{X}$ is a ringed space, coherent sheaves, invertible sheaves, and Cartier divisors are defined on $\widetilde{X}$. Then we can still define line bundles on $\widetilde{X}$ to be invertible sheaves. By the quasi-compactness of $\widetilde{X}$, we can prove that the natural maps

$$\varinjlim_{\mathcal{X}} \mathrm{Div}(\mathcal{X}) \longrightarrow \mathrm{Div}(\widetilde{X}),$$

$$\varinjlim_{\mathcal{X}} \mathrm{Pic}(\mathcal{X}) \longrightarrow \mathrm{Pic}(\widetilde{X})$$

are isomorphisms. If $X = \operatorname{Spec} F$, the first limit is the group of *Carter b-divisors* introduced by Shokurov [Sho] in the minimal model program.

To connect to our adelic divisors, we see that

$$\mathrm{Div}(X)_{\mathrm{mod},\mathbb{Q}} = \varinjlim_{\mathcal{X}} \mathrm{Div}(\mathcal{X})_{\mathbb{Q}}$$

is canonically isomorphic to $\mathrm{Div}(\widetilde{X})_{\mathbb{Q}}$. Then the group $\widehat{\mathrm{Div}}(X/k)_{\mathbb{Q}}$ is a suitable completion of $\mathrm{Div}(\widetilde{X})_{\mathbb{Q}}$. Similarly, the group $\widehat{\mathrm{Div}}(X/k)$ is a suitable completion of the mixed divisor group

$$\mathrm{Div}(\widetilde{X}, X) := \ker(\mathrm{Div}(\widetilde{X})_{\mathbb{Q}} \oplus \mathrm{Div}(X) \to \mathrm{Div}(X)_{\mathbb{Q}}).$$

Here the arrow sends $(\widetilde{D}, D)$ to $\widetilde{D}|_X - D$ as before.

2.7 Definitions over more general bases

The above theory of $\widehat{\mathrm{Div}}(X/k)$ and $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$, when k is either $\mathbb{Z}$ or a field, covers all the global situations we are interested in, but it does not include the local situation that k comes from a local field. Moreover, the definitions can also be generalized by replacing k by a general Dedekind scheme. The goal of this subsection is to sketch the treatment in these situations, which also includes the function field case.

The exposition here is very similar to the previous case, but we do not use it throughout this paper to avoid extra burden of terminology and potential confusion of cases. In fact, the setup here is only restricted to this subsection and partly to §3.6 and §4.6.2.

Valuations

By a *valuation* over a field K, we mean a map $|\cdot| : K \to \mathbb{R}$ satisfying the following properties:

(1) (positivity) $|0| = 0$, and $|a| > 0$ for any $a \in K^\times$.

(2) (triangle inequality) $|a + b| \leq |a| + |b|$ for any $a, b \in K$.

(3) (multiplicativity) $|ab| = |a| \cdot |b|$ for any $a, b \in K$.

The valuation is *trivial* if $|a| = 1$ for any $a \in K^\times$. The valuation is *archimedean* (resp. *non-archimedean*) if $|n|$ is unbounded (resp. bounded) for all $n \in \mathbb{Z}$ viewed as elements of K under the natural map $\mathbb{Z} \to K$. If $|\cdot|$ is non-archimedean, the *valuation ring* of K is $O_K := \{x \in K : |x| \leq 1\}$.

By a *non-archimedean field* $(K, |\cdot|)$, we mean a field K endowed with a complete non-archimedean non-trivial valuation $|\cdot|$.

Base valued schemes

Recall that an integral domain is called a *Prüfer domain* if all of its local rings are valuation rings. This is a classical term widely studied in commutative algebra. We refer to [Bou, p. 558, Ch. VII, §2, Ex. 12] for 14 equivalent definitions of Prüfer domains, and refer to [BG] for more properties and history of the concept.

It is easy to see that a Prüfer domain is a Dedekind domain if and only if it is noetherian. Thus Prüfer domains can be viewed as a non-noetherian generalization of Dedekind domains, and thus many nice properties of Dedekind domains also hold for Prüfer domains. For example, a module over a Prüfer domain is flat if and only if it is torsion-free, which can be checked by taking localizations.

A quasi-compact integral scheme is called a *Prüfer scheme* if all of of its local rings are valuation rings. We introduce this concept to include the following three important classes:

(1) $\operatorname{Spec} k$ for a field k;

(2) a Dedekind scheme, i.e, a regular and integral noetherian scheme of dimension 1;

(3) $\operatorname{Spec} O_K$, where K is a non-archimedean field and O_K is the valuation ring.

As a consequence of the flatness property, a reduced scheme X over a Prüfer scheme B is flat over B if and only if every irreducible component of X has a Zariski dense image in B.

By a *base valued scheme*, we mean a pair $\overline{B} = (B, \Sigma)$ consisting of a Prüfer scheme B and a subset Σ of $\operatorname{Hom}(K, \mathbb{C})$, where K denotes the function field of B. The set Σ is allowed to be empty, in which case we get a scheme $\overline{B} = B$.

Note that every $\sigma \in \Sigma$ induces an archimedean valuation $|\cdot|_\sigma$ over K. We may also think $\overline{B}$ as $(B, \{|\cdot|_\sigma\}_{\sigma\in\Sigma})$, but note that $|\cdot|_\sigma = |\cdot|_{\sigma'}$ if and only if $\sigma' = c \circ \sigma$ for the complex conjugation $c : \mathbb{C} \to \mathbb{C}$.

We introduce this definition to include the following important and natural types of base valued schemes:

(1) (geometric case) $\operatorname{Spec} k$, where k is a field;

(2) (number field case) $(\operatorname{Spec} O_K, \operatorname{Hom}(K, \mathbb{C}))$, where K is a number field;

(3) (function field case) a projective and geometrically integral regular curve B over a field k;

(4) (archimedean case) $(\operatorname{Spec} \mathbb{R}, i_{\mathrm{st}})$ or $(\operatorname{Spec} \mathbb{C}, \mathrm{id})$, where $i_{\mathrm{st}} : \mathbb{R} \to \mathbb{C}$ is the standard injection and $\mathrm{id} : \mathbb{C} \to \mathbb{C}$ is the identity map;

(5) (non-archimedean case) $\operatorname{Spec} O_K$, where K is a non-archimedean field and O_K is the valuation ring;

(6) (Dedekind case) a Dedekind scheme B.

Note that $\Sigma = \emptyset$ in cases (1), (3), (6), and Σ is finite in all the cases. We usually write K for the function field of B.

Case (1) with any k and case (2) with $K = \mathbb{Q}$ are exactly our original case $k = \mathbb{Z}$ or k is a field in §1.6. The Dedekind case actually includes the function field case, but we list the function case separately for its importance.

Let B be a Prüfer scheme. By an *arithmetic variety* over B, we mean an integral scheme which is flat, separated, and of finite type over B. By a *quasi-projective arithmetic variety* (resp. *projective arithmetic variety*) over B, we mean an arithmetic variety over B which is quasi-projective (resp. projective) over B. For a quasi-projective arithmetic variety $\mathcal{U}$ over B, a *projective model* means a projective arithmetic variety $\mathcal{X}$ over B endowed with an open immersion $\mathcal{U} \to \mathcal{X}$ over B.

As in §2.3, a flat integral scheme X over B is *essentially quasi-projective over* B if there is a *pro-open* immersion $X \to \mathcal{X}$ to a projective arithmetic variety $\mathcal{X}$ over B. A *quasi-projective model* (resp. *projective model*) of X means a quasi-projective (resp. projective) arithmetic variety $\mathcal{U}$ over B endowed with a pro-open immersion $X \to \mathcal{U}$ over B.

Model adelic divisors and adelic line bundles

Let $\overline{B} = (B, \Sigma)$ be a base valued scheme. Let $\mathcal{X}$ be a projective arithmetic variety over B. We define arithmetic divisors and hermitian line bundles on $\mathcal{X}$ as follows.

An *arithmetic divisor* on $\mathcal{X}$ is a pair $(\mathcal{D}, g_{\mathcal{D}})$, where $\mathcal{D}$ is a Cartier divisor on $\mathcal{X}$, and $g_{\mathcal{D}} : \mathcal{X}_\Sigma(\mathbb{C}) \setminus |\mathcal{D}_\Sigma(\mathbb{C})| \to \mathbb{R}$ is a continuous Green's function of $\mathcal{D}_\Sigma(\mathbb{C})$ on $\mathcal{X}_\Sigma(\mathbb{C})$ as in §2.1.2. Here $\mathcal{X}_\Sigma(\mathbb{C}) := \coprod_{\sigma\in\Sigma} \mathcal{X}_\sigma(\mathbb{C})$ is a projective analytic variety, and the Cartier divisor $\mathcal{D}_\Sigma(\mathbb{C})$ on $\mathcal{X}_\Sigma(\mathbb{C})$ is defined similarly. By restriction, we have a Green's function $g_{\mathcal{D},\sigma} : \mathcal{X}_\sigma(\mathbb{C}) \setminus |\mathcal{D}_\sigma(\mathbb{C})| \to \mathbb{R}$ of $\mathcal{D}_\sigma(\mathbb{C})$ on $\mathcal{X}_\sigma(\mathbb{C})$. Then we can also think $g_{\mathcal{D}}$ as a collection of $g_{\mathcal{D},\sigma}$ over $\sigma \in \Sigma$.

The Green's function $g_{\mathcal{D}}$ is further required to be *invariant under the complex conjugation* $c : \mathbb{C} \to \mathbb{C}$ in the sense that for any $\sigma \in \Sigma$ such that $\bar{\sigma} = c \circ \sigma \in \Sigma$, we require $g_{\mathcal{D},\bar{\sigma}} = g_{\mathcal{D},\sigma} \circ c$.

The divisor $(\mathcal{D}, g_{\mathcal{D}})$ is *effective* (resp. *strictly effective*) if $\mathcal{D}$ is an effective Cartier divisor on $\mathcal{X}$, and $g_{\mathcal{D}} \geq 0$ (resp. $g_{\mathcal{D}} > 0$).

A *principal arithmetic divisors* on $\mathcal{X}$ is an arithmetic divisor of the form

$$\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f|)$$

for some $f \in K(\mathcal{X})^\times$.

A *hermitian line bundle* on $\mathcal{X}$ is a pair $(\mathcal{L}, \|\cdot\|)$, where $\mathcal{L}$ is a line bundle on $\mathcal{X}$, and $\|\cdot\|$ is a continuous hermitian metric of $\mathcal{L}_\Sigma(\mathbb{C})$ on $\mathcal{X}_\Sigma(\mathbb{C})$ as in §2.1.1. As above, the metric $\|\cdot\|$ is equivalent to a collection of continuous metrics $\|\cdot\|_\sigma$ of the line bundle $\mathcal{L}_\sigma(\mathbb{C})$ on $\mathcal{X}_\sigma(\mathbb{C})$ over $\sigma \in \Sigma$. The metric is also required to be *invariant under the complex conjugation* in the above sense.

Now we have the following groups (or category)

$$\widehat{\mathrm{Div}}(\mathcal{X}), \quad \widehat{\mathrm{Pr}}(\mathcal{X}), \quad \widehat{\mathrm{Pic}}(\mathcal{X}), \quad \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}), \quad \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathbb{Q}}.$$

Here $\widehat{\mathrm{Div}}(\mathcal{X})$ (resp. $\widehat{\mathrm{Pr}}(\mathcal{X})$) is the group of arithmetic divisors (resp. principal arithmetic divisors) on $\mathcal{X}$. And $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ (resp. $\widehat{\mathrm{Pic}}(\mathcal{X})$) is the category (resp. group) of hermitian line bundles on $\mathcal{X}$ under isometry. The category $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathbb{Q}}$ is defined from $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ similar to that in §2.2.

Define

$$\widehat{\mathrm{CaCl}}(\mathcal{X}) := \widehat{\mathrm{Div}}(\mathcal{X})/\widehat{\mathrm{Pr}}(\mathcal{X}).$$

Then there is a canonical isomorphism

$$\widehat{\mathrm{CaCl}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{X}).$$

If $\Sigma = \emptyset$, and thus $\overline{B} = B$ is a scheme, then the above groups are just the usual ones

$$\mathrm{Div}(\mathcal{X}), \quad \mathrm{Pr}(\mathcal{X}), \quad \mathrm{CaCl}(\mathcal{X}), \quad \mathcal{P}\mathrm{ic}(\mathcal{X}), \quad \mathrm{Pic}(\mathcal{X}).$$

Let $\mathcal{U}$ be an open subscheme of $\mathcal{X}$. As in §2.2, we also have the groups of objects of $(\mathbb{Q}, \mathbb{Z})$-coefficients:

$$\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U}), \quad \widehat{\mathrm{CaCl}}(\mathcal{X}, \mathcal{U}).$$

For example, $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ is the fiber product of the natural map $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}} \to \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}$ with the natural map $\mathrm{Div}(\mathcal{U}) \to \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}$, whose elements are pairs $(\overline{\mathcal{D}}, \mathcal{D}')$, called *arithmetic* $(\mathbb{Q}, \mathbb{Z})$*-divisors on* $(\mathcal{X}, \mathcal{U})$, where $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ and $\mathcal{D}' \in \mathrm{Div}(\mathcal{U})$ have equal images in $\mathrm{Div}(\mathcal{U})_{\mathbb{Q}}$.

An element of $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ is called *effective* if its images in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ and $\mathrm{Div}(\mathcal{U})$ are both effective. The effectivity induces a *partial order* on $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ as before.

Adelic divisors on a quasi-projective variety

Let $\mathcal{U}$ be a quasi-projective arithmetic variety over B. Using pull-back morphisms, define

$$\widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U}), \qquad \widehat{\mathrm{Pr}}(\mathcal{U}/\overline{B})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Pr}}(\mathcal{X}).$$

Here the limits are over projective models $\mathcal{X}$ of $\mathcal{U}$ over B.

The direct limit is actually filtered, i.e., for any two projective models $\mathcal{X}_1, \mathcal{X}_2$ of $\mathcal{U}$ over B, there is a third projective model $\mathcal{Y}$ of $\mathcal{U}$ over B dominating $\mathcal{X}_1, \mathcal{X}_2$ in the sense that there are morphisms $\mathcal{Y} \to \mathcal{X}_1$ and $\mathcal{Y} \to \mathcal{X}_2$ of projective models of $\mathcal{U}$ over B. In fact, it suffices to take $\mathcal{Y}$ to be the Zariski closure of the image of the composition $\mathcal{U} \to \mathcal{U} \times_B \mathcal{U} \to \mathcal{X}_1 \times_B \mathcal{X}_2$, where the first map is the diagonal map. Here $\mathcal{Y}$ is flat over B by the Prüfer property.

The partial order in $\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ induces a partial order in $\widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})_{\mathrm{mod}}$ by the limit process.

Let $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ be *a boundary divisor of* $\mathcal{U}$ *over* B; that is, $\mathcal{X}_0$ is a projective model of $\mathcal{U}$ over B, $\overline{\mathcal{E}}_0$ is a strictly effective arithmetic divisor on $\mathcal{X}_0$ with support $|\mathcal{E}_0| = \mathcal{X}_0 \setminus \mathcal{U}$. This gives an *extended norm*

$$\|\cdot\|_{\overline{\mathcal{E}}_0} : \widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})_{\mathrm{mod}} \longrightarrow [0, \infty]$$

defined by

$$\|\overline{\mathcal{D}}\|_{\overline{\mathcal{E}}_0} := \inf\{\epsilon \in \mathbb{Q}_{>0} : \ -\epsilon\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}} \leq \epsilon\overline{\mathcal{E}}_0\}.$$

Here the inequalities are again defined by effectivity of divisors, and we take the convention that $\inf(\emptyset) = \infty$. The *boundary topology on* $\widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})_{\mathrm{mod}}$ is the topology over $\widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})_{\mathrm{mod}}$ induced by the extended norm $\|\cdot\|_{\overline{\mathcal{E}}_0}$. Thus a neighborhood basis at 0 of the topology is given by

$$B(\epsilon, \widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})_{\mathrm{mod}}) := \{\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})_{\mathrm{mod}} : \ -\epsilon\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}} \leq \epsilon\overline{\mathcal{E}}_0\}, \quad \epsilon \in \mathbb{Q}_{>0}.$$

By translation, it gives a neighborhood basis at any point.

Let $\widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})$ be the *completion* of $\widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})_{\mathrm{mod}}$ with respect to the boundary topology. An element of $\widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})$ is called an *adelic divisor* (or a *compactified divisor*) on $\mathcal{U}$. Define the *class group of adelic divisors* of $\mathcal{U}$ to be

$$\widehat{\mathrm{CaCl}}(\mathcal{U}/\overline{B}) := \widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})/\widehat{\mathrm{Pr}}(\mathcal{U}/\overline{B})_{\mathrm{mod}}.$$

Adelic line bundles on a quasi-projective variety

Let $\mathcal{U}$ be a quasi-projective variety over B. Let $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ be as above. Define the *category* $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\overline{B})$ *of adelic line bundles* on $\mathcal{U}$ as follows. An object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\overline{B})$ is a pair $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ where:

(1) $\mathcal{L}$ is an object of $\mathcal{P}\mathrm{ic}(\mathcal{U})$, i.e., a line bundle on $\mathcal{U}$;

(2) $\mathcal{X}_i$ is a projective model of $\mathcal{U}$ over B;

(3) $\overline{\mathcal{L}}_i$ is an object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}_i)_{\mathbb{Q}}$, i.e. a hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}_i$;

(4) $\ell_i : \mathcal{L} \to \mathcal{L}_i|_{\mathcal{U}}$ is an isomorphism in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$.

Similar to §2.5, the sequence is required to satisfy the *Cauchy condition* that the sequence $\{\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})\}_{i\geq 1}$ is a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})_{\mathrm{mod}}$ under the boundary topology.

A morphism from an object $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\overline{B})$ to another object $(\mathcal{L}', (\mathcal{X}_i', \overline{\mathcal{L}}_i', \ell_i')_{i\geq 1})$ of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\overline{B})$ is an isomorphism $\iota : \mathcal{L} \to \mathcal{L}'$ of the integral line bundles on $\mathcal{U}$ satisfying the following properties. Denote by $\iota_1 : \overline{\mathcal{L}}_1 \dashrightarrow \overline{\mathcal{L}}_1'$ the rational map on $\mathcal{U}$ induced by ι, which induces an element $\widehat{\mathrm{div}}(\iota_1)$ of $\widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})_{\mathrm{mod}}$. Then we require that the sequence $\{\widehat{\mathrm{div}}(\ell_i'\ell_1'^{-1}) - \widehat{\mathrm{div}}(\ell_i\ell_1^{-1}) + \widehat{\mathrm{div}}(\iota_1)\}_{i\geq 1}$ of $\widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})_{\mathrm{mod}}$ converges to 0 in $\widehat{\mathrm{Div}}(\mathcal{U}/\overline{B})$ under the boundary topology.

An object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\overline{B})$ is called an *adelic line bundle* (or a *compactified line bundle*) on $\mathcal{U}$. Define $\widehat{\mathrm{Pic}}(\mathcal{U}/\overline{B})$ to be the *group* of isomorphism classes of objects of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\overline{B})$. As before, there is a canonical isomorphism

$$\widehat{\mathrm{CaCl}}(\mathcal{U}/\overline{B}) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U}/\overline{B}).$$

Definitions on essentially quasi-projective schemes

Let $\overline{B} = (B, \Sigma)$ be a base valued scheme. Let X be a flat and essentially quasi-projective integral scheme over B. Define

$$\widehat{\mathrm{Div}}(X/\overline{B}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/\overline{B}),$$

$$\widehat{\mathrm{CaCl}}(X/\overline{B}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{CaCl}}(\mathcal{U}/\overline{B}),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X/\overline{B}) := \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\overline{B}),$$

$$\widehat{\mathrm{Pic}}(X/\overline{B}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/\overline{B}).$$

An element of $\widehat{\mathrm{Div}}(X/\overline{B})$ is called an *adelic divisor on* $X/\overline{B}$. An object of $\widehat{\mathcal{P}\mathrm{ic}}(X/\overline{B})$ is called an *adelic line bundle on* $X/\overline{B}$.

We take the following alternative notations:

(1) If $\Sigma = \emptyset$ and thus $\overline{B} = B$ is a Prüfer scheme, we may also write

$$\widehat{\mathrm{Div}}(X/\overline{B}), \quad \widehat{\mathrm{CaCl}}(X/\overline{B}), \quad \widehat{\mathcal{P}\mathrm{ic}}(X/\overline{B}), \quad \widehat{\mathrm{Pic}}(X/\overline{B})$$

as

$$\widetilde{\mathrm{Div}}(X/B), \quad \widetilde{\mathrm{CaCl}}(X/B), \quad \widetilde{\mathcal{P}\mathrm{ic}}(X/B), \quad \widetilde{\mathrm{Pic}}(X/B).$$

This is to emphasize that there is no archimedean component involved in the terms. If $B = \mathrm{Spec}\, R$ is affine, they are further written as

$$\widetilde{\mathrm{Div}}(X/R), \quad \widetilde{\mathrm{CaCl}}(X/R), \quad \widetilde{\mathcal{P}\mathrm{ic}}(X/R), \quad \widetilde{\mathrm{Pic}}(X/R).$$

(2) If $\overline{B} = (\mathrm{Spec}\, O_K, \mathrm{Hom}(K, \mathbb{C}))$ is in the arithmetic case, we may also write

$$\widehat{\mathrm{Div}}(X/\overline{B}), \quad \widehat{\mathrm{CaCl}}(X/\overline{B}), \quad \widehat{\mathcal{P}\mathrm{ic}}(X/\overline{B}), \quad \widehat{\mathrm{Pic}}(X/\overline{B})$$

as

$$\widehat{\mathrm{Div}}(X/O_K), \quad \widehat{\mathrm{CaCl}}(X/O_K), \quad \widehat{\mathcal{P}\mathrm{ic}}(X/O_K), \quad \widehat{\mathrm{Pic}}(X/O_K).$$

We take a similar notation for the archimedean case $\overline{B} = (\mathrm{Spec}\, \mathbb{R}, i_{\mathrm{st}})$ or $\overline{B} = (\mathrm{Spec}\, \mathbb{C}, \mathrm{id})$.

To compare the notations with the original setting ($k = \mathbb{Z}$ or a field), we have the following.

(a) If k is a field, the current term $\widehat{\mathrm{Div}}(X/B)$ with $B = \mathrm{Spec}\, k$ is the same as the original term $\widehat{\mathrm{Div}}(X/k)$.

(b) If $k = \mathbb{Z}$, the current term $\widehat{\mathrm{Div}}(X/\overline{B})$ with $\overline{B} = (\mathrm{Spec}\, \mathbb{Z}, \mathrm{Hom}(\mathbb{Q}, \mathbb{C}))$ is the same as the original term $\widehat{\mathrm{Div}}(X/\mathbb{Z})$. On the other hand, the term $\widetilde{\mathrm{Div}}(X/\mathbb{Z})$ removes the Green's functions from the arithmetic case. Then we have natural forgetful maps

$$\widehat{\mathrm{Div}}(X/\mathbb{Z}) \longrightarrow \widetilde{\mathrm{Div}}(X/\mathbb{Z}), \qquad \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z}) \longrightarrow \widetilde{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z}).$$

They are actually surjective (or essentially surjective).

(c) If K is a number field, for any flat and essentially quasi-projective integral scheme X over O_K, we have canonical isomorphisms

$$\widehat{\mathrm{Div}}(X/O_K) \longrightarrow \widehat{\mathrm{Div}}(X/\mathbb{Z}), \qquad \widehat{\mathcal{P}\mathrm{ic}}(X/O_K) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X/\mathbb{Z}).$$

In fact, this follows from the fact that a scheme over O_K is projective (resp. flat) over O_K if and only if it is projective (resp. flat) over $\mathbb{Z}$. Therefore, our original approach essentially includes this case.

For any base valued scheme $\overline{B} = (B, \Sigma)$, there are canonical forgetful maps
$$\widehat{\mathrm{Div}}(X/\overline{B}) \longrightarrow \widetilde{\mathrm{Div}}(X/B) \longrightarrow \mathrm{Div}(X),$$
and
$$\widehat{\mathcal{P}\mathrm{ic}}(X/\overline{B}) \longrightarrow \widetilde{\mathcal{P}\mathrm{ic}}(X/B) \longrightarrow \mathcal{P}\mathrm{ic}(X).$$
These are induced by the forgetful functor
$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\overline{B}) \longrightarrow \widetilde{\mathcal{P}\mathrm{ic}}(\mathcal{U}/B) \longrightarrow \mathcal{P}\mathrm{ic}(\mathcal{U}),$$
given by
$$(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)) \longmapsto (\mathcal{L}, (\mathcal{X}_i, \mathcal{L}_i, \ell_i)) \longmapsto \mathcal{L}.$$
As a convention, our notation for the three objects are usually denoted by
$$\overline{L} \longmapsto \widetilde{L} \longmapsto L.$$
We often refer L as the *underlying line bundle of* $\overline{L}$ and $\widetilde{L}$, and refer $\overline{L}$ as an *adelic extension of* L.

The theory over function fields

Arakelov geometry is analogous to algebraic geometry over fields, which is the reason why have the uniform terminology throughout this paper. However, Arakelov geometry is actually more analogous to algebraic geometry over a projective curve. Here we explore this analogue briefly.

Let k be a field, and B be a projective and regular curve over k. Denote by $K = k(B)$ the function field of B. In the above perspective, the counterpart of the arithmetic object $\widehat{\mathrm{Div}}(\cdot/\mathbb{Z})$ should be the geometric object $\widetilde{\mathrm{Div}}(\cdot/B)$.

Let X be flat and essentially quasi-projective integral scheme over B. Then X is also essentially quasi-projective over k. We claim that there are canonical isomorphisms
$$\widetilde{\mathrm{Div}}(X/B) \longrightarrow \widetilde{\mathrm{Div}}(X/k),$$
$$\widetilde{\mathrm{CaCl}}(X/B) \longrightarrow \widetilde{\mathrm{CaCl}}(X/k),$$
$$\widetilde{\mathrm{Pic}}(X/B) \longrightarrow \widetilde{\mathrm{Pic}}(X/k),$$
$$\widetilde{\mathcal{P}\mathrm{ic}}(X/B) \longrightarrow \widetilde{\mathcal{P}\mathrm{ic}}(X/k).$$

In this sense, we do not lose much in our original setup by considering the objects over the absolute base field k.

To see the isomorphisms, note that any quasi-projective (resp. projective) model of X over B is a quasi-projective (resp. projective) model of X over k. Moreover, for any quasi-projective (resp. projective) model $\mathcal{U}$ of X over k, the rational map $\mathcal{U} \dashrightarrow B$ is defined along X and can be turned to a morphism by shrinking $\mathcal{U}$ (resp. blowing-up $\mathcal{U}$ along a center disjoint from X). Therefore, the inverse systems of quasi-projective (resp. projective) models of X over B is cofinal to the inverse system of quasi-projective (resp. projective) models of X over k.

3 Interpretation by Berkovich spaces

Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k, as defined in §2.3. In §2.5, we have introduced the category $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ of adelic line bundles on X. The goal of this section is to introduce a category $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ of metrized line bundles on the Berkovich analytic space X^{an} associated to X, and study the analytification functor from $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ to $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$. The analytification functor is fully faithful, and thus gives a convenient interpretation of adelic line bundles. This generalizes the work of [Zha2] for projective varieties over number fields.

3.1 Berkovich spaces

In this section, we review definitions and some basic properties on Berkovich spaces. In the end, we introduce a density result which will be useful in analytification of adelic divisors and adelic line bundles.

3.1.1 Generality on Berkovich spaces

Berkovich spaces are best known as analytic spaces associated to varieties over non-archimedean fields, whose foundation was introduced by Berkovich [Ber1]. By Berkovich [Ber2, §1], the base fields are relaxed to be Banach rings, and the old construction works similarly. In the following, we recall the construction of [Ber2, §1] to adapt our setting that the schemes are not required to be of finite type.

Let k be a commutative Banach ring with 1. Let X be a scheme over k. In the following, we recall the definition and basic properties of the *Berkovich space* X^{an} *associated to* X, which is more rigorously written as $(X/k)^{\mathrm{an}}$ to emphasize the dependence on k.

(1) *Affine case.* If $X = \operatorname{Spec} A$, then X^{an} is defined to be the space $\mathcal{M}(A) = \mathcal{M}(A/k)$ of multiplicative semi-norms on A whose restriction to k is bounded by $|\cdot|_{\mathrm{Ban}}$. For each $x \in \mathcal{M}(A)$, denote its corresponding semi-norm on A by $|\cdot|_x : A \to \mathbb{R}$. For any $f \in A$, write $|f|_x$ as $|f(x)|$, which gives a real-valued function $|f|$ on $\mathcal{M}(A)$. The topology on $\mathcal{M}(A)$ is the weakest one such that the function $|f| : \mathcal{M}(A) \to \mathbb{R}$ is continuous for all $f \in A$.

(2) *General case.* If X is covered by an affine open cover $\{\operatorname{Spec} A_i\}_i$, then X^{an} is defined to be the union of $\mathcal{M}(A_i)$, glued in the canonical way. The topology of X^{an} is the weakest one such that each $\mathcal{M}(A_i)$ is an open subspace of X^{an}.

(3) *Residue field.* For each $x \in \mathcal{M}(A)$, the corresponding semi-norm $|\cdot|_x$ induces a norm on the integral domain $A/\ker(|\cdot|_x)$. The completion of the fraction field of $A/\ker(|\cdot|_x)$ is called the *residue field* of x, and denoted by H_x. Denote by $|\cdot|$ the valuation (multiplicative norm) on H_x induced by $|\cdot|_x$. Then $|\cdot|_x : A \to \mathbb{R}$ is equal to the composition

$$A \longrightarrow H_x \xrightarrow{|\cdot|} \mathbb{R}.$$

We write the first map as $f \mapsto f(x)$, which is compatible with the convention $|f|_x = |f(x)|$. The notation H_x generalizes to any scheme X over k.

(4) *Contraction.* There is a canonical contraction map $\kappa : X^{\mathrm{an}} \to X$. It suffices to describe it in the case $X = \operatorname{Spec} A$. For each $x \in \mathcal{M}(A)$, the kernel of the map $|\cdot|_x : A \to \mathbb{R}$ is a prime ideal of A, and thus defines an element $\kappa(x) \in \operatorname{Spec} A$.

(5) *Injection.* Assume that for any $x \in \operatorname{Spec} k$, the semi-norm $|\cdot|_{x,0}$ on k, induced by the trivial norm on the residue field k/x, is bounded by $|\cdot|_{\mathrm{Ban}}$. This gives a natural injection $\iota : \operatorname{Spec} k \to \mathcal{M}(k)$ by sending x to $|\cdot|_{x,0}$.

Under this assumption, there is a natural injection $\iota : X \to X^{\mathrm{an}}$ defined similarly. In fact, it suffices to describe it in the case $X = \operatorname{Spec} A$. For any $x \in \operatorname{Spec} A$, still denote by $|\cdot|_{x,0}$ the semi-norm on A induced by the trivial norm on the residue field A/x. Then $\iota : X \to X^{\mathrm{an}}$ sends x to $|\cdot|_{x,0}$. It is obvious that $\kappa \circ \iota$ is the identity map on X.

(6) *Functoriality.* Any morphism $f : X \to Y$ over k induces a continuous map $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. For any point $v \in Y^{\mathrm{an}}$, the *fiber* $X_v^{\mathrm{an}} = (f^{\mathrm{an}})^{-1}(v)$, defined as a subspace of X^{an}, is canonically homeomorphic to the Berkovich space $(X_{H_v}/H_v)^{\mathrm{an}}$. More generally, for any subset $T \subset Y^{\mathrm{an}}$, denote by X_T^{an} the preimage of T, viewed as a subspace of X^{an}. These notation automatically applies to the case $Y = \operatorname{Spec} k$ and $Y^{\mathrm{an}} = \mathcal{M}(k)$.

By [Ber2, Lem. 1.1, Lem. 1.2], we have the following basic topological properties:

(1) If X is separated and of finite type over k, then X^{an} is Hausdorff.

(2) If X is of finite type over k, then X^{an} is locally compact.

(3) If X is projective over k, then X^{an} is compact.

It is well-known that Berkovich spaces also include the complex analytic spaces coming from algebraic varieties.

(1) If $k = \mathbb{C}$ with the standard absolute value, and X is of finite type over $\mathbb{C}$, then X^{an} is homeomorphic to the analytic space $X(\mathbb{C})$.

(2) If $k = \mathbb{R}$ with the standard absolute value, and X is of finite type over $\mathbb{R}$, then X^{an} is homeomorphic to the quotient of the analytic space $X(\mathbb{C})$ by the action of the complex conjugation.

In general, we have a decomposition

$$X^{\mathrm{an}} = X^{\mathrm{an}}[\infty] \cup X^{\mathrm{an}}[\mathrm{f}],$$

where $X^{\mathrm{an}}[\infty]$ is the subset of all archimedean semi-norms in X^{an}, and $X^{\mathrm{an}}[\mathrm{f}]$ is the subset of all non-archimedean semi-norms in X^{an}.

3.1.2 Our choice of base ring

Let k be either $\mathbb{Z}$ or a field. Similar to §1.6, we introduce a uniform terminology for these two cases. Endow k with a norm $|\cdot|_{\mathrm{Ban}}$ as follows. If $k=\mathbb{Z}$, $|\cdot|_{\mathrm{Ban}}$ is the usual archimedean absolute value $|\cdot|_\infty$; if k is a field, $|\cdot|_{\mathrm{Ban}}$ is the trivial valuation $|\cdot|_0$. This makes k into a Banach ring.

Concerning our special situation, we have the following easy results and notations:

(1) If k is a field, then $\mathcal{M}(k)$ has only one element $v_0=|\cdot|_0$ by definition. In this case, if X is a finite type over k, then $X\mapsto X^{\mathrm{an}}$ is just the analytification functor constructed in [Ber1, §3.5].

(2) If k is a field, and X is a projective regular curve over k, then X^{an} is the union of the closed line segments $\{|\cdot|_v^t : 0\leq t\leq\infty\}$ for all closed points $v\in X$, by identifying $|\cdot|_v^0$ with the trivial norm $|\cdot|_0$ for all $v\in X$ as one point. Here $|\cdot|_v$ denotes the normalized valuation $\exp(-\mathrm{ord}_v)$. The space $\mathcal{M}(k(X))$ for the function field $k(X)$ is exactly the subspace of X^{an} obtained by removing the subset $\{|\cdot|_v^\infty : v\in X \text{ closed}\}$.

(3) In the arithmetic case ($k=\mathbb{Z}$), the space $\mathcal{M}(\mathbb{Z})$ is compact and path-connected. As described in [Ber1, 1.4.3], it is the union of the closed line segment
$$[0,1]_\infty:=\{|\cdot|_\infty^t : 0\leq t\leq 1\},$$
and the closed line segments
$$[0,\infty]_p:=\{|\cdot|_p^t : 0\leq t\leq\infty\}$$
for all finite primes p, by identifying the endpoints $|\cdot|_\infty^0$ and $|\cdot|_p^0$ for all finite primes p with the trivial norm $|\cdot|_0$ of $\mathbb{Z}$. Here $|\cdot|_\infty$ and $|\cdot|_p$ denote the usual normalized valuations. The canonical injection $\iota:\operatorname{Spec}\mathbb{Z}\to\mathcal{M}(\mathbb{Z})$ sends the generic point to the trivial norm $|\cdot|_0$, and sends a prime p to $|\cdot|_p^\infty$, the semi-norm of $\mathbb{Z}$ induced by the trivial norm of $\mathbb{F}_p$. The space $\mathcal{M}(\mathbb{Q})$ is exactly the subspace of $\mathcal{M}(\mathbb{Z})$ obtained by removing the subset $\{|\cdot|_p^\infty : p<\infty\}$. There is a very similar description for number fields.

For convenience, denote
$$v_0=|\cdot|_0,\quad v_\infty=|\cdot|_\infty,\quad v_\infty^t=|\cdot|_\infty^t,\quad v_p=|\cdot|_p,\quad v_p^t=|\cdot|_p^t.$$

We may also write ∞ and p for v_∞ and v_p, viewed as points of $\mathcal{M}(\mathbb{Z})$. For convenience, denote by

$$(0,1]_\infty, \quad (0,1)_\infty, \quad (0,\infty]_p, \quad [0,\infty)_p, \quad (0,\infty)_p$$

the sub-intervals of the line segments obtained by removing one or two endpoints; for example,

$$(0,\infty)_p := \{|\cdot|_p^t : 0 < t < \infty\}.$$

(4) In the arithmetic case ($k = \mathbb{Z}$), there is a structure map $X^{\mathrm{an}} \to \mathcal{M}(\mathbb{Z})$. This gives disjoint unions

$$X^{\mathrm{an}} = \bigcup_{v\in\mathcal{M}(\mathbb{Z})} X_v^{\mathrm{an}},$$

where X_v^{an} is the fiber of X^{an} above v. The most distinguished fibers are

$$X_\infty^{\mathrm{an}} = X_{v_\infty}^{\mathrm{an}} = X_{\mathbb{R}}^{\mathrm{an}}, \qquad X_p^{\mathrm{an}} = X_{v_p}^{\mathrm{an}} = X_{\mathbb{Q}_p}^{\mathrm{an}}.$$

According to the structure of $\mathcal{M}(\mathbb{Z})$, we can further write X^{an} as a disjoint union of the following subspaces:

(i) $X_{v_0}^{\mathrm{an}} = (X_{\mathbb{Q}}/\mathbb{Q})^{\mathrm{an}}$ under the trivial norm of $\mathbb{Q}$;

(ii) $X_{v_p^\infty}^{\mathrm{an}} = (X_{\mathbb{F}_p}/\mathbb{F}_p)^{\mathrm{an}}$ under the trivial norm of $\mathbb{F}_p$ for finite primes p;

(iii) $X_{(0,\infty)_p}^{\mathrm{an}}$, homeomorphic to $X_{\mathbb{Q}_p}^{\mathrm{an}} \times (0,\infty)$ for finite primes p;

(iv) $X_{(0,1]_\infty}^{\mathrm{an}}$, homeomorphic to $X_{\mathbb{R}}^{\mathrm{an}} \times (0,1]$.

(5) In both cases (that k is $\mathbb{Z}$ or a field), if X is connected and of finite type over k, then X^{an} is path-connected. In fact, we can assume that X is normal by passing to its normalization.

We first treat the geometric case that k is a field. By blowing-up X, there is a flat morphism $X \to C$ to a connected regular curve C over k. We can further assume that the fibers of $X \to C$ are connected by taking integral closure of C in X. The fibers of $X^{\mathrm{an}} \to C^{\mathrm{an}}$ are path-connected by induction and by the well-known case of non-trivial valuation fields. There are finitely many (connected) closed curves $C_1, \cdots, C_n$ in X such that $\mathrm{Im}(C_1 \to C), \cdots, \mathrm{Im}(C_n \to C)$ is a Zariski open cover of C. Note that $C_1^{\mathrm{an}}, \cdots, C_n^{\mathrm{an}}$ are path-connected by example (2) above, so X^{an} is path-connected.

In the arithmetic case, denote by O_K the integral closure of $\mathbb{Z}$ in $\mathcal{O}_X$. The fibers of $X^{\mathrm{an}} \to \mathcal{M}(O_K)$ are path-connected. For any finite extension K' of K, and any open subscheme C' of $\operatorname{Spec} O_{K'}$, the space C'^{an} is connected by an explicit description similar to (3). So X^{an} is path-connected as in the geometric case.

Note that any multiplicative semi-norm on $\mathbb{Z}$ is bounded by the standard archimedean absolute value, and thus belongs to $\mathcal{M}(\mathbb{Z})$. The space X^{an} in the arithmetic space is actually the "largest" Berkovich space associated to X defined in terms of multiplicative semi-norms.

Let k be either $\mathbb{Z}$ or a field. Let X be a proper scheme over k. There is a *specialization map* (or *reduction map*)

$$r : X^{\mathrm{an}} \longrightarrow X$$

defined as follows. For any point $x \in X^{\mathrm{an}}[\mathrm{f}]$, recall that H_x is the (complete) residue field of x in X^{an}, denote by R_x the valuation ring of H_x, and denote by m_x the maximal ideal of R_x. As X is proper over k, the valuative criterion gives a unique k-morphism $\operatorname{Spec} R_x \to X$ extending the k-morphism $\operatorname{Spec} H_x \to X$ associated to x. Define $r(x)$ to be the image of the unique closed point of $\operatorname{Spec} R_x$ in X.

For any point $x \in X^{\mathrm{an}}[\infty]$, we still have a morphism $\operatorname{Spec} H_x \to X$. Here H_x is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. Define $r(x)$ to be the image of $\operatorname{Spec} H_x$ in X.

3.1.3 Density result

We are interested in X^{an} for essentially quasi-projective scheme X over k. The following result asserts that the Berkovich spaces of essentially quasi-projective schemes do not lose "much points" from those of its quasi-projective models. In the arithmetic case, the space X^{an} is somehow determined by its fibers above the non-trivial absolute values of $\mathbb{Q}$.

Lemma 3.1.1. *Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k.*

(1) Let $X \to \mathcal{U}$ be a quasi-projective model of X over k. Then the induced map $X^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}}$ is continuous, injective, and with a dense image.

Moreover, the set of $v \in X^{\mathrm{an}}$ corresponding to discrete or archimedean valuations of H_v is dense in $\mathcal{U}^{\mathrm{an}}$.

(2) *If $k = \mathbb{Z}$, then $X^{\mathrm{an}} \setminus X^{\mathrm{an}}_{\iota(\operatorname{Spec}\mathbb{Z})}$ is dense in X^{an}. Here $\iota : \operatorname{Spec}\mathbb{Z} \to \mathcal{M}(\mathbb{Z})$ is the canonical injection whose image consists of v_0 and v_p^∞ for finite primes p.*

Proof. We first prove (1). Only the density is not automatic from the definitions. Denote by F the function field of X, which is also the function field of $\mathcal{U}$. There is a composition of injections $(\operatorname{Spec} F)^{\mathrm{an}} \to X^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}}$. It suffices to prove that $(\operatorname{Spec} F)^{\mathrm{an}}$ is dense in $\mathcal{U}^{\mathrm{an}}$. We can assume that $\mathcal{U}$ is projective over k by passing to a projective model.

A point $\xi \in \mathcal{U}^{\mathrm{an}}[\mathrm{f}]$ is called divisorial if there is a birational morphism $\mathcal{U}' \to \mathcal{U}$ from a normal integral scheme of finite type over k, together with a prime Weil divisor $D \subset \mathcal{U}'$, such that $|\cdot|_\xi = \exp(-t\operatorname{ord}_D)$ for some constant $t > 0$. Note that ξ actually lies in $(\operatorname{Spec} F)^{\mathrm{an}}[\mathrm{f}]$. It suffices to prove that the set of divisorial points is dense in $\mathcal{U}^{\mathrm{an}}[\mathrm{f}]$ for all projective variety $\mathcal{U}$ over k.

Note that the analogous statement for a non-archimedean field k (with a non-trivial valuation) is a well-known result. For example, in this case, Berkovich's theory implies that the analytic space $\mathcal{U}^{\mathrm{an}}$ has a topological basis consisting of strictly k-affinoid domains, and any k-affinoid domain has a (non-empty) Shilov boundary. By [GM, A.3, A.6, A.9], any point in the Shilov boundary of a strictly k-affinoid domain is actually a divisorial point.

Return to the original $(\mathcal{U}, k)$. We will prove the density by induction on the dimension of $\mathcal{U}$. Assume that $\mathcal{U}$ is normal by passing to its normalization. The case of dimension 1 essentially follows from the explicit descriptions, noting that the space $\mathcal{M}(O_F)$ for a number field F has a description analogous to that of $\mathcal{M}(\mathbb{Z})$. Assume that $\dim \mathcal{U} > 1$, and that the density statement holds for all lower dimensions.

We first prove the case $k = \mathbb{Z}$ (for the density of divisorial points). Let F be the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}(\mathcal{U})$, so that $\mathcal{U}$ has geometrically connected fibers over $\operatorname{Spec} O_F$. Write $\mathcal{U}^{\mathrm{an}}$ as unions of fibers $\mathcal{U}_v^{\mathrm{an}}$ above $v \in \mathcal{M}(O_F)$. By induction and by the case of non-archimedean fields, it suffices to prove that, for any $v \in \mathcal{M}(O_F) \setminus \mathcal{M}(F)$ corresponding to the trivial norm of the residue field $\mathbb{F}_\wp$ for some prime ideal $\wp$ of O_F, and any irreducible component U (endowed with the reduced scheme structure) of the special fiber $\mathcal{U}_{\mathbb{F}_\wp}$, any divisorial point $\xi \in (U/\mathbb{F}_\wp)^{\mathrm{an}}$ lies in the closure of the divisorial points of $\mathcal{U}^{\mathrm{an}}$.

To illustrate the key idea, we will first treat the nice case that $U = \mathcal{U}_{\mathbb{F}_\wp}$ is smooth over $\mathbb{F}_p$, $\xi \in (U/\mathbb{F}_\wp)^{\mathrm{an}}$ is a divisorial point corresponding to a prime divisor $D \subset U$, and that there is a section S of $\mathcal{U}_{O_{F_v}}$ over O_{F_v} such that the point $u = S_{\mathbb{F}_\wp}$ is regular in all $D, U, \mathcal{U}$. In that case, we can find a local system of parameters $\varpi, x_1, \cdots, x_d \in \mathcal{O}_{\mathcal{U},u}$, such that locally at $u \in \mathcal{U}$, U is defined by the ideal (ϖ), and D is defined by the ideal (ϖ, x_1). Here we require $\varpi \in O_F$ to be a generator of $\wp$. Any $f \in \mathcal{O}_{\mathcal{U},u}$ can be uniquely written as a power series

$$f = \sum_{n \geq 0} a_n x_1^n, \qquad a_n \in O_{F_\wp}[[x_2, \cdots, x_d]].$$

Let $0 < r < 1$ and $t > 0$ be real numbers. Set

$$|f|_{y_t} = \max_{n \geq 0} \left(|a_n|_\wp^t r^n\right).$$

Here $|\cdot|_\wp = \exp(-\mathrm{ord}_\wp)$. We can check that $y_t \in \mathcal{U}^{\mathrm{an}}$ with

$$\lim_{t \to \infty} |f|_{y_t} = r^{\min\{n : |a_n|_\wp = 1\}} = r^{\mathrm{ord}_{x_1}(f \bmod \wp)}.$$

As r varies, the right-hand side gives exactly all the divisorial points of $(U/\mathbb{F}_\wp)^{\mathrm{an}}$ corresponding to the divisor $D \subset U$. This finishes the nice case.

Now we need to make a few operations and replacements to convert the general case $(\mathcal{U}_{O_{F_\wp}}, \wp, U, \xi, D)$ to the nice case. Note that the situation is local at $\wp$ over O_F, so we are only concerned with $\mathcal{U}_{O_{F_\wp}}$ instead of $\mathcal{U}$. The process is mostly geometric, but a little tedious.

First, we convert to the case that U is normal and $\xi \in (U/\mathbb{F}_\wp)^{\mathrm{an}}$ is a divisorial point corresponding to a prime divisor $D \subset U$. In fact, assume that $\xi \in (U/\mathbb{F}_\wp)^{\mathrm{an}}$ is a divisorial point given by a prime divisor D' on a normal projective scheme U' over $\mathbb{F}_\wp$ with a birational morphism $U' \to U$. Then $U' \to U$ is obtained by blowing-up U along a closed subscheme Z of U. Let $\mathcal{U}' \to \mathcal{U}$ be the blowing-up of $\mathcal{U}$ along Z, the strict transform of U is exactly U'. Replace $(\mathcal{U}, U)$ by $(\mathcal{U}', U')$.

Second, there is a finite and flat morphism $\mathcal{U}_{O_{F_\wp}} \to \mathbb{P}^d_{O_{F_\wp}}$ with $d = \dim \mathcal{U}$. Take any ample line bundle $\mathcal{L}$ on $\mathcal{U}_{O_{F_\wp}}$. Denote by $\mathcal{L}_0$ its pull-back to $\mathcal{U}_{\mathbb{F}_\wp}$. There is a positive integer m, such that $\Gamma(\mathcal{U}_{\mathbb{F}_\wp}, m\mathcal{L}_0)$ contains a base-point-free subspace V of dimension $d+1$ such that the corresponding morphism $\mathcal{U}_{O_{F_\wp}} \to \mathbb{P}(V)$ is finite. This is done by the classical argument of embedding to a projective space of a high dimension and projecting to hyperplanes successively. For m large enough, the map $\Gamma(\mathcal{U}_{O_{F_\wp}}, m\mathcal{L}) \to \Gamma(\mathcal{U}_{\mathbb{F}_\wp}, m\mathcal{L}_0)$ is surjective, and thus we can lift V to an $O_{F_\wp}$-submodule $\widetilde{V}$ of $\Gamma(\mathcal{U}_{O_{F_\wp}}, m\mathcal{L})$ of the same rank. The morphism $\mathcal{U}_{O_{F_\wp}} \to \mathbb{P}(\widetilde{V})$ satisfies the requirement.

Third, in the morphism $\mathcal{U}_{O_{F_\wp}} \to \mathbb{P}^d_{O_{F_\wp}}$, denote by U_0 and D_0 the images of U and D respectively. Note that $U_0 = \mathbb{P}^d_{\mathbb{F}_\wp}$. We claim that the result for $(\mathbb{P}^d_{O_{F_\wp}}, U_0, D_0, \xi_0)$ implies that for $(\mathcal{U}_{O_{F_\wp}}, U, D, \xi)$. Here ξ_0 is the image of ξ in U_0^{an}. In fact, denote by $\mathcal{U}' \to \mathbb{P}^d_{O_{F_\wp}}$ the Galois closure of $\mathcal{U}_{O_{F_\wp}} \to \mathbb{P}^d_{O_{F_\wp}}$, defined to be the normalization of the Galois closure of the function fields. Denote by $\xi'_1, \cdots, \xi'_m$ the preimage of ξ in $(\mathcal{U}'/O_{F_\wp})^{\mathrm{an}}$. If the result for $(\mathbb{P}^d_{O_{F_\wp}}, U_0, D_0, \xi_0)$ holds, then by taking preimages of divisorial point, one of ξ'_i lies in the closure of divisorial points of $(\mathcal{U}'/O_{F_\wp})^{\mathrm{an}}$. This also uses compactness of $(\mathcal{U}'/O_{F_\wp})^{\mathrm{an}}$. As the Galois group acts transitively on $\xi'_1, \cdots, \xi'_m$, any ξ'_i lies in the closure of divisorial points of $(\mathcal{U}'/O_{F_\wp})^{\mathrm{an}}$. Taking images in $\mathcal{U}^{\mathrm{an}}$, we see that ξ lies in the closure of divisorial points $\mathcal{U}^{\mathrm{an}}$.

Replace $(\mathcal{U}_{O_{F_\wp}}, U, D, \xi)$ by $(\mathbb{P}^d_{O_{F_\wp}}, U_0, D_0, \xi_0)$. We have convert the problem to the case that $U = \mathcal{U}_{\mathbb{F}_\wp}$ is smooth over $\mathbb{F}_\wp$ and $\xi \in (U/\mathbb{F}_\wp)^{\mathrm{an}}$ is a divisorial point corresponding to a prime divisor $D \subset U$.

Fourth, it remains to construct a section S of $\mathcal{U}_{O_{F_\wp}}$ over $O_{F_\wp}$ passing through a regular point of D. This is easily done by the base change by a finite unramified extension of $F_\wp$.

Therefore, our proof of (1) for the arithmetic case $k = \mathbb{Z}$ is complete. If k is a field, by blowing-up $\mathcal{U}$, there is a fibration $\mathcal{U} \to \mathbb{P}^1_k$. Then the above induction argument still works.

Now we prove (2). Let $\mathcal{U}$ be a normal projective model of X as in (1). Still consider the composition $(\operatorname{Spec} F)^{\mathrm{an}} \to X^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}}$. The topologies of $(\operatorname{Spec} F)^{\mathrm{an}}$ and X^{an} are the same as the subspace topologies induced from $\mathcal{U}^{\mathrm{an}}$. It suffices to prove that $(\operatorname{Spec} F)^{\mathrm{an}} \setminus (\operatorname{Spec} F)^{\mathrm{an}}_{\iota(\operatorname{Spec}\mathbb{Z})}$ is dense in $\mathcal{U}^{\mathrm{an}}$. In (1), we have proved that $(\operatorname{Spec} F)^{\mathrm{an}}$ is dense in $\mathcal{U}^{\mathrm{an}}$, but the proof actually gives the statement that $(\operatorname{Spec} F)^{\mathrm{an}} \setminus (\operatorname{Spec} F)^{\mathrm{an}}_{\iota(\operatorname{Spec}\mathbb{Z})}$ is dense in $\mathcal{U}^{\mathrm{an}} \setminus \mathcal{U}^{\mathrm{an}}_{v_0}$. Here v_0 denotes the trivial norm of $\mathbb{Z}$. Therefore, it suffices to prove that, any divisorial point ξ of $\mathcal{U}^{\mathrm{an}}_{v_0}$ lies in the closure of $\cup_{p<\infty}\mathcal{U}^{\mathrm{an}}_{\mathbb{F}_p}$ in $\mathcal{U}^{\mathrm{an}}$.

As above, we have a geometrically connected morphism $\mathcal{U} \to \operatorname{Spec} O_F$. Then the divisorial point ξ lies in $\mathcal{U}^{\mathrm{an}}_{v_0} = (\mathcal{U}_F/F)^{\mathrm{an}}$ under the trivial norm of F. By replacing $\mathcal{U}$ by a blowing-up if necessary, we can assume that ξ corresponds to a prime divisor D of $\mathcal{U}_F$. Denote by $\mathcal{D}$ the Zariski closure of D in $\mathcal{U}$. For all but finitely many prime ideal $\wp$ of O_F, the reduction $\mathcal{U}_{\mathbb{F}_\wp}$ is normal and $\mathcal{D}_{\mathbb{F}_\wp}$ is a prime divisor of $\mathcal{U}_{\mathbb{F}_\wp}$. For any rational function f of $\mathcal{U}$, for all but finitely many prime ideal $\wp$ of O_F, the specializations above $\wp$ of the irreducible components of $\mathrm{div}(f|_{\mathcal{U}_F})$ are irreducible and distinct, which give $\mathrm{ord}_D(f|_{\mathcal{U}_F}) = \mathrm{ord}_{\mathcal{D}_{\mathbb{F}_\wp}}(f|_{\mathcal{U}_{\mathbb{F}_\wp}})$. This proves that $\exp(-\mathrm{ord}_D)$ is the limit of $\exp(-\mathrm{ord}_{\mathcal{D}_{\mathbb{F}_\wp}})$ as $\wp$ varies. Therefore, x lies in the closure of divisorial points of $(\mathrm{Spec}\, F)^{\mathrm{an}} \setminus (\mathrm{Spec}\, F)^{\mathrm{an}}_{v_0}$ in $\mathcal{U}^{\mathrm{an}}$. This proves (2). $\square$

3.2 Arithmetic divisors and metrized line bundles

In this subsection, we introduce arithmetic divisors and metrized line bundles on Berkovich spaces, which are analytic counterparts of the adelic divisors in §2.4 and the adelic line bundles in §2.5.

3.2.1 Arithmetic divisors

Let k be a commutative Banach ring, which is also an integral domain. Let X be an integral scheme over k. Let $X^{\mathrm{an}} = (X/k)^{\mathrm{an}}$ be the Berkovich space defined above.

Let D be a Cartier divisor on X. By a *Green's function* of the divisor D on X^{an}, we mean a continuous function $g : X^{\mathrm{an}} \setminus |D|^{\mathrm{an}} \to \mathbb{R}$ with logarithmic singularity along D in the sense that, for any rational function f on a Zariski open subset U of X satisfying $\mathrm{div}(f) = D|_U$, the function $g + \log|f|$ can be extended to a continuous function on U^{an}.

The pair $\overline{D} = (D, g)$ is called an *arithmetic divisor* on X^{an}. An arithmetic divisor is called *effective* if D is an effective Cartier divisor on X and $g \geq 0$ on $X^{\mathrm{an}} \setminus |D|^{\mathrm{an}}$. An arithmetic divisor is called *principal* if it is of the form

$$\widehat{\mathrm{div}}_{X^{\mathrm{an}}}(f) := (\mathrm{div}(f), -\log|f|)$$

for some nonzero rational function f on X.

An arithmetic divisor $\overline{D}$ or its Green's function g is called *norm-equivariant* if for any points $x, x_1 \in X^{\mathrm{an}} \setminus |D|^{\mathrm{an}}$ satisfying $|\cdot|_x = |\cdot|_{x_1}^t$ for some $0 \leq t < \infty$ locally on $\mathcal{O}_X$, we have $g(x) = t\, g(x_1)$. By definition, principal arithmetic divisors are norm-equivariant.

Denote by $\widehat{\mathrm{Div}}(X^{\mathrm{an}})$ the group of arithmetic divisors on X^{an}, by $\widehat{\mathrm{Pr}}(X^{\mathrm{an}})$ the group of principal arithmetic divisors on X^{an}, and by $\widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}}$ the group of norm-equivariant arithmetic divisors on X^{an}. Denote the class group of arithmetic divisors as

$$\widehat{\mathrm{CaCl}}(X^{\mathrm{an}}) := \widehat{\mathrm{Div}}(X^{\mathrm{an}})/\widehat{\mathrm{Pr}}(X^{\mathrm{an}}),$$

$$\widehat{\mathrm{CaCl}}(X^{\mathrm{an}})_{\mathrm{eqv}} := \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}}/\widehat{\mathrm{Pr}}(X^{\mathrm{an}}).$$

Notice that for any arithmetic divisor $\overline{D} = (D, g)$ on X^{an}, the algebraic part D is a Cartier divisor on X (instead of X^{an}) and g is a function on X^{an}. We take this ad hoc definition to avoid defining general Cartier divisors on X^{an}, by the lack of a good theory of analytic functions on X^{an}.

3.2.2 Metrized line bundles

Let k be a commutative Banach ring which is also an integral domain. Let X be an integral scheme over k. Let $X^{\mathrm{an}} = (X/k)^{\mathrm{an}}$ be the Berkovich space defined above.

Let L be a line bundle on X. At each point $x \in X^{\mathrm{an}}$, denote by $\bar{x}$ the image of x in X. The *fiber* $L^{\mathrm{an}}(x)$ of L at x is defined to be the H_x-line $L(\bar{x}) \otimes_{k(\bar{x})} H_x$, or equivalently the completion of the fiber $L(\bar{x})$ of L on $\bar{x}$ with respect to the semi-norm $|\cdot|_x$. By a *metric* $\|\cdot\|$ of L on X^{an} we mean a continuous metric on $\coprod_{x \in X^{\mathrm{an}}} L^{\mathrm{an}}(x)$ compatible with the semi-norms on $\mathcal{O}_X$. More precisely, to each point $x \in X^{\mathrm{an}}$, we assign a norm $\|\cdot\|_x$ on the H_x-line $L^{\mathrm{an}}(x)$ which is compatible with the norm $|\cdot|_x$ of H_x in the sense that

$$\|f\ell\|_x = |f|_x \cdot \|\ell\|_x, \qquad f \in H_x, \quad \ell \in L^{\mathrm{an}}(x).$$

We always assume that the metric $\|\cdot\|$ on L is *continuous* in the sense that, for any section ℓ of L on a Zariski open subset U of X, the function $\|\ell(x)\| = \|\ell(x)\|_x$ is continuous in $x \in U^{\mathrm{an}}$.

The pair $(L, \|\cdot\|)$ above is called a *metrized line bundle* on X^{an}. An *isometry* from a metrized line bundles $(L, \|\cdot\|)$ to another one $(L', \|\cdot\|')$ is an isomorphism $i : L \to L'$ of line bundles on X such that $\|\cdot\| = i^*\|\cdot\|'$.

A metrized line bundle $\overline{L} = (L, \|\cdot\|)$ or its metric $\|\cdot\|$ is called *norm-equivariant* if for any rational section s of L on X, and any points $x, x_1 \in X^{\mathrm{an}} \setminus |\mathrm{div}(s)|^{\mathrm{an}}$ satisfying $|\cdot|_x = |\cdot|_{x_1}^t$ for some $0 \leq t < \infty$ locally on $\mathcal{O}_X$, we have $\|s\|_x = \|s\|_{x_1}^t$.

Denote by $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ the *category* of metrized line bundles on X^{an}, where the morphisms are isometries. Denote by $\widehat{\mathrm{Pic}}(X^{\mathrm{an}})$ the *group* of isometry classes of metrized line bundles on X^{an}. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$ (resp. $\widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$) the full subcategory (resp. the subgroup) of norm-equivariant line bundles in $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})$ (resp. $\widehat{\mathrm{Pic}}(X^{\mathrm{an}})$).

Similar to $\widehat{\mathrm{Div}}(X^{\mathrm{an}})$, elements of $\widehat{\mathrm{Pic}}(X^{\mathrm{an}})$ are of the form $(L, \|\cdot\|)$, where L is a line bundle on X (instead of X^{an}) and $\|\cdot\|$ is a metric on X^{an}. We have forgetful maps

$$\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}}) \longrightarrow \mathcal{P}\mathrm{ic}(X), \quad \widehat{\mathrm{Pic}}(X^{\mathrm{an}}) \longrightarrow \mathrm{Pic}(X).$$

The fibers of the second map are homogeneous spaces of the group of metrics on $\mathcal{O}_X$.

There are canonical isomorphisms

$$\widehat{\mathrm{CaCl}}(X^{\mathrm{an}}) \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}}),$$

$$\widehat{\mathrm{CaCl}}(X^{\mathrm{an}})_{\mathrm{eqv}} \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}.$$

In fact, given any arithmetic divisor (D, g) on X^{an}, the term $e^{-g/2}$ defines a metric on $\mathcal{O}(D)$, and thus we obtain a metrized line bundle on X^{an}. Conversely, for any metrized line bundle $(L, \|\cdot\|)$ on X^{an}, if s is a rational section of L, then

$$\widehat{\mathrm{div}}_{X^{\mathrm{an}}}(s) := (\mathrm{div}(s), -\log \|s\|)$$

defines an arithmetic divisor on X^{an}. Both processes keep the properties of being norm-equivariant.

In the case $k = \mathbb{Z}$, a norm-equivariant Green's function or a norm-equivariant metric on a line bundle on X^{an} is uniquely determined by its restriction to the disjoint union of the distinguished fibers $X_v^{\mathrm{an}} = X_{\mathbb{Q}_v}^{\mathrm{an}}$ over all places $v \leq \infty$. This follows from Lemma 3.1.1(2). Later one, all Green's functions and metrics in our consideration will be norm-equivariant.

3.3 Analytification of adelic divisors

The adelic divisors in §2.4 induce norm-equivariant arithmetic divisors on Berkovich spaces. The goal of this section is to study this analytification process. The main result is as follows:

Proposition 3.3.1. *Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. There are canonical injective maps*

$$\widehat{\mathrm{Div}}(X/k) \longrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}},$$

$$\widehat{\mathrm{CaCl}}(X/k) \longrightarrow \widehat{\mathrm{CaCl}}(X^{\mathrm{an}})_{\mathrm{eqv}}.$$

In the following, we will construct the maps and prove the injectivity in the order of projective case, quasi-projective case and essentially quasi-projective case.

3.3.1 Projective case

Let k be either $\mathbb{Z}$ or a field. Take the uniform terminology in §1.6.

Let $\mathcal{X}$ be a projective variety over k. Then there is a canonical map

$$\widehat{\mathrm{Div}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{X}^{\mathrm{an}})_{\mathrm{eqv}}.$$

In the following, for any $\overline{\mathcal{D}} = (\mathcal{D}, g) \in \widehat{\mathrm{Div}}(\mathcal{X})$, we will introduce a Green's function $\widetilde{g}$ of $\mathcal{D}$ on $\mathcal{X}^{\mathrm{an}}$, and define the map by $(\mathcal{D}, g) \mapsto (\mathcal{D}, \widetilde{g})$.

We will define $\widetilde{g}$ according to the decomposition $\mathcal{X}^{\mathrm{an}} = \mathcal{X}^{\mathrm{an}}[\mathrm{f}] \cup \mathcal{X}^{\mathrm{an}}[\infty]$, and then check the continuity.

For any point $x \in \mathcal{X}^{\mathrm{an}}[\mathrm{f}]$, recall that there is a specialization map $r : \mathcal{X}^{\mathrm{an}}[\mathrm{f}] \to \mathcal{X}$ by the properness of $\mathcal{X}$ over k. Let $\mathcal{U}$ be a Zariski open subscheme of $\mathcal{X}$ containing $r(x)$ such that $\mathcal{D}|_{\mathcal{U}}$ is defined by a single equation $f \in k(\mathcal{U})^{\times}$ on $\mathcal{U}$. By $r(x) \in \mathcal{U}$, the image of $\operatorname{Spec} R_x \to X$ lies in $\mathcal{U}$ and thus $x \in \mathcal{U}^{\mathrm{an}}$. Define $\widetilde{g}(x) = -\log|f(x)|$. This definition is independent of the choice of $(\mathcal{U}, f)$.

It is easy to define $\widetilde{g}$ on $\mathcal{X}^{\mathrm{an}}[\infty]$ (in the arithmetic case). In fact, the Green's function g on $\mathcal{X}(\mathbb{C})$ descends to the fiber $\mathcal{X}^{\mathrm{an}}_{\infty} = \mathcal{X}^{\mathrm{an}}_{\mathbb{R}}$. This gives the definition of $\widetilde{g}$ on $\mathcal{X}^{\mathrm{an}}_{\infty}$. It extends to $\mathcal{X}^{\mathrm{an}}[\infty]$ by requiring $\widetilde{g}$ to be norm-equivariant. In fact, for any point $x \in \mathcal{X}^{\mathrm{an}}[\infty]$, there is a unique point $x_1 \in \mathcal{X}^{\mathrm{an}}_{\infty}$ such that $|\cdot|_x = |\cdot|_{x_1}^t$ for some $0 < t < 1$, and then we set $\widetilde{g}(x) = t\,\widetilde{g}(x_1)$.

Now we prove that $\widetilde{g}$ is indeed a Green's function; i.e., $\widetilde{g}$ is continuous on $\mathcal{X}^{\mathrm{an}} \setminus |\mathcal{D}|^{\mathrm{an}}$, and has logarithmic singularity along $\mathcal{D}$.

Lemma 3.3.2. *The function $\widetilde{g}$ is a Green's function of $\mathcal{D}$ on $\mathcal{X}^{\mathrm{an}}$.*

Proof. We first note that the continuity of $\widetilde{g}$ on $\mathcal{X}^{\mathrm{an}} \setminus |\mathcal{D}|^{\mathrm{an}}$ (for all such $\overline{\mathcal{D}}$) implies that $\widetilde{g}$ has logarithmic singularity along $\mathcal{D}$. In fact, let f be a local equation of $\mathcal{D}$ on an open subscheme $\mathcal{U}$ of $\mathcal{X}$. Assume that the continuity holds for the arithmetic divisor $(\mathcal{D} - \mathrm{div}_{\mathcal{X}}(f), g + \log|f|_{\infty})$, i.e., $\widetilde{g} + \log|f|$ is continuous on $\mathcal{X}^{\mathrm{an}} \setminus |\mathcal{D} - \mathrm{div}(f)|^{\mathrm{an}}$, Then $\widetilde{g}$ has the correct logarithmic singularity on $\mathcal{U}^{\mathrm{an}}$. Vary $(\mathcal{U}, f)$ to cover $\mathcal{X}$.

Now we prove the continuity of $\widetilde{g}$ on $\mathcal{X}^{\mathrm{an}}[\mathrm{f}] \setminus |\mathcal{D}|^{\mathrm{an}}$. Let $r : \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ be the specialization map. Let $\{x_m\}_{m \geq 1}$ be a sequence in $\mathcal{X}^{\mathrm{an}}[\mathrm{f}] \setminus |\mathcal{D}|^{\mathrm{an}}$ converging to $x \in \mathcal{X}^{\mathrm{an}}[\mathrm{f}] \setminus |\mathcal{D}|^{\mathrm{an}}$. We need to prove that $\widetilde{g}(x_m)$ converges to $\widetilde{g}(x)$. Let $\mathcal{U}_1, \cdots, \mathcal{U}_n$ be an open cover of $\mathcal{X}$ such that, for any $i = 1, \cdots, n$, $\mathcal{U}_i$ contains $r(x)$ and $\mathcal{D}$ is defined by a single equation f_i on $\mathcal{U}_i$.

To see the existence of the open cover, by quasi-compactness, it suffices to prove that for any point $y \in \mathcal{X}$, there is an open neighborhood of $\{r(x), y\}$ in $\mathcal{X}$ such that $\mathcal{D}$ is principal on $\mathcal{U}$. Note that $\mathcal{D}$ is principal on $\mathcal{U}$ if and only if the line bundle $\mathcal{O}(\mathcal{D})$ is trivial in $\mathrm{Pic}(\mathcal{U})$. We can further assume that $\mathcal{D}$ is very ample, by writing $\mathcal{D}$ as the difference of two very ample Cartier divisors on $\mathcal{X}$. Then there is an embedding $\mathcal{X} \to \mathbb{P}_k^N$ using global sections of $\mathcal{O}(\mathcal{D})$. For any hyperplane $\mathcal{H}$ of $\mathbb{P}_k^N$, the line bundle $\mathcal{O}(\mathcal{D})$ is trivial in $\mathrm{Pic}(\mathcal{X} \setminus \mathcal{H})$, since $\mathcal{O}_{\mathbb{P}_k^N}(1)$ is trivial in $\mathrm{Pic}(\mathbb{P}_k^N \setminus \mathcal{H})$. Now it suffices to choose a hyperplane $\mathcal{H}$ disjoint with $\{r(x), y\}$. This is easy if k is infinite. If k is finite, it is also easy if N is large.

Now we have the open cover $\mathcal{U}_1, \cdots, \mathcal{U}_n$. Denote by I_i the set of $m \geq 1$ such that $r(x_m) \in \mathcal{U}_i$. Then we have $I_1 \cup \cdots \cup I_n = \{1, 2, \cdots\}$. It suffices to prove $\lim_{m \in I_i} \widetilde{g}(x_m) = \widetilde{g}(x)$ for each $i = 1, \cdots, n$. By definition, $r(x) \in \mathcal{U}$ implies that the image of the closed point of $\mathrm{Spec}\, R_x \to \mathcal{X}$ lies in $\mathcal{U}$, where $R_x \subset H_x$ is the valuation ring. This implies that the image of $\mathrm{Spec}\, R_x \to \mathcal{X}$ lies in $\mathcal{U}$, and thus $x \in \mathcal{U}^{\mathrm{an}}$. Thus $\widetilde{g}(x) = -\log|f(x)|$ by definition. Similarly, $m \in I_i$ implies $x_m \in \mathcal{U}^{\mathrm{an}}$ and $\widetilde{g}(x_m) = -\log|f(x_m)|$. It follows that $\lim_{m \in I_i} \widetilde{g}(x_m) = \widetilde{g}(x)$. This proves that $\widetilde{g}$ is continuous on $\mathcal{X}^{\mathrm{an}}[\mathrm{f}] \setminus |\mathcal{D}|^{\mathrm{an}}$.

If $k = \mathbb{Z}$, we need to make extra arguments to extend the continuity of $\widetilde{g}$ from $\mathcal{X}^{\mathrm{an}}[\mathrm{f}] \setminus |\mathcal{D}|^{\mathrm{an}}$ to $\mathcal{X}^{\mathrm{an}} \setminus |\mathcal{D}|^{\mathrm{an}}$. By definition, $\widetilde{g}$ is continuous on $\mathcal{X}^{\mathrm{an}}[\infty] \setminus |\mathcal{D}|^{\mathrm{an}}$. It remains to prove that $\widetilde{g}$ is continuous when $\mathcal{X}^{\mathrm{an}}[\infty] \setminus |\mathcal{D}|^{\mathrm{an}}$ approaches $\mathcal{X}_{v_0}^{\mathrm{an}} \setminus |\mathcal{D}|^{\mathrm{an}}$, where $v_0 \in \mathcal{M}(\mathbb{Z})$ is the trivial norm of $\mathbb{Z}$. Namely, let $\{x_m\}_{m \geq 1}$ be a sequence in $\mathcal{X}^{\mathrm{an}}[\infty] \setminus |\mathcal{D}|^{\mathrm{an}}$ converging to a point x in $\mathcal{X}_{v_0}^{\mathrm{an}} \setminus |\mathcal{D}|^{\mathrm{an}}$. We need to prove that $\widetilde{g}(x_m)$ converges to $\widetilde{g}(x)$.

The canonical homeomorphism $\mathcal{X}^{\mathrm{an}}[\infty] \to \mathcal{X}_\infty^{\mathrm{an}} \times (0, 1]$ induces a projection $\pi : \mathcal{X}^{\mathrm{an}}[\infty] \to \mathcal{X}_\infty^{\mathrm{an}}$. Here $\mathcal{X}_\infty^{\mathrm{an}} = \mathcal{X}_{\mathbb{R}}^{\mathrm{an}} = \mathcal{X}(\mathbb{C})/\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is compact. To prove $\lim_{m \geq 1} \widetilde{g}(x_m) = \widetilde{g}(x)$, by proof by contradiction, it suffices to prove $\lim_{m \in I} \widetilde{g}(x_m) = \widetilde{g}(x)$ for all subsequences I of $\{1, 2, \cdots\}$ such that $\{\pi(x_m)\}_{m \in I}$ converges in $\mathcal{X}_\infty^{\mathrm{an}}$.

For such a subsequence I, denote by $z = \lim_{m \in I} \pi(x_m)$ in $\mathcal{X}_\infty^{\mathrm{an}}$. There is an open neighborhood $\mathcal{U}$ of $\{r(x), r(z)\}$ in $\mathcal{X}$ such that $\mathcal{D}$ is defined by a single equation f on $\mathcal{U}$. The existence of $\mathcal{U}$ has already been proved in the above the non-archimedean case.

Similarly, the condition $r(x), r(z) \in \mathcal{U}$ implies $x, z \in \mathcal{U}^{\mathrm{an}}$. This holds for x as in the above non-archimedean case, and holds for z since $r(z)$ is the image of $\mathrm{Spec}\, H_z \to \mathcal{X}$.

By removing finitely many elements of I, we can assume that x_m lies in

$\mathcal{U}^{\mathrm{an}}$ for every $m \in I$. Then $-\log |f|(x_m)$ for $m \in I$ converges to $-\log |f|(x) = \widetilde{g}(x)$. Denote $h(y) = \widetilde{g}(y) + \log |f|(y)$, as a function on $\mathcal{U}^{\mathrm{an}}$. It suffices to prove $\lim_{m\in I} h(x_m) = 0$.

Note that h is norm-equivariant on $\mathcal{U}^{\mathrm{an}}$. For $m \in I$, denote by t_m the image of x_m under the canonical projection $\mathcal{X}^{\mathrm{an}}[\infty] \to (0,1]_\infty = (0,1]$. We have $\lim_{m\in I} t_m \to 0$. It follows that $\lim_{m\in I} h(x_m) = \lim_{m\in I} t_m\, h(\pi(x_m)) = 0$ as $\lim_{m\in I} \pi(x_m) = z$ in $\mathcal{U}^{\mathrm{an}}_\infty$. This finishes the proof. $\square$

The following effectivity result will be very useful to prove injectivity of the analytification map.

Lemma 3.3.3. *Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{X}$ be a projective variety over k and let $i : X \to \mathcal{X}$ be a pro-open immersion. Let $\overline{\mathcal{D}} = (\mathcal{D}, g)$ be an arithmetic divisor on $\mathcal{X}$, and denote by $\tilde{g}$ the Green's function of $\mathcal{D}$ on $\mathcal{X}^{\mathrm{an}}$ induced by $\overline{\mathcal{D}}$. Assume one of the following two conditions:*

(1) $\mathcal{X}$ is normal;

(2) the scheme $\mathcal{X}$ is integrally closed in X, and the Cartier divisor $\mathcal{D}|_X$ is effective on X.

Then $\overline{\mathcal{D}}$ is effective if and only if $\tilde{g} \geq 0$ on $\mathcal{X}^{\mathrm{an}} \setminus |\mathcal{D}|^{\mathrm{an}}$.

Proof. Note that (1) is a special case of (2) with X equal to the generic point of $\mathcal{X}$, but we list it separately for its independent importance. It suffices to prove the "if" part. Assuming $\tilde{g} \geq 0$, we need to prove that $\mathcal{D}$ is effective. This is an analytic version of Lemma 2.3.6. It suffices to prove that for any $v \in \mathcal{X} \setminus X$ of codimension one in $\mathcal{X}$, the valuation $\mathrm{ord}_v(\mathcal{D})$ in the local ring $\mathcal{O}_{\mathcal{X},v}$ is non-negative. Consider the divisorial point $\xi = \exp(-\mathrm{ord}_v)$ of $\mathcal{X}^{\mathrm{an}}$. Let f be a local equation of $\mathcal{D}$ in an open neighborhood of v in $\mathcal{X}$. By definition,

$$\widetilde{g}(\xi) = -\log |f(\xi)| = -\log(\exp(-\mathrm{ord}_v f)) = \mathrm{ord}_v f = \mathrm{ord}_v(\mathcal{D}).$$

It follows that $\mathrm{ord}_v(\mathcal{D}) \geq 0$. $\square$

3.3.2 Quasi-projective case

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{U}$ be a quasi-projective variety over k, and $\mathcal{X}$ be a projective model of $\mathcal{U}$. The analytification map

$$\widehat{\mathrm{Div}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{X}^{\mathrm{an}})_{\mathrm{eqv}}$$

defined above induces a map

$$\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}},$$

which sends $\overline{\mathcal{D}} = (\mathcal{D}, g_{\mathcal{D}})$ to $\overline{\mathcal{D}}^{\mathrm{an}} := (\mathcal{D}|_{\mathcal{U}}, \tilde{g}_{\mathcal{D}})$. Here $\mathcal{D}|_{\mathcal{U}}$ is the integral part of $\overline{\mathcal{D}}$, which is an integral Cartier divisor on $\mathcal{U}$. By direct limit, the map gives a map

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}} \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}.$$

In the following, we prove that the map can be extended to adelic divisors of quasi-projective varieties by taking limits.

Proof of Proposition 3.3.1: quasi-projective case. We need to define and prove the injectivity of

$$\widehat{\mathrm{Div}}(\mathcal{U}/k) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}},$$

$$\widehat{\mathrm{CaCl}}(\mathcal{U}/k) \longrightarrow \widehat{\mathrm{CaCl}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}.$$

The injectivity of the first map implies that of the second map. In fact, if $\widehat{\mathrm{Div}}(\mathcal{U}/k) \to \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$ is defined and injective, then the map $\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{\mathrm{mod}} \to \widehat{\mathrm{Pr}}(\mathcal{U}^{\mathrm{an}})$ is also injective. Thus $\widehat{\mathrm{Pr}}(\mathcal{U}/k)_{\mathrm{mod}} \to \widehat{\mathrm{Pr}}(\mathcal{U}^{\mathrm{an}})$ is bijective as both groups are quotients of $k(\mathcal{U})^{\times}$. The quotients give a well-defined and injective map $\widehat{\mathrm{CaCl}}(\mathcal{U}/k) \to \widehat{\mathrm{CaCl}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$.

To treat the first map, we will extend the map

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}} \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$$

to a map

$$\widehat{\mathrm{Div}}(\mathcal{U}/k) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$$

by continuity. Recall that the left-hand side is endowed with the boundary topology using $\overline{\mathcal{E}}_0$; similarly, we endow the right-hand side with the boundary topology using the divisor $\overline{\mathcal{E}}_0^{\mathrm{an}}$. Here

$$\overline{\mathcal{E}}_0^{\mathrm{an}} = (\mathcal{E}_0|_U, \tilde{g}_0) = (0, \tilde{g}_0)$$

is the image of $\overline{\mathcal{E}}_0 = (\mathcal{E}_0, g_0)$ in $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$.

Note that that the map $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}} \to \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$ keeps the partial order of effectivity, so it sends Cauchy sequences to Cauchy sequences. To prove that the map is well-defined, it suffices to prove that $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$ is complete under the boundary topology. Let $\{(\mathcal{D}_i, \tilde{g}_i)\}_{i\geq 1}$ be a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$. Then we have $\mathcal{D}_i = \mathcal{D}_1$ for all i, and there is a sequence $\{\epsilon_i\}_i$ of positive rational numbers converging to 0 such that

$$-\epsilon_i \tilde{g}_0 \leq \tilde{g}_i - \tilde{g}_j \leq \epsilon_i \tilde{g}_0, \quad \forall\, j \geq i \geq 1.$$

Note that $\tilde{g}_0$ is continuous on $\mathcal{U}^{\mathrm{an}}$ and thus bounded on any compact subset of $\mathcal{U}^{\mathrm{an}}$. Then $\{\tilde{g}_i - \tilde{g}_1\}_i$ is uniformly convergent (to a continuous function) on any compact subset of $\mathcal{U}^{\mathrm{an}}$. As $\mathcal{U}^{\mathrm{an}}$ is locally compact, the sequence $\{\tilde{g}_i - \tilde{g}_1\}_i$ is pointwise convergent to a continuous function on $\mathcal{U}^{\mathrm{an}}$. Then $\tilde{g}_i = \tilde{g}_1 + (\tilde{g}_i - \tilde{g}_1)$ converges to a Green's function of $\mathcal{D}_1$ on $\mathcal{U}^{\mathrm{an}}$. This gives the limit of $\{(\mathcal{D}_i, \tilde{g}_i)\}_{i\geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$. Therefore, $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$ is complete, and the first map of the proposition is well-defined.

In the definition

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}} = \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U}),$$

we can replace each $\mathcal{X}$ by its normalization in $\mathcal{U}$, so that $\mathcal{X}$ is integrally closed in $\mathcal{U}$. By Lemma 3.3.3, an element of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ is effective if and only if its image in $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$ is effective. As a consequence, for any sequence $\{\overline{\mathcal{D}}_i\}_i$ of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$, if the image of $\{\overline{\mathcal{D}}_i\}_i$ in $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$ is a Cauchy sequence equivalent to 0, then $\{\overline{\mathcal{D}}_i\}_i$ is a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ equivalent to 0. This proves the injectivity. $\square$

3.3.3 Essentially quasi-projective case

Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. Recall

$$\widehat{\mathrm{Div}}(X/k) = \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/k),$$

$$\widehat{\mathrm{CaCl}}(X/k) = \varinjlim_{\mathcal{U}} \widehat{\mathrm{CaCl}}(\mathcal{U}/k).$$

Here the limits are over quasi-projective models $\mathcal{U}$ of X over k.

Note that we have already had an injection

$$\widehat{\mathrm{Div}}(\mathcal{U}/k) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$$

for quasi-projective models $\mathcal{U}$ of X. Its direct limit gives an injection

$$\widehat{\mathrm{Div}}(X/k) \longrightarrow \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}.$$

Composing with the map

$$\varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \longrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}},$$

we get a map

$$\widehat{\mathrm{Div}}(X/k) \longrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}}.$$

This is the map in Proposition 3.3.1. Now we are ready to finish the proof of the proposition.

Proof of Proposition 3.3.1: essentially quasi-projective case. Similar to the quasi-projective case, it suffices to prove the injectivity of

$$\widehat{\mathrm{Div}}(X/k) \longrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}}.$$

By the above composition, it suffices to prove that the map

$$\varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}} \longrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}}$$

is injective. By Lemma 3.1.1, $X^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}}$ is injective with a dense image. Then it suffices to prove that the map

$$\Phi : \varinjlim_{\mathcal{U}} \mathrm{Div}(\mathcal{U}) \longrightarrow \mathrm{Div}(X)$$

is injective.

Fix a quasi-projective model $\mathcal{U}_0$ of X. By Lemma 2.3.4, in the above limits, we can take $\{\mathcal{U}\}$ to be the inverse system of open subschemes of $\mathcal{U}_0$ containing X. If $\mathcal{D}$ is an element in the kernel of Φ, then we can assume that $\mathcal{D}$ lies in $\mathrm{Div}(\mathcal{U})$ for some $\mathcal{U}$. At any point $x \in X$, $\mathcal{D}$ is defined by a single equation f in a neighborhood of x in $\mathcal{U}$. By assumption, f is invertible in $\mathcal{O}_{\mathcal{U},x}$, so f is invertible on a neighborhood $\mathcal{V}_x$ of x in $\mathcal{U}$, or equivalently $\mathcal{D}$ is 0 on $\mathcal{V}_x$. Taking unions of $\mathcal{V}_x$ for all $x \in X$, we see that $\mathcal{D}$ is 0 on an open neighborhood of X in $\mathcal{U}_0$. Thus $\mathcal{D} = 0$. This proves the injectivity of Φ. The proof is complete. $\square$

3.4 Analytification of adelic line bundles

The adelic line bundles in §2.5 induce norm-equivariant metrized line bundles on Berkovich spaces. The goal of this section is to study this analytification process. The main result is as follows:

Proposition 3.4.1. *Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. There is a canonical fully faithful functor*

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{eqv}},$$

which induces an injective map

$$\widehat{\mathrm{Pic}}(X/k) \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathrm{eqv}}.$$

Most of the process is parallel and implied by the analytification of adelic divisors in §3.3. We will include it for the sake of readers. As in the case of adelic divisors, we will construct the maps and prove the injectivity in the order of projective case, quasi-projective case and essentially quasi-projective case. In the end, we will consider the canonical measures on Berkovich spaces induced by this process.

3.4.1 Projective case

Let k be either $\mathbb{Z}$ or a field. Take the uniform terminology in §1.6. Let $\mathcal{X}$ be a projective variety over k. Then there is a canonical functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}^{\mathrm{an}})_{\mathrm{eqv}}$$

and a canonical map

$$\widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{X}^{\mathrm{an}})_{\mathrm{eqv}}.$$

This is very similar to that construction in §3.3, and it is actually a consequence of the latter for choosing a rational section s of a line bundle on $\mathcal{X}$ and convert metrics $\|s\|$ to Green's functions $-\log\|s\|$.

For importance, we sketch the definition here. Let $\overline{\mathcal{L}}$ be a hermitian line bundle on $\mathcal{X}$. We need to define a metric of $\mathcal{L}$ on $\mathcal{X}^{\mathrm{an}}$. The metric of the fibers of $\mathcal{L}$ on $\mathcal{X}^{\mathrm{an}}_{\infty} = \mathcal{X}^{\mathrm{an}}_{\mathbb{R}}$ are given by the original hermitian metric, and it extends to $\mathcal{X}^{\mathrm{an}}[\infty]$ by norm-equivariance. For the metric of $\mathcal{L}$ at a point $x \in \mathcal{X}^{\mathrm{an}}[\mathrm{f}]$, let $\phi_x^{\circ} : \mathrm{Spec}\, R_x \to \mathcal{X}$ be the k-morphism extending the k-morphism $\phi_x : \mathrm{Spec}\, H_x \to \mathcal{X}$ under the valuative criterion. Then $(\phi_x^{\circ})^*\mathcal{L}$

is a free module over R_x of rank 1. Let s_x be a basis of this free module. Define the metric of $\mathcal{L}(x) = \phi_x^*\mathcal{L}$ by setting $\|s_x\| = 1$. The continuity of the metric is a consequence of Lemma 3.3.2.

3.4.2 Quasi-projective case

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{U}$ be a quasi-projective variety over k. We are going to have a canonical functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$$

and a canonical map

$$\widehat{\mathrm{Pic}}(\mathcal{U}/k) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}.$$

The functor is described as follows. Recall from §2.5 that an object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ is a sequence $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$. Resume the other notations for this sequence in §2.5. Note that each $\overline{\mathcal{L}}_i$ induces a metric $\|\cdot\|_i^*$ of $\mathcal{L}_i$ on $\mathcal{X}_i^{\mathrm{an}}$. By the isomorphism $\ell_i : \mathcal{L} \to \mathcal{L}_i|_{\mathcal{U}}$, and by restriction, we get a metric $\|\cdot\|_i$ of $\mathcal{L}$ on $\mathcal{U}^{\mathrm{an}}$. We will see that the Cauchy condition implies that these metrics converge pointwise to a continuous metric $\|\cdot\|$ of $\mathcal{L}$ on $\mathcal{U}^{\mathrm{an}}$. Then $\overline{\mathcal{L}}^{\mathrm{an}} := (\mathcal{L}, \|\cdot\|)$ defines an element of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$, which is the desired image of the functor.

By the above idea, we prove Proposition 3.4.1 for quasi-projective varieties.

Proof of Proposition 3.4.1: quasi-projective case. We need to prove that the above construction gives a functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}},$$

and prove that the functor is fully faithful. This is more or less a consequence of the quasi-projective case of Proposition 3.3.1. We will write some parts of the proof, and convert some other parts to the proposition.

Resume the above notations in the construction of the functor. Let $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ be an object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$. Denote by $\|\cdot\|_i$ the metric of $\mathcal{L}$ on $\mathcal{U}^{\mathrm{an}}$ induced by $(\mathcal{X}_i, \overline{\mathcal{L}}_i)$.

We first check that the metric $\|\cdot\|_i$ converges pointwise to a metric of $\|\cdot\|_i$ of $\mathcal{L}$ on $\mathcal{U}^{\mathrm{an}}$. This is very similar to the quasi-projective case of Proposition 3.3.1. In fact, from §2.5, the Cauchy condition means that there is a sequence $\{\epsilon_j\}_{j\geq 1}$ of positive rational numbers converging to 0 such that in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$,

$$-\epsilon_j\overline{\mathcal{E}}_0 \leq \widehat{\mathrm{div}}(\ell_i\ell_1^{-1}) - \widehat{\mathrm{div}}(\ell_j\ell_1^{-1}) \leq \epsilon_j\overline{\mathcal{E}}_0, \quad i \geq j \geq 1.$$

This implies

$$-\epsilon_j\tilde{g}_0 \leq \log(\|\cdot\|_i/\|\cdot\|_j) \leq \epsilon_j\tilde{g}_0, \quad i \geq j \geq 1.$$

Here $\tilde{g}_0$ is the Green's function of $\mathcal{E}_0$ on $\mathcal{X}_0^{\mathrm{an}}$ induced by $\overline{\mathcal{E}}_0$, which is continuous on $\mathcal{U}^{\mathrm{an}}$. Write $f_i = \log(\|\cdot\|_i/\|\cdot\|_1)$ as a continuous function on $\mathcal{U}^{\mathrm{an}}$. Then the above condition gives

$$-\epsilon_j\tilde{g}_0 \leq f_i - f_j \leq \epsilon_j\tilde{g}_0, \quad i \geq j \geq 1.$$

As in the proof of Proposition 3.3.1, since $\mathcal{U}^{\mathrm{an}}$ is locally compact, f_i converges pointwise to a continuous function f on $\mathcal{U}^{\mathrm{an}}$. As a consequence, $\|\cdot\|_i$ converges pointwise to a continuous metric $\|\cdot\|$, and

$$-\epsilon_j\tilde{g}_0 \leq \log(\|\cdot\|/\|\cdot\|_j) \leq \epsilon_j\tilde{g}_0, \quad j \geq 1.$$

This gives the functor image $\overline{\mathcal{L}}^{\mathrm{an}} = (\mathcal{L}, \|\cdot\|)$.

To check that it is indeed a fully faithful functor, let $\overline{\mathcal{L}}'$ be another object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ with image $\overline{\mathcal{L}}'^{\mathrm{an}}$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$. We need to prove that there is a canonical isomorphism

$$\mathrm{Hom}(\overline{\mathcal{L}}', \overline{\mathcal{L}}) \longrightarrow \mathrm{Hom}(\overline{\mathcal{L}}'^{\mathrm{an}}, \overline{\mathcal{L}}^{\mathrm{an}}).$$

This is equivalent to a canonical isomorphism

$$\mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{X}_0}, \overline{\mathcal{L}}'^{\vee} \otimes \overline{\mathcal{L}}) \longrightarrow \mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{U}}, (\overline{\mathcal{L}}'^{\mathrm{an}})^{\vee} \otimes \overline{\mathcal{L}}^{\mathrm{an}}).$$

Here $\overline{\mathcal{O}}_{\mathcal{X}_0} = (\mathcal{O}_{\mathcal{U}}, (\mathcal{X}_0, \overline{\mathcal{O}}_{\mathcal{X}_0}, 1))$ and $\overline{\mathcal{O}}_{\mathcal{U}} = (\mathcal{O}_{\mathcal{U}}, \|\cdot\|_0)$ are the neural elements, where $\|\cdot\|_0$ is defined by $\|1\|_0 = 1$.

Replacing $\overline{\mathcal{L}}'^{\vee} \otimes \overline{\mathcal{L}}$ by $\overline{\mathcal{L}}$, it suffices to prove that there is a canonical isomorphism

$$\Phi : \mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{X}_0}, \overline{\mathcal{L}}) \longrightarrow \mathrm{Hom}(\overline{\mathcal{O}}_{\mathcal{U}}, \overline{\mathcal{L}}^{\mathrm{an}}).$$

Write $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ as above.

Elements of both sides of Φ are represented by regular sections s of $\mathcal{L}$ everywhere non-vanishing on $\mathcal{U}$. Such a section s gives an element of the right-hand side if $\|s\| = 1$ on $\mathcal{U}^{\mathrm{an}}$. The section s gives an element of the left-hand side if the Cauchy sequence $\{\widehat{\mathrm{div}}(\ell_i\ell_1^{-1})+\widehat{\mathrm{div}}_{(\mathcal{X}_1,\overline{\mathcal{L}}_1)}(s)\}_{i\geq 1}$ of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ converges to 0 in $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ under the boundary topology. These two conditions on s are equivalent, since

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)\longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$$

is injective by Proposition 3.3.1. The proof is complete. $\square$

3.4.3 Essentially quasi-projective case

Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. Recall that

$$\widehat{\mathrm{Pic}}(X/k)=\varinjlim_{\mathcal{U}}\widehat{\mathrm{Pic}}(\mathcal{U}/k),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)=\varinjlim_{\mathcal{U}}\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k).$$

Here the limits are over quasi-projective models $\mathcal{U}$ of X over k. Note that we have a fully faithful functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)\longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}$$

for quasi-projective models $\mathcal{U}$ of X. Its direct limit gives a fully faithful functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)\longrightarrow \varinjlim_{\mathcal{U}}\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}.$$

Composing with the functor

$$\varinjlim_{\mathcal{U}}\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}\longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{eqv}},$$

we get a functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)\longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{eqv}}.$$

This is the functor in Proposition 3.4.1. Now we are ready to finish the proof of the proposition.

Proof of Proposition 3.4.1: essentially quasi-projective case. It suffices to prove that the functor

$$\varinjlim_{\mathcal{U}}\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})_{\mathrm{eqv}}\longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{eqv}}$$

is fully faithful. Similarly, by Lemma 3.1.1, $X^{\mathrm{an}} \to \mathcal{U}^{\mathrm{an}}$ is injective with a dense image, so it suffices to prove that the functor

$$\Psi : \varinjlim_{\mathcal{U}} \mathcal{P}\mathrm{ic}(\mathcal{U}) \longrightarrow \mathcal{P}\mathrm{ic}(X)$$

is fully faithful.

Fix a quasi-projective model $\mathcal{U}_0$ of X. By Lemma 2.3.4, in the above limits, we can take $\{\mathcal{U}\}$ to be the inverse system of open subschemes of $\mathcal{U}_0$ containing X.

To prove that the functor Ψ is fully faithful, it suffices to prove that for any line bundles $\mathcal{L}, \mathcal{L}'$ on some open neighborhood of X in $\mathcal{U}_0$, the canonical map

$$\varinjlim_{\mathcal{U}} \mathrm{Hom}(\mathcal{L}|_{\mathcal{U}}, \mathcal{L}'|_{\mathcal{U}}) \longrightarrow \mathrm{Hom}(\mathcal{L}|_X, \mathcal{L}'|_X)$$

is an isomorphism. The map is isomorphic to

$$\varinjlim_{\mathcal{U}} \Gamma(\mathcal{U}, \mathcal{L}^\vee \otimes \mathcal{L}') \longrightarrow \Gamma(X, \mathcal{L}^\vee \otimes \mathcal{L}').$$

The injectivity is clear as both sides are subgroups of rational sections of $\mathcal{L}^\vee \otimes \mathcal{L}'$ on X. For the surjectivity, it suffices to prove that, if a rational section s of $\mathcal{L}^\vee \otimes \mathcal{L}'$ is regular and nowhere vanishing on X, then it is regular and nowhere vanishing on a neighborhood of X in $\mathcal{U}_0$. In fact, for any $x \in X$, as s is regular and non-vanishing at x, it is so at an open neighborhood $\mathcal{V}_x$ of x in $\mathcal{U}_0$. Take unions of $\mathcal{V}_x$ for all $x \in X$. It gives an open neighborhood of X in $\mathcal{U}_0$ satisfying the requirement. This finishes the proof. $\square$

3.4.4 Consequence on shrinking the underlying scheme

A quick consequence of Proposition 3.3.1 and Proposition 3.4.1 is the following injectivity result.

Corollary 3.4.2. *Let k be either $\mathbb{Z}$ or a field. Let $f : X \to Y$ be a morphism of flat and essentially quasi-projective integral schemes over k. Assume that X and Y are normal, and f induces an isomorphism $k(Y) \to k(X)$ of the function fields. Then the canonical maps*

$$\widehat{\mathrm{Div}}(Y/k) \longrightarrow \widehat{\mathrm{Div}}(X/k),$$

$$\widehat{\mathrm{Pic}}(Y/k) \longrightarrow \widehat{\mathrm{Pic}}(X/k),$$

$$\widehat{\mathrm{Div}}(Y^{\mathrm{an}}) \longrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{an}}),$$
$$\widehat{\mathrm{Pic}}(Y^{\mathrm{an}}) \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}})$$

are injective.

Proof. By Proposition 3.3.1 and Proposition 3.4.1, it suffices to prove the last two maps are injective.

We claim that the injectivity of the fourth map is implied by that of the third map. In fact, by the isomorphism between $\widehat{\mathrm{Pic}}$ and $\widehat{\mathrm{CaCl}}$, the fourth map is isomorphic to the canonical map

$$\widehat{\mathrm{CaCl}}(Y^{\mathrm{an}}) \longrightarrow \widehat{\mathrm{CaCl}}(X^{\mathrm{an}}).$$

As $k(Y) \to k(X)$ is an isomorphism, the canonical map

$$\widehat{\mathrm{Pr}}(Y^{\mathrm{an}}) \longrightarrow \widehat{\mathrm{Pr}}(X^{\mathrm{an}})$$

is surjective. Then the injectivity of the third map implies that of the fourth.

Now we prove the injectivity of the third map

$$\widehat{\mathrm{Div}}(Y^{\mathrm{an}}) \longrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{an}}).$$

Assume that an arithmetic divisor (D, g_D) on Y^{an} lies in the kernel of this map. Then g_D is zero on $(\mathrm{Spec}\, k(Y))^{\mathrm{an}}$. By Lemma 3.1.1, g_D is zero on Y^{an}. Note that g_D has logarithmic singularity along $|D|^{\mathrm{an}}$ in Y^{an}. This implies that $|D|^{\mathrm{an}}$ is empty, and thus $D = 0$. It finishes the proof. □

3.5 Restricted analytic spaces

Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. The Berkovich space $X^{\mathrm{an}} = (X/k)^{\mathrm{an}}$ is intrinsic and functorial. Moreover, the analytification map

$$\widehat{\mathrm{Pic}}(X/k) \longrightarrow \widehat{\mathrm{Pic}}((X/k)^{\mathrm{an}})_{\mathrm{eqv}}$$

is functorial in the sense that it is compatible with the functoriality maps listed in §2.5.5.

However, a disadvantage is that the space is too large and too abstract to work on, mainly because it contains "too many" redundant points. The goal here is to consider a smaller subspace of $(X/k)^{\mathrm{an}}$, as the union of some distinguished fibers, which is sufficient for many applications. In this new setting, metrized line bundles are closer to the adelic line bundles of Zhang [Zha2] for projective varieties over number fields (cf. Proposition 3.5.2). We will first write the arithmetic case, and give a sketch for the geometric case.

3.5.1 Arithmetic case

Recall that $(\operatorname{Spec}\mathbb{Z})^{\mathrm{an}} = \mathcal{M}(\mathbb{Z})$ is the set of all multiplicative semi-norms of $\mathbb{Z}$. Define $(\operatorname{Spec}\mathbb{Z})^{\text{r-an}}$ to be the subspace of $(\operatorname{Spec}\mathbb{Z})^{\mathrm{an}}$ of non-trivial standard absolute values $|\cdot|_v$ of $\mathbb{Z}$. Hence $(\operatorname{Spec}\mathbb{Z})^{\text{r-an}}$ is bijective to the set $\{\infty, 2, 3, 5, 7, \cdots\}$ and endowed with the discrete topology.

Let X be a scheme over $\mathbb{Z}$. There is a structure map $X^{\mathrm{an}} \to (\operatorname{Spec}\mathbb{Z})^{\mathrm{an}}$. Define the *restricted analytic space* $X^{\text{r-an}} = (X/\mathbb{Z})^{\text{r-an}}$ *associated to* $X/\mathbb{Z}$ to be the preimage of $(\operatorname{Spec}\mathbb{Z})^{\text{r-an}}$ under the map $X^{\mathrm{an}} \to (\operatorname{Spec}\mathbb{Z})^{\mathrm{an}}$. It follows that

$$X^{\text{r-an}} = \coprod_{v\in(\operatorname{Spec}\mathbb{Z})^{\text{r-an}}} X_v^{\mathrm{an}},$$

where X_v^{an} is the fiber of X^{an} above v. Then X_v^{an} is canonically homeomorphic to $X_{\mathbb{Q}_v}^{\mathrm{an}} = (X_{\mathbb{Q}_v}/\mathbb{Q}_v)^{\mathrm{an}}$, the Berkovich space associated to $X_{\mathbb{Q}_v}$ over the complete field $\mathbb{Q}_v$. The topology on $X^{\text{r-an}}$ is induced by the disjoint union, so that each X_v^{an} is both open and closed in $X^{\text{r-an}}$.

Define an *arithmetic divisor* on $X^{\text{r-an}}$ to be a pair (D, g_D), where D is a Cartier divisor on X, and g_D is a *Green's function* of D on $X^{\text{r-an}}$, i.e. a continuous function $g : X^{\text{r-an}} \setminus |D|^{\text{r-an}} \to \mathbb{R}$ with logarithmic singularity along D in the sense that, for any rational function f on a Zariski open subset U of X satisfying $\operatorname{div}(f) = D|_U$, the function $g + \log|f|$ can be extended to a continuous function on $U^{\text{r-an}}$.

An arithmetic divisor on $X^{\text{r-an}}$ is called *principal* if it is of the form

$$\widehat{\operatorname{div}}_{X^{\text{r-an}}}(f) := (\operatorname{div}(f), -\log|f|)$$

for some nonzero rational function f on X.

Denote by $\widehat{\operatorname{Div}}(X^{\text{r-an}})$ (resp. $\widehat{\operatorname{Pr}}(X^{\text{r-an}})$) the *group* of arithmetic divisors (resp. principal arithmetic divisor) on X. Define

$$\widehat{\operatorname{CaCl}}(X^{\text{r-an}}) := \widehat{\operatorname{Div}}(X^{\text{r-an}})/\widehat{\operatorname{Pr}}(X^{\text{r-an}}).$$

Define a *metrized line bundle* on $X^{\text{r-an}}$ to be a pair $(L, \|\cdot\|)$, where L is a line bundle on X, and $\|\cdot\|$ is a continuous metric of fibers of L on $X^{\text{r-an}}$. This is similar to the original case, we omit the details.

Denote by $\widehat{\operatorname{Pic}}(X^{\text{r-an}})$ the *group* of isometry classes of metrized line bundle on X, and $\widehat{\mathcal{P}\mathrm{ic}}(X^{\text{r-an}})$ the *category* of metrized line bundle on X whose morphisms are isometries. There is a canonical isomorphism

$$\widehat{\operatorname{CaCl}}(X^{\text{r-an}}) \longrightarrow \widehat{\operatorname{Pic}}(X^{\text{r-an}}).$$

Finally, the motivation for restricted analytic spaces is as follows.

Proposition 3.5.1. *Let X be a flat and essentially quasi-projective integral scheme over $\mathbb{Z}$. There are canonical injective maps*

$$\widehat{\mathrm{Div}}(X) \longrightarrow \widehat{\mathrm{Div}}(X^{r\text{-}an}),$$

$$\widehat{\mathrm{Pic}}(X) \longrightarrow \widehat{\mathrm{Pic}}(X^{r\text{-}an}),$$

and a canonical fully faithful functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{r\text{-}an}).$$

Proof. This is a consequence of Proposition 3.3.1 and Proposition 3.4.1. For example, the first map is obtained as the composition

$$\widehat{\mathrm{Div}}(X) \longrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathrm{eqv}} \longrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{r\text{-}an}}).$$

Here the first arrow is injective by Proposition 3.3.1. The second arrow is injective, since a norm-equivalent Green's function on X^{an} is determined by its restriction to $X^{\mathrm{an}} \setminus X^{\mathrm{an}}_{\iota(\mathrm{Spec}\,\mathbb{Z})}$ by Lemma 3.1.1, and the latter is determined by its restriction to $X^{\mathrm{r\text{-}an}}$ by norm-equivariance. □

3.5.2 Comparison with the definition of [Zha2]

Let X be a projective variety over $\mathbb{Q}$, which is not assumed to be geometrically integral.

The *coherence condition* of an adelic line bundle $\overline{L}$ on $X^{\mathrm{r\text{-}an}}$ is the existence of an open subscheme $\mathcal{V}$ of $\mathrm{Spec}\,\mathbb{Z}$, a projective and flat morphism $\mathcal{U} \to \mathcal{V}$ whose generic fiber is isomorphic to $X \to \mathrm{Spec}\,\mathbb{Q}$, and a line bundle $\mathcal{L}$ on $\mathcal{U}$ endowed with an isomorphism $\mathcal{L}_{\mathbb{Q}} \to L$ over X, such that the metric of $\overline{L}$ on $X_p^{\mathrm{an}} \subset X^{\mathrm{r\text{-}an}}$ is equal to the metric of L on X_p^{an} induced by $\mathcal{L}$ for all primes $p \in \mathcal{V}$.

Denote by $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{r\text{-}an}})_{\mathrm{coh}}$ the full subcategory of $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{r\text{-}an}})$ whose objects are adelic line bundles on X satisfying the coherence condition.

Note that the coherence condition is part of the definition of the adelic line bundle in [Zha2, (1.2)]. Objects of $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{r\text{-}an}})_{\mathrm{coh}}$ is very close to the adelic line bundles in the loc. cit., except that the loc. cit. uses metrics on algebraic points of X over local fields instead of Berkovich spaces.

Proposition 3.5.2. *Let X be a projective variety over $\mathbb{Q}$. Then the functor*

$$\widehat{\mathcal{P}\mathrm{ic}}(X) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{r\text{-}an})$$

induces an equivalence

$$\widehat{\mathcal{P}\mathrm{ic}}(X) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{r\text{-}an})_{\mathrm{coh}}$$

of categories.

Proof. Note that the first functor is fully faithful, so it suffices to prove that its essential image is $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{r\text{-}an}})_{\mathrm{coh}}$. By definition,

$$\widehat{\mathcal{P}\mathrm{ic}}(X) = \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}),$$

where the limit is over quasi-projective models $\mathcal{U}$ of X. Replacing $\mathcal{U}$ by an open subscheme if necessary, we can assume that there is a projective and flat morphism $\mathcal{U} \to \mathcal{V}$ for some open subscheme $\mathcal{V}$ of $\mathrm{Spec}\,\mathbb{Z}$. It is reduced to treat the functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{r\text{-}an}}).$$

Let $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ be an adelic line bundle on $\mathcal{U}$, with underlying line bundle $\mathcal{L}$ on $\mathcal{U}$. Denote by $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ the image of $\overline{\mathcal{L}}$ in $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{r\text{-}an}})$, with underlying line bundle $L = \mathcal{L}|_X$ on X. Denote by $\|\cdot\|_{i,v}$ (resp. $\|\cdot\|_v^\circ$) the metric of L on X_v^{an} induced by $(\mathcal{X}_i, \overline{\mathcal{L}}_i)$ (resp. $(\mathcal{U}, \mathcal{L})$) for $v \leq \infty$ (resp. $v \in \mathcal{V}$). By definition, $\|\cdot\|_v$ is the pointwise limit of $\|\cdot\|_{i,v}$ on X_v^{an}.

As the base change of $(\mathcal{X}_i, \overline{\mathcal{L}}_i)$ to $\mathcal{V}$ is isomorphic to $(\mathcal{U}, \mathcal{L})$, they induce the same metrics of L at any closed point $v \in \mathcal{V}$. It follows that $\|\cdot\|_{i,v} = \|\cdot\|_v^\circ = \|\cdot\|_v$ for $v \in \mathcal{V}$. This proves that the metric of $\overline{L}$ satisfies the coherence condition over $\mathcal{V}$.

We claim that the convergence of $\|\cdot\|_{i,v}$ to $\|\cdot\|_v$ is the uniform convergence on X_v^{an} for any place $v \leq \infty$. In fact, take any projective model $\mathcal{X}_0$ of $\mathcal{U}$. Take the arithmetic divisor $\overline{\mathcal{E}}_0 = (\mathcal{E}_0, 1)$ over $\mathrm{Spec}\,\mathbb{Z}$, where $\mathcal{E}_0 = (\mathrm{Spec}\,\mathbb{Z}) \setminus \mathcal{V}$ is endowed with the reduced structure. Take $\overline{\mathcal{F}}_0$ to be the pull-back of $\overline{\mathcal{E}}_0$ to $\mathcal{X}_0$. Use $\overline{\mathcal{F}}_0$ to define the boundary topology of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}}$. There is also a boundary topology of $\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{r\text{-}an}})$ defined by $\overline{\mathcal{F}}_0^{\mathrm{an}}$. Note that the Green's function $\widetilde{g}$ of $\overline{\mathcal{F}}_0^{\mathrm{an}}$ is 0 on X_v^{an} for any $v \in \mathcal{V}$ and a positive constant on X_v^{an} for $v \notin \mathcal{V}$ (including $v = \infty$). As a consequence, the convergence of $\|\cdot\|_{i,v}$ to $\|\cdot\|_v$ is the uniform convergence.

Conversely, given any metrized line bundle $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ on $X^{\text{r-an}}$ satisfying the coherence condition over an open subscheme $\mathcal{V}$ of $\operatorname{Spec}\mathbb{Z}$, we will prove that $\overline{L}$ is isomorphic to the image of some adelic line bundle $\overline{\mathcal{L}} = \{(\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)\}_{i\geq 1}$ on some quasi-projective model $\mathcal{U}$ of X. In fact, by the coherence condition, there is a model $(\mathcal{U} \to \mathcal{V}, \mathcal{L})$ of $(X \to \operatorname{Spec}\mathbb{Q}, L)$ inducing the metric of $\overline{L}$ above $\mathcal{V}$. For any $v \notin \mathcal{V}$, the metric $\|\cdot\|_v$ is continuous. By a theorem of Gubler (cf. [Gub3, Thm. 7.12] and [Yua1, Lem. 3.5]), $\|\cdot\|_v$ is a uniform limit of model metrics $\{\|\cdot\|_{i,v}\}_{i\geq 1}$ if v is finite. For any $i \geq 1$, we can find an integral model $(\mathcal{X}_i, \overline{\mathcal{L}}_i)$ of (X, L) which extends $(\mathcal{U}, \mathcal{L})$ and induces the metric $\{\|\cdot\|_{i,v}\}_v$ of L. This gives the adelic line bundle $\overline{\mathcal{L}}$. It finishes the proof. $\square$

Let X be a projective variety over $\mathbb{Q}$. One can also check that our notion of nef adelic line bundles on $X/\mathbb{Z}$ agrees with our notion of strongly nef adelic line bundles on $X/\mathbb{Z}$. They also agree with the notion of "semipositive" in [Zha2].

3.5.3 Function field case

Let k be any field. Let X be a scheme over k. Recall that the analytic space $X^{\text{an}} = (X/k)^{\text{an}}$ is (Zariski locally) given by multiplicative semi-norms trivial over k. To define a restricted subspace of X^{an} as in the arithmetic case, we need an extra data to get a global field. This fits the setting at the end of §2.7.

Let B be a projective regular curve over k. Denote by $K = k(B)$ the function field. Any closed point $v \in B$ gives a normalized absolute value $|\cdot|_v = \exp(-\operatorname{ord}_v)$ of K. Define $B^{\text{r-an}} = (B/k)^{\text{r-an}}$ to be the subspace of $B^{\text{an}} = (B/k)^{\text{an}}$ of non-trivial normalized absolute values of K. Therefore, $B^{\text{r-an}}$ is bijective to the set of closed points of B and endowed with the discrete topology.

Let X be a scheme over B (instead of just over k). There is a natural map $X^{\text{an}} \to B^{\text{an}}$. Define the *restricted analytic space*

$$X^{r\text{-}an} = (X/B)^{r\text{-}an} = (X/B/k)^{r\text{-}an}$$

associated to $X/B/k$ to be the preimage of $B^{\text{r-an}}$ under the map $X^{\text{an}} \to B^{\text{an}}$. It follows that

$$X^{\text{r-an}} = \coprod_{v\in B^{\text{r-an}}} X_v^{\text{an}},$$

where X_v^{an} is the fiber of X^{an} above v. Then X_v^{an} is canonically homeomorphic to $X_{K_v}^{\mathrm{an}} = (X_{K_v}/K_v)^{\mathrm{an}}$, the Berkovich space associated to X_{K_v} over the complete field K_v. The topology on $X^{\text{r-an}}$ is induced by the disjoint union, so that each X_v^{an} is both open and closed in $X^{\text{r-an}}$.

Similar to the arithmetic case, we can define arithmetic divisors and metrized line bundles over $X^{\text{r-an}}$. Then Proposition 3.5.1 also holds for any flat and essentially quasi-projective integral scheme X over B. More precisely, there are canonical injective homomorphisms

$$\widehat{\mathrm{Div}}(X/k) \longrightarrow \widehat{\mathrm{Div}}(X^{\text{r-an}}),$$

$$\widehat{\mathrm{Pic}}(X/k) \longrightarrow \widehat{\mathrm{Pic}}(X^{\text{r-an}}),$$

and a canonical fully faithful functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\text{r-an}}).$$

Recall that from §2.7, we also have canonical isomorphisms

$$\widehat{\mathrm{Div}}(X/B) \longrightarrow \widehat{\mathrm{Div}}(X/k),$$

$$\widehat{\mathrm{Pic}}(X/B) \longrightarrow \widehat{\mathrm{Pic}}(X/k),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X/B) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X/k).$$

In the end, we remark that the theory depends on the structure $X/B/k$. In general, if we are only given X/k, then we may use some geometric operations to construct the curve B in the middle. For example, if X is quasi-projective over k of dimension at least 2, take $K = k(t)$ for some transcendental element $t \in k(X)$, which gives a rational map $X \dashrightarrow B$ with $B = \mathbb{P}_k^1$, and then blow-up X to get a morphism $X \to B$.

3.6 Local theory

Let X be a quasi-projective variety over $\mathbb{Q}$. We want to know more about the essential image of the functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\text{r-an}}).$$

If X is projective, Proposition 3.5.2 gives a satisfactory answer. If X is not projective, such a task might be impossible, but the situation simplifies when restricted to fibers of $X^{\text{r-an}}$, i.e., considering the essential image of

$$\widehat{\mathcal{P}\mathrm{ic}}(X) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X_v^{\mathrm{an}})$$

for any place v of $\mathbb{Q}$. This is the motivation of the theory in this subsection.

In this subsection, we will first study the completion process of adelic divisors on Berkovich spaces over complete fields, and introduce the Chambert-Loir measure in this situation.

3.6.1 The analytification functor

Let K be a field complete with respect to a non-trivial valuation $|\cdot|$. If K is non-archimedean, denote by O_K the valuation ring of K. If K is archimedean ($K = \mathbb{C}$ or $\mathbb{R}$), write $O_K = K$ for convenience.

Let U be a quasi-projective variety over K. By §2.7, we have introduced the groups

$$\widehat{\mathrm{Div}}(U/\overline{B}), \quad \widehat{\mathrm{CaCl}}(U/\overline{B}), \quad \widehat{\mathcal{P}\mathrm{ic}}(U/\overline{B}), \quad \widehat{\mathrm{Pic}}(U/\overline{B}).$$

Here we understand that the base valued scheme $\overline{B}$ to be $\mathrm{Spec}\, O_K$ in the non-archimedean case, and to be $(\mathrm{Spec}\, \mathbb{R}, i_{\mathrm{st}})$ or $(\mathrm{Spec}\, \mathbb{C}, \mathrm{id})$ in the archimedean case. By abuse of notations, we will write the groups uniformly by

$$\widehat{\mathrm{Div}}(U/O_K), \quad \widehat{\mathrm{CaCl}}(U/O_K), \quad \widehat{\mathcal{P}\mathrm{ic}}(U/O_K), \quad \widehat{\mathrm{Pic}}(U/O_K).$$

Let U^{an} be the (usual) Berkovich analytic space associated to U over K. As in §3.2, an *arithmetic divisor* on U^{an} is pair $\overline{D} = (D, g_D)$, where D is a Cartier divisor on U, and $g_D : U^{\mathrm{an}} \setminus |D|^{\mathrm{an}} \to \mathbb{R}$ is a continuous Green's function of D on U^{an}. Similarly, a *metrized line bundle* on U^{an} is pair $\overline{L} = (L, \|\cdot\|)$, where L is a line bundle on U, and $\|\cdot\|$ is a continuous metric of L on U^{an}.

Therefore, we have the following groups

$$\widehat{\mathrm{Div}}(U^{\mathrm{an}}), \quad \widehat{\mathrm{Pr}}(U^{\mathrm{an}}), \quad \widehat{\mathrm{CaCl}}(U^{\mathrm{an}}), \quad \widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}}), \quad \widehat{\mathrm{Pic}}(U^{\mathrm{an}}).$$

Here $\widehat{\mathrm{Div}}(U^{\mathrm{an}})$ (resp. $\widehat{\mathrm{Pr}}(U^{\mathrm{an}})$) is the group of arithmetic divisors (resp. principal arithmetic divisors) on U^{an}. And $\widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}})$ (resp. $\widehat{\mathrm{Pic}}(U^{\mathrm{an}})$) is the category (resp. group) of metrized line bundles on U^{an} under isometry. The group

$$\widehat{\mathrm{CaCl}}(U^{\mathrm{an}}) := \widehat{\mathrm{Div}}(U^{\mathrm{an}})/\widehat{\mathrm{Pr}}(U^{\mathrm{an}})$$

is canonically isomorphic to $\widehat{\mathrm{Pic}}(U^{\mathrm{an}})$.

The local counterparts of Proposition 3.3.1 and Proposition 3.4.1 are as follows. The proof is similar to and easier than the global case, so we omit them.

Proposition 3.6.1. *Let K be a field complete with respect to a non-trivial valuation. Let U be a quasi-projective variety over K. There are canonical injective maps*

$$\widehat{\mathrm{Div}}(U/O_K) \longrightarrow \widehat{\mathrm{Div}}(U^{\mathrm{an}}),$$
$$\widehat{\mathrm{CaCl}}(U/O_K) \longrightarrow \widehat{\mathrm{CaCl}}(U^{\mathrm{an}}),$$
$$\widehat{\mathcal{P}\mathrm{ic}}(U/O_K) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}}),$$
$$\widehat{\mathrm{Pic}}(U/O_K) \longrightarrow \widehat{\mathrm{Pic}}(U^{\mathrm{an}}).$$

In the following, we are going to study the images of the analytification functors. Denote

$$\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}} = \mathrm{Im}(\widehat{\mathrm{Div}}(U/O_K) \to \widehat{\mathrm{Div}}(U^{\mathrm{an}})),$$
$$\widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}})_{\mathrm{cptf}} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(U/O_K) \to \widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}})),$$
$$\widehat{\mathrm{Pic}}(U^{\mathrm{an}})_{\mathrm{cptf}} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(U/O_K) \to \widehat{\mathrm{Pic}}(U^{\mathrm{an}})).$$

They are compactifications of

$$\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{mod}} = \mathrm{Im}(\widehat{\mathrm{Div}}(U/O_K)_{\mathrm{mod}} \to \widehat{\mathrm{Div}}(U^{\mathrm{an}})),$$
$$\widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}})_{\mathrm{mod}} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(U/O_K)_{\mathrm{mod}} \to \widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}})),$$
$$\widehat{\mathrm{Pic}}(U^{\mathrm{an}})_{\mathrm{mod}} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(U/O_K)_{\mathrm{mod}} \to \widehat{\mathrm{Pic}}(U^{\mathrm{an}})).$$

We will first describe the compactification process directly on U^{an}.

3.6.2 Compactified arithmetic divisors

Let K be a field complete with respect to a non-trivial valuation $|\cdot|$. Let U be a quasi-projective variety over K. Recall that *projective model* of U over O_K is a flat and projective integral scheme $\mathcal{X}$ over O_K together with an open immersion $U \to \mathcal{X}_K$.

If K is non-archimedean, an *arithmetic model* (or *integral model*) of a Cartier divisor D of U is a pair $(\mathcal{X}, \mathcal{D})$, where $\mathcal{X}$ is a projective model of U over O_K, and $\mathcal{D}$ is a Cartier $\mathbb{Q}$-divisor on $\mathcal{X}$ extending D in that $\mathcal{D}$ and D have the same image in $\mathrm{Div}(U)_{\mathbb{Q}}$.

If K is archimedean, an *arithmetic model* of a Cartier divisor D of U is a pair $(\mathcal{X}, \mathcal{D})$, where $\mathcal{X}$ is a projective model of U over $O_K = K$, and

$\mathcal{D} = (\widetilde{D}, g_{\widetilde{D}})$ consisting of a $\mathbb{Q}$-divisor $\widetilde{D}$ on $\mathcal{X}$ extending D and a continuous Green's function $g_{\widetilde{D}} : \mathcal{X}^{\mathrm{an}} \setminus |\widetilde{D}|^{\mathrm{an}} \to \mathbb{R}$ of $\widetilde{D}$ on $\mathcal{X}^{\mathrm{an}}$.

In both cases, the arithmetic model $(\mathcal{X}, \mathcal{D})$ induces a Green's function $g_{\mathcal{D}}$ of $\mathcal{D}_K$ on $\mathcal{X}_K^{\mathrm{an}}$, and thus a Green's function $g_{\mathcal{D}}|_{U^{\mathrm{an}}}$ of D on U^{an} by restriction. The process is essentially the same as the global case described in §3.3. The Green's function $g_{\mathcal{D}}|_{U^{\mathrm{an}}}$ is called a *model Green's function*, and the arithmetic divisor $(D, g_{\mathcal{D}}|_{U^{\mathrm{an}}})$ is called a *model arithmetic divisor* on U^{an}. The model Green's function or the model arithmetic divisor is called *nef* (or *semipositive*) if either $\mathcal{D}$ is nef on $\mathcal{X}$ in the non-archimedean case, or $\mathcal{D}$ has a semipositive Chern current on $\mathcal{X}(\mathbb{C})$ in the archimedean case (cf. §2.1).

By definition, the image

$$\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{mod}} = \mathrm{Im}(\widehat{\mathrm{Div}}(U/O_K)_{\mathrm{mod}} \to \widehat{\mathrm{Div}}(U^{\mathrm{an}}))$$

is the group of all model arithmetic divisors on U^{an}. It is a natural subgroup of $\widehat{\mathrm{Div}}(U^{\mathrm{an}})$. Now we endow it with a boundary topology as in §2.4.

By a *boundary divisor* of U over O_K, we mean an arithmetic model $(\mathcal{X}_0, \mathcal{E}_0)$ over O_K of the divisor 0 on U such that the support of $\mathcal{E}_{0,K}$ on $\mathcal{X}_{0,K}$ is exactly $\mathcal{X}_{0,K} \setminus U$, and such that the induced Green's function $g_{\mathcal{E}_0} > 0$ on $\mathcal{X}_{0,K}^{\mathrm{an}}$. Then $(\mathcal{X}_0, \mathcal{E}_0)$ induces an arithmetic divisor $\overline{E}_0 = (0, g_0)$ on U^{an}. Here $g_0 = g_{\mathcal{E}_0}|_{U^{\mathrm{an}}}$ is a continuous function on U^{an}. Moreover, $g_{\mathcal{E}_0}$ has a strictly positive lower bound on X_0^{an}. We call $\overline{E}_0 = (0, g_0)$ a *boundary divisor* of U^{an}.

Now have a *boundary norm*

$$\| \cdot \|_{\overline{E}_0} : \widehat{\mathrm{Div}}(U^{\mathrm{an}}) \longrightarrow [0, \infty]$$

defined by

$$\|\overline{D}\|_{\overline{E}_0} := \inf\{\epsilon \in \mathbb{Q}_{>0} : \ -\epsilon\overline{E}_0 \le \overline{D} \le \epsilon\overline{E}_0\}.$$

Here we take the convention that $\inf(\emptyset) = \infty$. Then $\| \cdot \|_{\overline{E}_0}$ is an extended norm. Now we have a *boundary topology on* $\widehat{\mathrm{Div}}(U^{\mathrm{an}})$ induced by the boundary norm, for which a neighborhood basis at 0 is formed by

$$B(\epsilon, \widehat{\mathrm{Div}}(U^{\mathrm{an}})) := \{\overline{D} \in \widehat{\mathrm{Div}}(U^{\mathrm{an}}) : \ -\epsilon\overline{E}_0 \le \overline{D} \le \epsilon\overline{E}_0\}, \quad \epsilon \in \mathbb{Q}_{>0}.$$

Here "$\le$" is still given by effectivity. By translation, it gives a neighborhood basis at any point. The topology does not depend on the choice of $\overline{E}_0$.

By a similar method, we have a boundary topology over $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{mod}}$, which is the same as the subspace topology induced from $\widehat{\mathrm{Div}}(U^{\mathrm{an}})$.

By construction,

$$\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}} = \mathrm{Im}(\widehat{\mathrm{Div}}(U/O_K) \to \widehat{\mathrm{Div}}(U^{\mathrm{an}}))$$

is equal to the completion of $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{mod}}$ with respect to the boundary topology. An element of $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}}$ is called a *compactified divisor* on U^{an}. An compactified divisor or its Green's function is called *strongly nef* (or *strongly semipositive*) if it is a limit of nef model arithmetic divisors under the boundary topology. A compactified divisor $\overline{D}$ of $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}}$ or its Green's function is called *nef* (or *semipositive*) if there exists a strongly nef element $\overline{D}_0$ of $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}}$ such that $a\overline{D} + \overline{D}_0$ is strongly nef for all positive integers a.

Lemma 3.6.2. *The space $\widehat{\mathrm{Div}}(U^{\mathrm{an}})$ is complete with respect to the boundary topology, and contains $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}}$ as a subspace.*

Proof. The second statement is by definition, while the first statement is similar to and easier than Lemma 3.6.3 below. We omit the proof. □

3.6.3 Singularity of Green's functions

It turns out that there is a surprisingly explicit description of $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}}$, which is determined by the space of Green's functions in it. For that purpose, we start with the following spaces of real-valued functions on U^{an}.

(1) $C(U^{\mathrm{an}})$ denotes the space of real-valued continuous functions on U^{an};

(2) $G(U^{\mathrm{an}})$ denotes the space of Green's functions on U^{an} associated to Cartier divisors of U.

By definition, there is a natural injection

$$\widehat{\mathrm{Div}}(U^{\mathrm{an}}) \longrightarrow \mathrm{Div}(U) \oplus G(U^{\mathrm{an}})$$

and a canonical exact sequence

$$0 \longrightarrow C(U^{\mathrm{an}}) \longrightarrow \widehat{\mathrm{Div}}(U^{\mathrm{an}}) \longrightarrow \mathrm{Div}(U) \longrightarrow 0.$$

In terms of the boundary divisor $(0, g_0)$ with $g_0 = g_{\mathcal{E}_0}|_{U^{\mathrm{an}}}$, we have a boundary topology on $C(U^{\mathrm{an}})$ and $G(U^{\mathrm{an}})$. The topologies are compatible under inclusion. For example, the *boundary topology on* $G(U^{\mathrm{an}})$ is induced by the *boundary norm on* $G(U^{\mathrm{an}})$ given by

$$\|g\|_{g_0} := \|g/g_0\|_{\sup} = \sup\{|g(x)/g_0(x)| : x \in U^{\mathrm{an}}\}.$$

Then a neighborhood basis at 0 to be formed by

$$B(\epsilon, G(U^{\mathrm{an}})) := \{g \in G(U^{\mathrm{an}}) : \ -\epsilon g_0 < g < \epsilon g_0\}, \quad \epsilon \in \mathbb{Q}_{>0}.$$

Here the inequalities are understood to hold pointwise away from the loci of the logarithmic singularities. By translation, it gives a neighborhood basis at any point.

We have the following basic result, which is essentially contained in our previous treatments.

Lemma 3.6.3. *The space* $C(U^{\mathrm{an}})$ *is complete with respect to the boundary topology. If* U *is normal, the space* $G(U^{\mathrm{an}})$ *is complete with respect to the boundary topology.*

Proof. This is similar to the quasi-projective model case of the proof of Proposition 3.3.1. We only treat $G(U^{\mathrm{an}})$, as $C(U^{\mathrm{an}})$ is similar. In fact, let $\{f_i\}_{i\geq 1}$ be a Cauchy sequence in $G(U^{\mathrm{an}})$. Then there is a sequence $\{\epsilon_j\}_{j\geq 1}$ of positive rational numbers converging to 0 such that

$$-\epsilon_j g_0 < f_i - f_j < \epsilon_j g_0, \quad i \geq j \geq 1.$$

Note that g_0 is continuous on U^{an}, so $h_i = f_i - f_1$ is bounded on any compact subset of U^{an}. Note that $h_i = f_i - f_1$ has logarithmic singularity along a Cartier divisor D_i on U. By assumption, U is normal and we can view D_i as a Weil divisor on U. The boundedness of h_i implies $D_i = 0$, and thus implies that h_i is actually continuous on U^{an}.

Hence, $\{h_i\}_{i\geq 1}$ is a sequence of continuous function on U^{an}. By the boundary norm, $\{h_i/g_0\}_{i\geq 1}$ is uniformly convergent, and thus the limit is a continuous function. Then $\{h_i\}_{i\geq 1}$ pointwise converges to a continuous function h on U^{an}. Then $f_1 + h$ is the limit of $\{f_i\}_{i\geq 1}$ in $G(U^{\mathrm{an}})$. □

In order to study $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}}$, we introduce the following spaces:

(3) $C(U^{\mathrm{an}})_{\mathrm{mod}}$ denotes the space of *model functions* on U^{an}, i.e., model Green's functions induced by a pair $(\mathcal{X}, \mathcal{D})$, where $\mathcal{X}$ is a projective model of X over O_K, and $\mathcal{D}$ is an arithmetic $\mathbb{Q}$-divisor on $\mathcal{X}$ such that the generic fiber $\mathcal{D}_K = 0$ on $\mathcal{X}_K$ (instead of just on U);

(4) $G(U^{\mathrm{an}})_{\mathrm{mod}}$ denotes the space of model Green's functions on U^{an} associated to Cartier divisors of U.

(5) $C(U^{\mathrm{an}})_{\mathrm{cptf}}$ denotes the completion of $C(U^{\mathrm{an}})_{\mathrm{mod}}$ with respect to the boundary topology;

(6) $G(U^{\mathrm{an}})_{\mathrm{cptf}}$ denotes the completion of $G(U^{\mathrm{an}})_{\mathrm{mod}}$ with respect to the boundary topology.

As $C(U^{\mathrm{an}})$ and $G(U^{\mathrm{an}})$ are complete, we have inclusions

$$C(U^{\mathrm{an}})_{\mathrm{mod}} \longrightarrow C(U^{\mathrm{an}})_{\mathrm{cptf}} \longrightarrow C(U^{\mathrm{an}}),$$

$$G(U^{\mathrm{an}})_{\mathrm{mod}} \longrightarrow G(U^{\mathrm{an}})_{\mathrm{cptf}} \longrightarrow G(U^{\mathrm{an}}).$$

By the direct limit defining $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{mod}}$ (commuting with exact sequences), we have a canonical exact sequence

$$0 \longrightarrow C(U^{\mathrm{an}})_{\mathrm{mod}} \longrightarrow \widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{mod}} \longrightarrow \widetilde{\mathrm{Div}}(U/K)_{\mathrm{mod}} \longrightarrow 0.$$

Here

$$\widetilde{\mathrm{Div}}(U/K)_{\mathrm{mod}} = \widehat{\mathrm{Div}}(U/K)_{\mathrm{mod}} = \varinjlim_{\mathcal{X}} \mathrm{Div}(X, U),$$

where the limit is over all projective models X of U over K. We use $\widetilde{\mathrm{Div}}$ instead of $\widehat{\mathrm{Div}}$ to avoid confusion. We further have a canonical injection

$$\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{mod}} \longrightarrow \widetilde{\mathrm{Div}}(U/K)_{\mathrm{mod}} \oplus G(U^{\mathrm{an}})_{\mathrm{mod}}.$$

Taking completions, we have a canonical injection

$$\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}} \longrightarrow \widetilde{\mathrm{Div}}(U/K) \oplus G(U^{\mathrm{an}})_{\mathrm{cptf}},$$

and a sequence

$$0 \longrightarrow C(U^{\mathrm{an}})_{\mathrm{cptf}} \longrightarrow \widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}} \longrightarrow \widetilde{\mathrm{Div}}(U/K) \longrightarrow 0.$$

Our main result below claims that the sequence is exact, and gives an explicit description of $C(U^{\mathrm{an}})_{\mathrm{cptf}}$.

Recall that the boundary divisor $(0, g_0)$ on U^{an} induced by the boundary divisor $(\mathcal{X}_0, \mathcal{E}_0)$ on U. We further denote $X_0 = \mathcal{X}_{0,K}$ and $E_0 = \mathcal{E}_{0,K}$.

Theorem 3.6.4. *Let K be a field complete with respect to a non-trivial valuation. Let U be a quasi-projective variety over K. The following are true:*

(1) The canonical sequence

$$0 \longrightarrow C(U^{\mathrm{an}})_{\mathrm{cptf}} \longrightarrow \widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}} \longrightarrow \widetilde{\mathrm{Div}}(U/K) \longrightarrow 0$$

is exact.

(2) For any projective model X of U over K, denote by $C(U^{\mathrm{an}}, X^{\mathrm{an}})$ the space of continuous functions $h : X^{\mathrm{an}} \to \mathbb{R}$ supported on U^{an}; i.e., $h(x) = 0$ for any $x \in X^{\mathrm{an}} \setminus U^{\mathrm{an}}$. Denote by $C(U^{\mathrm{an}})_0$ the image of the injection

$$C(U^{\mathrm{an}}, X^{\mathrm{an}}) \longrightarrow C(U^{\mathrm{an}}), \quad h \longmapsto h|_{U^{\mathrm{an}}}.$$

Then $C(U^{\mathrm{an}})_0$ is independent of the choice of X as a projective model of U over K. Moreover,

$$C(U^{\mathrm{an}})_{\mathrm{cptf}} = g_0 \cdot C(U^{\mathrm{an}})_0 = \{g_0 h : h \in C(U^{\mathrm{an}})_0\}.$$

A major feature of the theorem is that functions in $C(U^{\mathrm{an}})_{\mathrm{cptf}}$ grow as $o(g_0)$ along the boundary, which is a very broad type of singularity. For a natural example of a compactified divisor on $\mathbb{C}$ with a Green's function of singularity $O(\log g_0)$, see §5.5 for the Hodge bundle of the moduli space of principally polarized abelian varieties.

In the literature, there are many research on Green's functions with singularity. We first note that Burgos–Kramer–Kühn [BKK] has introduced a general arithmetic intersection theory of arithmetic Chow cycles with pre-log-log currents. This theory treats particularly singularities of type $O(\log g_0)$ and thus includes the above example of Hodge bundles. We also refer to Bost [Bos] and Moriwaki [Mor1] for an arithmetic intersection of arithmetic Chow cycles with L_1^2-currents. To compare with our current theory, all these references treat intersection theory on a fixed projective arithmetic variety and focus on singularities of Green currents, while we treat intersection theory of suitable limits of hermitian line bundles (which corresponds to arithmetic Chow cycles of co-dimension one). In the limit process, our underlying line bundles also vary.

3.6.4 Proof of Theorem 3.6.4

The proof of Theorem 3.6.4 is long as it contains many different parts. We include a detailed proof in the following.

We first prove that the space $C(U^{\mathrm{an}})_0$ in Theorem 3.6.4(2) is independent of the choice of X. Namely, if X' is another projective model of U over K, then

$$\mathrm{Im}(C(U^{\mathrm{an}}, X^{\mathrm{an}}) \to C(U^{\mathrm{an}})) = \mathrm{Im}(C(U^{\mathrm{an}}, X'^{\mathrm{an}}) \to C(U^{\mathrm{an}})).$$

We can assume that there is a birational morphism $\pi : X' \to X$ extending the identity map of U. By pull-back via $\pi^{\mathrm{an}} : X'^{\mathrm{an}} \to X^{\mathrm{an}}$, we have an inclusion

$$\mathrm{Im}(C(U^{\mathrm{an}}, X^{\mathrm{an}}) \to C(U^{\mathrm{an}})) \subset \mathrm{Im}(C(U^{\mathrm{an}}, X'^{\mathrm{an}}) \to C(U^{\mathrm{an}})).$$

It suffices to prove the inverse direction. For any $h' \in C(U^{\mathrm{an}}, X'^{\mathrm{an}})$, we want to descend it to the left-hand side. Define $h : X^{\mathrm{an}} \to \mathbb{R}$ by setting $h|_{U^{\mathrm{an}}} = h'|_{U^{\mathrm{an}}}$ and $h|_{X^{\mathrm{an}} \setminus U^{\mathrm{an}}} = 0$. Then the pull-back of h via $\pi^{\mathrm{an}} : X'^{\mathrm{an}} \to X^{\mathrm{an}}$ is exactly h'. It suffices to prove that h is continuous on X^{an}. Since X'^{an} and X^{an} are both Hausdorff and compact, their closed subsets are the same as compact subsets, so π^{an} is a closed map. Then the continuity of h' implies that of h by the basic result listed in Lemma 3.6.5. This proves the independence on X.

Now we prove the second statement of Theorem 3.6.4(2), i.e. $C(U^{\mathrm{an}})_{\mathrm{cptf}} = g_0 \cdot C(U^{\mathrm{an}})_0$. For any projective model X of U over K dominating X_0, the image of the natural map $C(X^{\mathrm{an}}) \to C(U^{\mathrm{an}})$ is contained in $g_0 \cdot C(U^{\mathrm{an}})_0$. As a consequence, we have a composition of injections

$$C(U^{\mathrm{an}})_{\mathrm{mod}} \longrightarrow \varinjlim_{X} C(X^{\mathrm{an}}) \longrightarrow g_0 \cdot C(U^{\mathrm{an}})_0.$$

Here the limit is over all projective models X of U over K. To prove the result, it suffices to prove that $g_0 \cdot C(U^{\mathrm{an}})_0$ is complete and that $C(U^{\mathrm{an}})_{\mathrm{mod}}$ is dense in $g_0 \cdot C(U^{\mathrm{an}})_0$ under the boundary topology.

It is easy to prove that $g_0 \cdot C(U^{\mathrm{an}})_0$ is complete. In fact, under the bijection $C(U^{\mathrm{an}})_0 \to g_0 \cdot C(U^{\mathrm{an}})_0$, the boundary topology on $g_0 \cdot C(U^{\mathrm{an}})_0$ corresponds to the uniform topology on $C(U^{\mathrm{an}})_0$, which also corresponds to the uniform topology on $C(U^{\mathrm{an}}, X_0^{\mathrm{an}}) \subset C(X_0^{\mathrm{an}})$. The last space is complete.

Now we prove that $C(U^{\mathrm{an}})_{\mathrm{mod}}$ is dense in $g_0 \cdot C(U^{\mathrm{an}})_0$. Note that $C(X_0^{\mathrm{an}})_{\mathrm{mod}}$ is dense in $C(X_0^{\mathrm{an}})$ under the uniform topology. This is already used in the proof of Proposition 3.5.2, as a theorem of Gubler (cf. [Gub3, Thm. 7.12] and [Yua1, Lem. 3.5]). As the uniform topology is stronger than the boundary topology, we see that $C(X_0^{\mathrm{an}})$ lies in the closure of $C(U^{\mathrm{an}})_{\mathrm{mod}}$ in $g_0 \cdot C(U^{\mathrm{an}})_0$. Thus it is reduced to prove that $C(X_0^{\mathrm{an}})$ is dense in $g_0 \cdot C(U^{\mathrm{an}}, X_0^{\mathrm{an}})$ under the boundary topology.

Let $f = g_0 h$ be an element of $g_0 \cdot C(U^{\mathrm{an}}, X_0^{\mathrm{an}})$. Define

$$f_n := \min\{g_0, n\} \cdot h.$$

One checks that $\min\{g_0, n\} \in C(X_0^{\mathrm{an}})$ and thus $f_n \in C(X_0^{\mathrm{an}})$. Denote

$$Z_n = \{x \in X_0^{\mathrm{an}} : g_0(x) \geq n\}, \quad \epsilon_n = \max\{|h(x)| : x \in Z_n\}.$$

Note that $\{Z_n\}_n$ decreases to $|E_0|^{\mathrm{an}}$, so ϵ_n decreases to 0. Then we have

$$|f - f_n| = \max\{0, g_0 - n\} \cdot |h| \leq \epsilon_n g_0.$$

Thus $\{f_n\}_n$ converges to f. This proves Theorem 3.6.4(2).

Now we prove Theorem 3.6.4(1), i.e., the exactness of

$$0 \longrightarrow C(U^{\mathrm{an}})_{\mathrm{cptf}} \longrightarrow \widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}} \longrightarrow \widetilde{\mathrm{Div}}(U/K) \longrightarrow 0.$$

We first prove the exactness in the middle. Let $\overline{D}$ be an element in the kernel of $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}} \to \widetilde{\mathrm{Div}}(U/K)$. So $\overline{D}$ is the limit of a sequence $\overline{D}_i = (D_i, g_i)$ (with $i \geq 1$) in $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{mod}}$ with $\lim_i D_i = 0$ in $\widetilde{\mathrm{Div}}(U/K)$. We need to prove that $g_\infty := \lim_i g_i$ lies in $C(U^{\mathrm{an}})_{\mathrm{cptf}}$. Denote $h_i = g_i/g_0$, viewed as a continuous function on U^{an}. It suffices to prove that $h_\infty := \lim_i h_i$ defines an element of $C(U^{\mathrm{an}})_0$ naturally. We will use the following properties:

(a) The sequence h_i converges uniformly to h_∞ on U^{an}.

(b) There is a compact subset W_i of U^{an} for each $i \geq 1$, such that

$$\|h_i\|_{U^{\mathrm{an}} \setminus W_i, \sup} := \sup\{|h_i(x)| : x \in U^{\mathrm{an}} \setminus W_i\}$$

converges to 0 as $i \to \infty$.

Property (b) comes from the condition $\lim_i D_i = 0$ in $\widetilde{\mathrm{Div}}(U/K)$. In fact, the condition gives $-\epsilon_i E_0 \leq D_i \leq \epsilon_i E_0$ with $\epsilon_i \to 0$. In terms of Green's functions, this implies that $\epsilon_i g_0 \pm g_i$ is bounded below on U^{an}. As $\epsilon_i g_0$ goes to infinity along the boundary of U^{an}, we see that $2\epsilon_i g_0 \pm g_i = \epsilon_i g_0 + (\epsilon_i g_0 \pm g_i) \geq 0$

in a neighborhood of $\mathcal{X}_{i,K}^{\mathrm{an}} \setminus U^{\mathrm{an}}$ in $\mathcal{X}_{i,K}^{\mathrm{an}}$, where $(\mathcal{X}_i, \mathcal{D}_i)$ is a projective model of $(X, 0)$ over O_K inducing $\overline{D}_i$.

We claim that (a) and (b) imply that $h_\infty = \lim_i h_i$ lies in $C(U^{\mathrm{an}})_0$. This is basic in topology. In fact, the function h_∞ lies in $C(U^{\mathrm{an}})$ by (a). It suffices to prove that h_∞ converges to 0 along the boundary $X_0^{\mathrm{an}} \setminus U^{\mathrm{an}}$. Assume the contrary. Then there is a sequence $\{x_j\}_{j\geq 1}$ in U^{an} converging to a point $x_\infty \in X_0^{\mathrm{an}} \setminus U^{\mathrm{an}}$ such that $|h_\infty(x_j)| > c$ for a constant $c > 0$. By (a), we can assume that there is i_0 such that $|h_i(x_j)| > c/2$ for all $i \geq i_0$ and $j \geq 1$. This implies that $\|h_i\|_{U^{\mathrm{an}} \setminus W_i, \sup} \geq c/2$ for all $i \geq i_0$, which contradicts to (b). This proves the exactness in the middle.

It remains to prove the right exactness in Theorem 3.6.4(1), i.e. the surjectivity of $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}} \to \widetilde{\mathrm{Div}}(U/K)$. Let $\widetilde{D}$ be an element of $\widetilde{\mathrm{Div}}(U/K)$, represented by a Cauchy sequence $\{D_i\}_i$ in $\widetilde{\mathrm{Div}}(U/K)_{\mathrm{mod}}$. We need to find a preimage of $\widetilde{D}$ in $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}}$. There is a sequence ϵ_i of positive rational numbers such that

$$-\epsilon_i E_0 \leq D_i - D_{i+1} \leq \epsilon_i E_0.$$

By the Cauchy property, replacing $\{D_i\}_i$ by a subsequence if necessary, we can assume that $\sum_{i\geq 1} \epsilon_i$ converges. We claim that for any $i \geq 1$, there is a model Green's function g_i of D_i on U^{an} such that the sequence $\{g_i\}_i$ satisfies

$$-\epsilon_i g_0 \leq g_i - g_{i+1} \leq \epsilon_i g_0, \quad i \geq 1.$$

If the claim holds, the sequence $\{(D_i, g_i)\}_i$ is a Cauchy sequence, and represents a preimage of $\widetilde{D}$ in $\widehat{\mathrm{Div}}(U^{\mathrm{an}})_{\mathrm{cptf}}$.

It remains to prove the claim. We will construct g_i inductively. Assume that $g_1, \cdots, g_i$ is constructed, and we need to construct g_{i+1} satisfying the requirement. Let g'_{i+1} be a model Green's function of D_{i+1} on U^{an}. Assume that (D_i, g_i) and (D_{i+1}, g_{i+1}) can be realized as a model arithmetic divisors of mixed coefficients on (X_{i+1}, U) for some projective model X_{i+1} of U over K. Set $g_{i+1} = g'_{i+1} - f$ for $f \in C(X_{i+1}^{\mathrm{an}})_{\mathrm{mod}}$. It suffices to find $f \in C(X_{i+1}^{\mathrm{an}})_{\mathrm{mod}}$ satisfying

$$-\epsilon_i g_0 \leq g_i - g'_{i+1} + f \leq \epsilon_i g_0.$$

As before, $C(X_{i+1}^{\mathrm{an}})_{\mathrm{mod}}$ is dense in $C(X_{i+1}^{\mathrm{an}})$ under uniform convergence. So it suffices to find $f \in C(X_{i+1}^{\mathrm{an}})$ satisfying

$$-\epsilon_i (g_0 - c_0) \leq g_i - g'_{i+1} + f \leq \epsilon_i (g_0 - c_0),$$

where $c_0 > 0$ is a constant with $g_0 > c_0$ on $U^{\rm an}$. The condition is equivalent to

$$g'_{i+1} - g_i - \epsilon_i(g_0 - c_0) \leq f \leq g'_{i+1} - g_i + \epsilon_i(g_0 - c_0).$$

This is an inequality of Green's functions on $X_{i+1}^{\rm an}$ corresponding to the divisor relation

$$D_{i+1} - D_i - \epsilon_i E_0 \leq 0 \leq D_{i+1} - D_i + \epsilon_i E_0.$$

By the first inequality of divisors, $g'_{i+1} - g_i - \epsilon_i(g_0 - c_0)$ has a finite upper bound c on $X_{i+1}^{\rm an}$. Then we can take

$$f = \min\{c,\ g'_{i+1} - g_i + \epsilon_i(g_0 - c_0)\},$$

which is continuous by the second inequality of divisors. This finishes the proof of Theorem 3.6.4.

In the above proof, the following basic result was used. We list it separately, since it will be used again later.

Lemma 3.6.5. *Let $\pi : M \to N$ be a surjective, closed and continuous map of topological spaces. Let $f : N \to \mathbb{R}$ be a map, and $\pi^* f = f \circ \pi : M \to \mathbb{R}$ the pull-back. Then f is continuous if and only if $\pi^* f$ is continuous.*

Proof. For the "if" part, prove that inverse images of closed sets under f are closed. □

3.6.5 Compactified metrics

Here we briefly introduce the corresponding notions of compactified line bundles. Resume the above notations. Namely, let K be a field complete with respect to a non-trivial absolute value $|\cdot|$. Set O_K to be K in the archimedean case, and to be the valuation ring in the non-archimedean case. Let U be a quasi-projective variety over K.

Recall from Proposition 3.6.1 that there are canonical injective maps

$$\widehat{\mathcal{P}\mathrm{ic}}(U/O_K) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(U^{\rm an}),$$

$$\widehat{\mathrm{Pic}}(U/O_K) \longrightarrow \widehat{\mathrm{Pic}}(U^{\rm an}).$$

Denote

$$\widehat{\mathcal{P}\mathrm{ic}}(U^{\rm an})_{\rm cptf} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(U/O_K) \to \widehat{\mathcal{P}\mathrm{ic}}(U^{\rm an})),$$

$$\widehat{\mathrm{Pic}}(U^{\rm an})_{\rm cptf} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(U/O_K) \to \widehat{\mathrm{Pic}}(U^{\rm an})).$$

They are compactifications of

$$\widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}})_{\mathrm{mod}} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(U/O_K)_{\mathrm{mod}} \to \widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}})),$$
$$\widehat{\mathrm{Pic}}(U^{\mathrm{an}})_{\mathrm{mod}} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(U/O_K)_{\mathrm{mod}} \to \widehat{\mathrm{Pic}}(U^{\mathrm{an}})).$$

As in the case of arithmetic divisors, we are going to describe these groups or categories directly on U^{an}.

If K is non-archimedean, an *arithmetic model* (or *integral model*) of a line bundle L on U is a pair $(\mathcal{X}, \mathcal{L})$, where $\mathcal{X}$ is a projective model of U over O_K, and $\mathcal{L}$ is a $\mathbb{Q}$-line bundle on $\mathcal{X}$ extending L.

If K is archimedean, an *arithmetic model* of a line bundle L on U is a pair $(\mathcal{X}, \mathcal{L})$, where $\mathcal{X}$ is a projective model of U over $O_K = K$, and $\mathcal{L} = (\widetilde{L}, \|\cdot\|_{\widetilde{L}})$ consisting of a $\mathbb{Q}$-line bundle $\widetilde{L}$ on $\mathcal{X}$ extending L and a continuous metric $\|\cdot\|_{\widetilde{L}}$ of $\widetilde{L}$ on $\mathcal{X}^{\mathrm{an}}$.

In both cases, the arithmetic model $(\mathcal{X}, \mathcal{L})$ induces a metric $\|\cdot\|_{\mathcal{L}}$ of $\mathcal{L}_K$ on $\mathcal{X}_K^{\mathrm{an}}$, and thus a metric of L on U^{an} by restriction. The process is essentially the same as the global case described in §3.4. The metric $\|\cdot\|_{\mathcal{L}}$ of L on U^{an} is called a *model metric*, and the metrized line bundle $(L, \|\cdot\|_{\mathcal{L}})$ on U^{an} is called a *model metrized line bundle*.

The model metric or the model metrized line bundle is called *nef* (or *semipositive*) if either $\mathcal{L}$ is nef on $\mathcal{X}$ in the non-archimedean case or the metric of $\mathcal{L}$ is semipositive on $\mathcal{X}(\mathbb{C})$ (cf. §2.1).

Recall that we have boundary topologies on $\widehat{\mathrm{Div}}(X^{\mathrm{an}})$ and $C(X^{\mathrm{an}})$ in terms of $g_0 = g_{\mathcal{E}_0}|_{U^{\mathrm{an}}}$ obtained by the choice of a pair $(\mathcal{X}_0, \mathcal{E}_0)$.

A metrized line bundle $\overline{L} = (L, \|\cdot\|)$ on U^{an} or its metric is called *compactified* if there is a sequence of model metrics $\{\|\cdot\|_i\}_{i\geq 1}$ of L on U^{an}, such that the continuous function $\log(\|\cdot\|_i/\|\cdot\|)$ on U^{an} converges to 0 under the boundary topology on $C(X^{\mathrm{an}})$.

The metrized line bundle $\overline{L}$ or its metric $\|\cdot\|$ is said to be *strongly nef* (or *strongly semipositive*) if there exists such a sequence such that every model metric $\|\cdot\|_i$ is nef. The metrized line bundle $\overline{L}$ or its metric $\|\cdot\|$ is said to be *nef* (or *semipositive*) if there exists a strongly nef metrized line bundle $\overline{M}$ such that $a\overline{L} + \overline{M}$ is strongly nef for all positive integers a. The metrized line bundle $\overline{L}$ or its metric $\|\cdot\|$ is said to be *integrable* if $\overline{L}$ is isometric to the difference of two strongly nef metrized line bundles.

Finally, our result is as follows:

(1) $\widehat{\mathrm{Pic}}(U^{\mathrm{an}})_{\mathrm{mod}}$ (resp. $\widehat{\mathrm{Pic}}(U^{\mathrm{an}})_{\mathrm{cptf}}$) is the subgroup of $\widehat{\mathrm{Pic}}(U^{\mathrm{an}})$ consisting of model metrized (resp. compactified) line bundles on U^{an}.

(2) $\widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}})_{\mathrm{mod}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}})_{\mathrm{cptf}}$) is equivalent to the full subcategory of $\widehat{\mathcal{P}\mathrm{ic}}(U^{\mathrm{an}})$ consisting of model (resp. compactified) metrized line bundles on U^{an}.

3.6.6 Chambert-Loir measures

Let U be a quasi-projective variety over a complete field K with a non-trivial valuation as above. Denote $n = \dim U$. Let $\overline{L}_1, \overline{L}_2, \cdots, \overline{L}_n$ be strongly nef (compactified) metrized line bundles on U^{an}. We will see that there is a canonical Radon measure $c_1(\overline{L}_1)c_1(\overline{L}_2)\cdots c_1(\overline{L}_n)$ on the Berkovich space U^{an}, which generalizes the Monge–Ampère measure in the complex case. We will call this measure *the Chambert-Loir measure*.

If K is archimedean, this is easily done by classical analysis in [BT, Thm. 2.1] (or [Dem1, Cor. 1.6]). If K is non-archimedean and U is projective, this is constructed by Chambert-Loir [CL] when K has a dense and countable subfield, and extended to general K by Gubler [Gub1]. If K is non-archimedean and U is quasi-projective, we will follow the theory of Chambert-Loir and Ducros in [CD], a vast generalization of the construction of [CL] via a local analytic approach.

In the following, assume that K is non-archimedean and that U is quasi-projective over K. We are going to apply [CD, Cor. 5.6.5] to $\overline{L}_1, \overline{L}_2, \cdots, \overline{L}_n$. Before that, we claim that the metric of a strongly nef metrized line bundle satisfies the condition of [CD, Cor. 5.6.5]; i.e. it is locally psh-approachable on U^{an} in the sense of [CD, 6.3.1, Def. 5.6.3, Def. 5.5.1].

In fact, let $\overline{L} = (L, \|\cdot\|)$ be a strongly nef metrized line bundle on U^{an}. By definition, the metric $\|\cdot\|$ is the limit of a sequence of nef model metrics $\|\cdot\|_i$ under the boundary topology of $C(U^{\mathrm{an}})$. Since U^{an} is locally compact, the convergence is locally uniform as in Lemma 3.6.3. Therefore, it suffices to prove the nef model case. So we assume that the metric $\|\cdot\|$ is a nef model metric, and we need to prove that it is locally psh-approachable. This is a consequence of [CD, Cor. 6.3.4], since the metric $\|\cdot\|$ is induced by a nef line bundle $\mathcal{L}$ on a projective model $\mathcal{X}$ of U over O_K. Note that the loc. cit. is only stated for the ample case, but can be extended to the nef case. In fact, take any ample line bundle $\mathcal{M}$ on $\mathcal{X}$, which induces a metric $\|\cdot\|_{\mathcal{M}}$ of $M = \mathcal{M}|_U$ on U^{an}. For any local sections s and t of L and M regular and everywhere non-vanishing on a Zariski open set W of U, the function $-\log\|s\| - \epsilon\log\|t\|_{\mathcal{M}}$ is globally psh-approachable on W^{an} for any positive rational numbers $\epsilon > 0$. As $\epsilon \to 0$, the function converges to $-\log\|s\|$, which is uniform on any compact subset of W^{an}. This proves that $\|\cdot\|$ is locally psh-approachable, and finishes the quasi-projective case.

Finally, by [CD, Cor. 5.6.5], there is a canonical measure

$$c_1(\overline{L}_1)\cdots c_1(\overline{L}_n) = d'd''(-\log\|\cdot\|_1)\wedge\cdots\wedge d'd''(-\log\|\cdot\|_n)$$

over U^{an}. Here the right-hand side is understood as follows. If W is a Zariski open subset of X and s is a regular and everywhere non-vanishing section of L on W, then we set $d'd''(-\log\|\cdot\|_i) = d'd''(-\log\|s\|_i)$ on W^{an}. This is independent of the choice of s by the Poincaré-Lelong formula in [CD, Thm. 4.6.5].

The measure is defined by a weak convergence process. We describe it as follows. For any $i = 1,\cdots,n$, the metric of $\overline{L}_i$ is the limit of model metrics induced by (projective) arithmetic models $(\mathcal{X}_{i,j},\mathcal{L}_{i,j})$ of (U,L) over O_K. We can assume that $\mathcal{X}_{i,j}$ is independent of i, and write it as $\mathcal{X}_j$. Denote $X_j = \mathcal{X}_{j,K}$, which is a projective model of U over K. Denote by $\overline{L}_{i,j} = (L_{i,j},\|\cdot\|_{i,j})$ the metrized line bundle on the compact space X_j^{an}, induced by the model $(\mathcal{X}_{i,j},\mathcal{L}_{i,j})$. Denote by $C_c(U^{\mathrm{an}})$ the space of real-valued, continuous and compactly supported function on U^{an}. Then the construction gives, for any $f\in C_c(U^{\mathrm{an}})$,

$$\int_{U^{\mathrm{an}}} f c_1(\overline{L}_1)\cdots c_1(\overline{L}_n) = \lim_{j\to\infty}\int_{X_j^{\mathrm{an}}} f c_1(\overline{L}_{1,j})\cdots c_1(\overline{L}_{n,j}).$$

As X_j is projective over K, the right-hand side is equal to the integration defined by global intersection numbers by [CL, Gub1].

It is worth noting that by [CT, Cor. 4.2], the integral of $c_1(\overline{L}_{1,j})\cdots c_1(\overline{L}_{n,j})$ on any Zariski closed subset of X_j^{an} of positive codimension is 0.

3.6.7 Application to finitely generated fields

Let k be either $\mathbb{Z}$ or a field. Take the uniform terminology in §1.6. Let F be a finitely generated field over k. Let v be a point of $(\operatorname{Spec} F)^{\mathrm{an}} = \mathcal{M}(F/k)$ that is *not* the trivial valuation over F. Denote by F_v the completion of F with respect to v. It can be either archimedean or non-archimedean.

Let X be a *quasi-projective* variety of dimension n over F. Let $\overline{L}$ be a *strongly nef* adelic line bundle on X with an underlying line bundle L on X. Let X_v^{an} be the Berkovich space associated to the variety X_{F_v} over the complete field F_v, which is the fiber of $X^{\mathrm{an}}\to(\operatorname{Spec} F)^{\mathrm{an}}$ above v. By

Proposition 3.4.1, $\overline{L}$ induces a metric $\|\cdot\|$ of L on X^{an}, which restricts to a F_v-metric $\|\cdot\|_v$ of L on X_v^{an}.

Lemma 3.6.6. *Assume that X is quasi-projective over F and that $\overline{L}$ is strongly nef on X. Then the metric $\|\cdot\|_v$ of L on X_v^{an} induced by $\overline{L}$ is strongly nef.*

With the lemma, there is a Chambert-Loir measure

$$c_1(\overline{L})_v^n := c_1(L_{F_v}, \|\cdot\|_v)^n$$

over the Berkovich space X_v^{an} for any point $v \in \mathcal{M}(F/k)$ which is non-trivial over F. This measure will be used in our equidistribution conjectures and theorems.

By multi-linearity, for any integrable line bundles $\overline{L}_1, \cdots, \overline{L}_n$ on X, there is a (signed) Chambert-Loir measure

$$c_1(\overline{L}_1)_v \cdots c_1(\overline{L}_n)_v := c_1(L_{1,F_v}, \|\cdot\|_v) \cdots c_1(L_{n,F_v}, \|\cdot\|_v)$$

over the Berkovich space X_v^{an} for any point $v \in \mathcal{M}(F/k)$ which is non-trivial over F. Now we prove the lemma.

Proof of Lemma 3.6.6. We only treat the case that v is non-archimedean, since the archimedean case is easier.

Assume that $\overline{L}$ is represented by a Cauchy sequence $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{mod}}$. Here $\mathcal{U}$ is a quasi-projective model of X, and each $\overline{\mathcal{L}}_i$ is nef on $\mathcal{X}_i$. By Lemma 2.3.3, we can assume that $\mathcal{U}$ is equipped with a flat morphism $\mathcal{U} \to \mathcal{V}$ to a quasi-projective variety $\mathcal{V}$, whose generic fiber is isomorphic to $X \to \operatorname{Spec} F$. Let $\mathcal{S}$ be a fixed projective model of $\mathcal{V}$. By blowing-up $\mathcal{X}_i$ if necessary, we can assume that $\mathcal{U} \to \mathcal{V}$ extends to a morphism $\mathcal{X}_i \to \mathcal{S}$.

The point $v \in (\operatorname{Spec} F)^{\mathrm{an}} \subset \mathcal{S}^{\mathrm{an}}$ has a residue field F_v and a valuation ring $R_v \subset F_v$. By the valuative criterion, the morphism $\operatorname{Spec} F_v \to \mathcal{S}$ extends to a morphism $\operatorname{Spec} R_v \to \mathcal{S}$. The base change of $\mathcal{X}_i \to \mathcal{S}$ gives a morphism $\mathcal{X}_{i,R_v} \to \operatorname{Spec} R_v$ whose generic fiber contains X_{F_v} as an open subvariety. Denote by $\mathcal{X}'_{i,R_v}$ the Zariski closure of X_{F_v} in $\mathcal{X}_{i,R_v}$, so that $\mathcal{X}'_{i,R_v}$ is the unique irreducible component of $\mathcal{X}_{i,R_v}$ flat over R_v. By pull-back, we get a sequence of $\mathbb{Q}$-line bundles $\mathcal{L}_i|_{\mathcal{X}'_{i,R_v}}$ on $\mathcal{X}'_{i,R_v}$, which induces a sequence of model metrics $\|\cdot\|_i$ of L on X_v^{an}. The limit of these metrics is exactly the desired metric $\|\cdot\|_v$. Moreover, the convergence of $\{\|\cdot\|_i\}_i$ to $\|\cdot\|_v$ is with respect to the boundary topology, as we can see in the proof of Proposition 3.4.1 for quasi-projective varieties. $\square$

Remark 3.6.7. By the lemma, if X is projective over F, then $\|\cdot\|_v$ is semipositive in the sense that it is a uniform limit of metrics induced by nef models. In this case, we can also use the construction of [CL, Gub1] to define the measure.

4 Intersection theory

In this section, we develop an intersection theory of integrable adelic line bundles. There are two types of intersection pairings. The first type gives an absolute intersection number; the second type is an intersection pairing in a relative setting in terms of the Deligne pairing. While the absolute intersection number is very easy to obtain, the construction of the relative intersection pairing takes most of this section.

4.1 Intersection theory

In this subsection, we state both intersection pairings, prove the existence of the absolute version, and leave the proof the relative version to the rest of this section.

4.1.1 Absolute intersection numbers

In algebraic geometry, for a projective variety $\mathcal{X}$ of dimension d over a base field, there is an intersection pairing $\mathrm{Pic}(\mathcal{X})^d \to \mathbb{Z}$.

In Arakelov geometry, by the theory of Gillet–Soué [GS1], for each projective variety $\mathcal{X}$ of absolute dimension d over $\mathbb{Z}$, there is an intersection pairing $\widehat{\mathrm{Pic}}(\mathcal{X})^d_{\mathrm{smth}} \to \mathbb{R}$, which was extended to a pairing $\widehat{\mathrm{Pic}}(\mathcal{X})^d_{\mathrm{int}} \to \mathbb{R}$ as recalled in §2.1.

We are going to extend these pairings to adelic line bundles. As in the case of [Zha2, Mor4], we cannot expect the intersection to be defined for all adelic line bundles, but only for the integrable ones.

Proposition 4.1.1. *Let k be either $\mathbb{Z}$ or a field. For any flat and essentially quasi-projective integral scheme X over k, the intersection pairing above extends to a canonical multi-linear homomorphism*

$$\widehat{\mathrm{Pic}}(X/k)^d_{\mathrm{int}} \longrightarrow \mathbb{R}.$$

Here d is the absolute dimension of a quasi-projective model of X over k.

Moreover, if $\overline{L}_1, \cdots, \overline{L}_d$ are nef adelic line bundles on X, then their intersection number $\overline{L}_1 \cdot \overline{L}_2 \cdots \overline{L}_d \geq 0$.

Proof. It suffices to treat the case that $X = \mathcal{U}$ is a quasi-projective variety over k. We need to define $\langle \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_d \rangle$ for any $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_d \in \widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{snef}}$.

Let $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ be a boundary divisor of $\mathcal{U}$ over k, and we will use it to define the boundary topology of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$. We can further assume that $\overline{\mathcal{E}}_0$ is nef, which is possible by simply replacing $\overline{\mathcal{E}}_0$ by a nef arithmetic divisor $\overline{\mathcal{E}}_0'$ on $\mathcal{X}_0$ satisfying $\mathcal{E}_0' \geq \mathcal{E}_0$ and replacing $\mathcal{U}$ by $\mathcal{X}_0 \setminus |\mathcal{E}_0'|$.

For $j = 1, \cdots, d$, assume that $\overline{\mathcal{L}}_j$ is represented by a Cauchy sequence $(\mathcal{L}_j, (\mathcal{X}_i, \overline{\mathcal{L}}_{j,i}, \ell_{j,i})_{i \geq 1})$ with each $\overline{\mathcal{L}}_{j,i}$ nef on a projective model $\mathcal{X}_i$ dominating $\mathcal{X}_0$. Here we assume that the model $\mathcal{X}_i$ is independent of j, which is always possible. There is a sequence $\{\epsilon_i\}_{i \geq 1}$ of positive rational numbers converging to 0 such that

$$-\epsilon_i \overline{\mathcal{E}}_0 \leq \widehat{\mathrm{div}}(\ell_{j,i'} \ell_{j,i}^{-1}) \leq \epsilon_i \overline{\mathcal{E}}_0$$

for any $j = 1, \cdots, d$ and any $i' > i$.

For any subset $J \subset \{1, \cdots, d\}$, consider the sequence

$$\alpha_{J,i} := \overline{\mathcal{E}}_0^{d-|J|} \prod_{j \in J} \overline{\mathcal{L}}_{j,i}.$$

We will prove by induction that $\{\alpha_{J,i}\}_{i \geq 1}$ is a Cauchy sequence and thus convergent in $\mathbb{R}$. When J is the full set, we have the proposition.

There is nothing to prove if J is the empty set. Assume the claim is true for any $|J| < r$ for some $r > 0$. We need to prove the result for any J with $|J| = r$. Without loss of generality, assume $J = \{1, 2, \cdots, r\}$. Then

$$\begin{aligned}
\alpha_{J,i'} - \alpha_{J,i} &= \overline{\mathcal{E}}_0^{d-r} \overline{\mathcal{L}}_{1,i'} \cdots \overline{\mathcal{L}}_{r,i'} - \overline{\mathcal{E}}_0^{d-r} \overline{\mathcal{L}}_{1,i} \cdots \overline{\mathcal{L}}_{r,i} \\
&\leq \overline{\mathcal{E}}_0^{d-r} (\overline{\mathcal{L}}_{1,i} + \epsilon_i \overline{\mathcal{E}}_0) \cdots (\overline{\mathcal{L}}_{r,i} + \epsilon_i \overline{\mathcal{E}}_0) - \overline{\mathcal{E}}_0^{d-r} \overline{\mathcal{L}}_{1,i} \cdots \overline{\mathcal{L}}_{r,i} \\
&= \sum_{J' \subsetneq J} \epsilon_i^{r-|J'|} \alpha_{J',i}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\alpha_{J,i} - \alpha_{J,i'} &\leq \overline{\mathcal{E}}_0^{d-r} (\overline{\mathcal{L}}_{1,i'} + \epsilon_i \overline{\mathcal{E}}_0) \cdots (\overline{\mathcal{L}}_{r,i'} + \epsilon_i \overline{\mathcal{E}}_0) - \overline{\mathcal{E}}_0^{d-r} \overline{\mathcal{L}}_{1,i'} \cdots \overline{\mathcal{L}}_{r,i'} \\
&= \sum_{J' \subsetneq J} \epsilon_i^{r-|J'|} \alpha_{J',i'}.
\end{aligned}$$

It follows that $\{\alpha_{J,i}\}_i$ is a Cauchy sequence. This finishes the proof. $\square$

A basic property of the intersection number is the following projection formula.

Proposition 4.1.2 (projection formula). *Let k be either $\mathbb{Z}$ or a field. Let $f : X' \to X$ be a morphism of flat and essentially quasi-projective integral schemes over k. Assume that the absolute dimensions of quasi-projective models of X' and X over k are all equal to d. Let $\overline{L}_1, \cdots, \overline{L}_d$ be integrable adelic line bundles on X. Then*

$$f^*\overline{L}_1 \cdot f^*\overline{L}_2 \cdots f^*\overline{L}_d = \deg(f)\,(\overline{L}_1 \cdot \overline{L}_2 \cdots \overline{L}_d).$$

Here if f is dominant in that it maps the generic point of X' to the generic point of X, then $\deg(f)$ is the degree of the extension between the function fields; otherwise, we take the convention $\deg(f) = 0$.

Proof. By the limit process, it is reduced to the well-known formula projective case. □

4.1.2 Deligne Pairing: main theorem

Let $f : X \to Y$ be a projective and flat morphism of noetherian schemes of pure relative dimension n. The Deligne pairing is a multi-linear functor

$$\mathcal{P}\mathrm{ic}(X)^{n+1} \longrightarrow \mathcal{P}\mathrm{ic}(Y), \quad (L_1, \cdots, L_{n+1}) \longmapsto \langle L_1, \cdots, L_{n+1}\rangle.$$

The functor refines the intersection of the Chern classes of the line bundles, and satisfies many natural functorial properties, including the base change property, the multi-linearity, the symmetry, and the induction formula.

For a brief history of pairing, the case $n = 0$ is just the norm functor $N_{X/Y}$. Deligne [Del] constructed the functor for $n = 1$, and speculates a similar pairing for general n. For general n, the pairing was constructed by Elkik [Elk1] for any f which is projective, flat and further Cohen–Macaulay, and by Munoz Garcia [MG] for any f which is projective, equi-dimensional and of finite Tor-dimension (which implies the projective and flat case). Moreover, Ducrot [Duc] had a different treatment of the projective and flat case.

If X and Y are smooth varieties over $\mathbb{C}$ and f is smooth, and if $L_1, \cdots, L_{n+1}$ are endowed with smooth hermitian metrics, then the metrics transfer to a canonical smooth hermitian metric on $\langle L_1, \cdots, L_{n+1}\rangle$, as constructed by Deligne [Del] and Elkik [Elk2]. As we will prove later, the metric construction can be generalized to the projective and flat case, and in this case the Deligne pairing transfer continuous metrics to continuous metrics.

Our goal here is to extend the Deligne pairing to adelic line bundles. The main result of this section is as follows.

Theorem 4.1.3. *Let k be either $\mathbb{Z}$ or a field. Let Y be a flat and essentially quasi-projective integral scheme over k. Let $f : X \to Y$ be a projective and flat morphism of relative dimension n. Assume that X is integral and Y is normal. Then the Deligne pairing induces a symmetric and multilinear functor*

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(Y/k)_{\mathrm{int}},$$

When restricted to strongly nef or nef adelic line bundles, the functor induces functors

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{snef}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(Y/k)_{\mathrm{snef}},$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{nef}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(Y/k)_{\mathrm{nef}}.$$

Moreover, the maps are compatible with base changes of the form $Y' \to Y$, where Y' is any normal integral scheme, flat and essentially quasi-projective over k, such that $X' = X \times_Y Y'$ is integral.

The proof of this theorem will take up the rest of this section. After some preparations about metrics of the Deligne pairings and basic properties in the model case, the proof of the theorem will be given in §4.5.

4.2 Metrics of the Deligne pairing: statements

The goal of this subsection is two-fold. First, we review the treatment of the Deligne pairing of [MG] to setup a framework for our treatment. Second, we state some results on natural metrics of the Deligne pairing from metrics of the original line bundles, which generalizes the result of [Del, Elk2] from the smooth case to the general case. Note that the treatments of [Zha3, Mor2] on the metrics have gaps due to mis-interpretations of the definition of the canonical section $\langle s_1, \cdots, s_{n+1}\rangle$ of $\langle L_1, \cdots, L_{n+1}\rangle$ on Y.

4.2.1 Deligne Pairing: review

Here we recall some results of the Deligne pairing in [MG]. Our main interest is the Deligne pairing for projective and flat morphisms, but it seems inevitable to treat non-flat morphisms of finite Tor-dimension if we want to pass to generic hyperplane sections by an induction formula. Therefore, we will follow the generality of [MG] to treat morphisms of finite Tor-dimension.

Recall that a morphism $f : X \to Y$ of noetherian schemes is *of pure relative dimension* n if for every $y \in Y$, every irreducible component of X_y (if non-empty) has dimension n.

Recall that a morphism $f : X \to Y$ of noetherian schemes is *of finite Tor-dimension* if one of the following two equivalent conditions holds:

(a) there is an integer d_0 such that $\mathrm{Tor}_d^B(A, M) = 0$ for any $d > d_0$, for any affine open subscheme $\mathrm{Spec}\, A$ of X whose image under f lies in an affine open subscheme $\mathrm{Spec}\, B$ of Y, and for any B-module M.

(b) there is an integer d_0 such that $\mathrm{Tor}_d^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, M) = 0$ for any $d > d_0$, for any point $x \in X$ with $y = f(x) \in Y$, and for any $\mathcal{O}_{Y,y}$-module M.

See [SGA6, III, §3, Def. 3.2, Prop. 3.3] for more information. Note that this holds automatically if f is flat or Y is regular. Moreover, if $f : X \to Y$ is of finite Tor-dimension, and Z is an effective Cartier divisor of X, then $Z \to Y$ is also of finite Tor-dimension.

Let $f : X \to Y$ be a projective morphism of noetherian schemes of finite Tor-dimension and pure relative dimension n. Let $s_1, \cdots, s_{n+1}$ be global sections of $L_1, \cdots, L_{n+1}$ on X respectively. For any $i = 1, \cdots, n+1$, denote by $Z_i = \mathrm{div}(s_1) \cap \cdots \cap \mathrm{div}(s_i)$ the schematic intersection in X. Set $Z_0 = X$ for convenience. Following [MG, Def. 4.3.2], we say that the sequence $(s_1, \cdots, s_{n+1})$ is *strongly regular* if the following conditions hold:

(1) for any $i = 1, \cdots, n+1$, the section s_i is not a zero-divisor on Z_{i-1} in the sense that the morphism $\mathcal{O}_{Z_{i-1}} \to L_i|_{Z_{i-1}}$ induced by s_i is injective;

(2) for any $i = 1, \cdots, n$, the scheme Z_i is purely of relative dimension $n-i$ over Y.

If (1) and (2) hold for $i = 1, \cdots, n$, then we say that the sequence $(s_1, \cdots, s_n)$ is *strongly regular*. The condition (1) is actually symmetric in $s_1, \cdots, s_{n+1}$ by a basic property of regular sequences in local rings. Note that the notion of *strongly regular* is stronger than the notion of *very regular* in [MG, Def. 3.2.1, Def. 3.2.4], and is more convenient in applications.

The following existence of strongly regular sequence will be frequently used in our treatment. If $L_1, \cdots, L_{n+1}$ are f-ample on X, then there is a finite Zariski open cover V of Y and a positive integer m, such that the base change $(L_1^{\otimes m})_V, \cdots, (L_{n+1}^{\otimes m})_V$ has a strongly regular sequence of sections for the morphism $f_V : X_V \to V$. In fact, for any closed point $y \in Y$, let V_y be an affine open neighborhood of y in Y. Then we can find a global section s_1 of $(L_1^{\otimes m})_{V_y}$ on X_{V_y} for some positive integer m such that s_1 is non-vanishing at any associated point of X_{V_y} or X_y. This can be guaranteed by requiring s_1 to be non-vanishing at a prescribed closed point of every irreducible component and every embedded component of X_{V_y} and X_y. Then s_1 is not a zero-divisor on X_{V_y} or X_y, and thus $\mathrm{div}(s_1) \cap X_y$ is pure of dimension $n-1$. By semi-continuity of dimensions of fibers (cf. [EGA, IV-3, Cor, 13.1.5]), $\mathrm{div}(s_1)$ is of pure relative dimension $n-1$ over a neighborhood of y in V_y. Replace V_y by this open neighborhood. By induction, this gives the strongly regular sequence.

Let $(s_1, \cdots, s_{n+1})$ be a strongly regular sequence of sections of $(L_1, \cdots, L_{n+1})$ on X. By [MG, Prop. 3.2.6], there is a canonical global section $\langle s_1, \cdots, s_{n+1}\rangle$ of $\langle L_1, \cdots, L_{n+1}\rangle$ on Y. In fact, there is a canonical isomorphism

$$r : \langle L_1, \cdots, L_{n+1}\rangle \longrightarrow N_{Z_n/Y}(L_{n+1}).$$

The global section s_{n+1} gives a global section $N_{Z_n/Y}(s_{n+1})$ of $N_{Z_n/Y}(L_{n+1})$. Set

$$\langle s_1, \cdots, s_{n+1}\rangle = r^{-1}(N_{Z_n/Y}(s_{n+1})).$$

Note that $Z_n \to Y$ is finite but not necessarily flat over Y, the norm functor

$$N_{Z_n/Y} : \mathcal{P}\mathrm{ic}(Z_n) \longrightarrow \mathcal{P}\mathrm{ic}(Y)$$

is defined in [MG, §1.2], as a natural generalization of the finite and flat case. The section $\langle s_1, \cdots, s_{n+1}\rangle$ of $\langle L_1, \cdots, L_{n+1}\rangle$ behaves well if switching the orders of $L_1, \cdots, L_{n+1}$; see [MG, Thm. 3.4.2].

This essentially gives construction of $\langle L_1, \cdots, L_{n+1}\rangle$ for relatively ample line bundles $L_1, \cdots, L_{n+1}$ on X. By linearity, it generalizes to arbitrary line bundles $L_1, \cdots, L_{n+1}$ on X.

As a convention, the Deligne pairing $\langle L_1, \cdots, L_{n+1}\rangle$ for the morphism $f : X \to Y$ will also be written as

$$f_*\langle L_1, \cdots, L_{n+1}\rangle, \quad \langle L_1, \cdots, L_{n+1}\rangle_{X/Y}, \quad \langle L_1, \cdots, L_{n+1}\rangle_X.$$

This may be used when we vary $f : X \to Y$ to avoid confusion. We take this convention for all similar pairings introduced later.

4.2.2 Deligne Pairing: metric at a point

Let $Y = \operatorname{Spec}\mathbb{C}$ and $f : X \to Y$ be a projective morphism of pure relative dimension n. Let $\overline{L}_1, \cdots, \overline{L}_{n+1}$ be line bundles on X, endowed with integrable metrics. The goal is to endow a metric of $\langle L_1, \cdots, L_{n+1}\rangle$ on Y in this general setting.

Note that we do not assume that X is integral, so we need to extend the definition of metrics and integrations to this setting. Denote by X_{red} the reduced structure of X. Denote by $X_1, \cdots, X_r$ the irreducible components of of X_{red}, endowed with the reduced structures. For $i = 1, \cdots, r$, denote by η_i the generic point of X_i.

Define *the multiplicity of X_i in X* to be

$$\delta(X_i) = \delta(X_i, X) = \mathrm{length}_{\mathcal{O}_{X,\eta_i}}(\mathcal{O}_{X,\eta_i}).$$

See [BLR, §9.1, Def. 3] for example. For integrations, we define

$$\int_X \alpha := \sum_{i=1}^{r} \delta(X_i) \int_{X_i} \alpha|_{X_i}$$

in reasonable settings to be used later. For example, if X is a finite scheme over $\mathbb{C}$ (so $n = 0$), then for any function $\alpha : X_{\mathrm{red}} \to \mathbb{R}$, we take the convention

$$\int_X \alpha = \sum_{i=1}^{r} \delta(X_i)\alpha(X_i).$$

Most notions in §2.1.1-2.1.2 can be generalized to the current setting. By a *continuous function* on X, we mean a continuous function on X_{red}. By a *smooth function* on X, mean a continuous function $g : X_{\mathrm{red}} \to \mathbb{R}$, such that for any closed point $x \in X$, there is an open subscheme U of X containing x together with a closed immersion $U \to M$ to a complex manifold M such that $g|_U$ can be extended to a smooth function on M. Let L be a line bundle on X. By a *continuous metric* of L on X, we mean a continuous metric of $L|_{X_{\mathrm{red}}}$ on X_{red}. By a *smooth metric* L on X, we mean a continuous metric $\|\cdot\|$ of $L|_{X_{\mathrm{red}}}$ on X_{red} such that $\|s\|^2$ is a smooth function for any local section s of X, which is not a zero-divisor Zariski locally. Define *Chern currents*, *semipositive metrics*, *integrable metrics* similarly. In terms of integration, we essentially only care about the pull-back of these terms to $X_1, \cdots, X_n$.

Let $\overline{L}_1, \cdots, \overline{L}_{n+1}$ be line bundles on X, endowed with integrable metrics. Then $\langle L_1, \cdots, L_{n+1}\rangle$ is a 1-dimensional complex vector space. We endow a metric of $\langle L_1, \cdots, L_{n+1}\rangle$ as follows.

We assume that all L_i are very ample by linearity. For any nonzero section s_1 of L_1 on X, which is a regular sequence in that s_1 is not a zero-divisor Zariski locally on X, we have a natural isomorphism

$$[s_1] : \langle L_1, \cdots, L_{n+1}\rangle \longrightarrow \langle L_2, \cdots, L_{n+1}\rangle_{Z_1}.$$

Here $Z_1 = \mathrm{div}(s_1)$, and the right-hand side is the Deligne pairing with respect to the morphism $Z_1 \to Y$. Define the norm of the map $[s_1]$ by

$$\log \|[s_1]\| = -\int_X \log \|s_1\| c_1(\overline{L}_2) \cdots c_1(\overline{L}_{n+1}).$$

This defines the metric of $\langle L_1, \cdots, L_{n+1}\rangle$ by induction on $\dim X$.

We have a few remarks to justify this definition. First, the integral is a sum of integrals on X_i with weight $\delta(X_i)$, so we only need to consider the pull-back of the measure $c_1(\overline{L}_2) \cdots c_1(\overline{L}_{n+1})$ to X_i. Second, the measure $c_1(\overline{L}_2) \cdots c_1(\overline{L}_{n+1})$ (over X_i) is regularized by [BT, Thm. 2.1] (or [Dem1, Cor. 1.6]), and the integral on the right-hand side is convergent by [CT, Thm. 4.1]. Third, there is a Stokes formula as follows.

Lemma 4.2.1 (Stokes formula). *If (s_1, s_2) is a strongly regular sequence of sections of (L_1, L_2) on X. Then*

$$\int_X \log \|s_1\| c_1(\overline{L}_2) c_1(\overline{L}_3) \cdots c_1(\overline{L}_{n+1}) - \int_X \log \|s_2\| c_1(\overline{L}_1) c_1(\overline{L}_3) \cdots c_1(\overline{L}_{n+1})$$
$$= \int_{\mathrm{div}(s_2)} \log \|s_1\| c_1(\overline{L}_3) \cdots c_1(\overline{L}_{n+1}) - \int_{\mathrm{div}(s_1)} \log \|s_2\| c_1(\overline{L}_3) \cdots c_1(\overline{L}_{n+1}).$$

Proof. This is a generalization of [Elk2, I.1.3]. If X is integral, the formula holds for integrable metrics by a regularization process, or as an easy consequence of [CT, Thm. 4.1]. If X is not integral, by the case of integral schemes, it suffices to check that for any irreducible component V of $\mathrm{div}(s_1)$, endowed with the reduced structure,

$$\delta(V, \mathrm{div}(s_1)) = \sum_{i=1}^{r} \delta(V, \mathrm{div}(s_1|_{X_i}))\delta(X_i, X).$$

This is a consequence of [BLR, §9.1, Lem. 6] by setting $A = \mathcal{O}_{X,\eta_V}$, $M = A$ and a to be a defining equation of V in A. Here η_V denotes the generic point of V. $\square$

With the Stokes formula, as in [Elk2, Thm. I.1.1(c)], we can prove that the definition of the metric is independent of the choices of the induction process, and the Deligne pairing with the metric is symmetric and multilinear.

In a single formula, if $(s_1, \cdots, s_{n+1})$ is a strongly regular sequence of sections of $(L_1, \cdots, L_{n+1})$ on X, then the metric of $\langle L_1, \cdots, L_{n+1}\rangle$ is given by

$$-\log \|\langle s_1, \cdots, s_{n+1}\rangle\| = -\sum_{i=1}^{n+1} \int_{Z_{i-1}} \log \|s_i\| c_1(\overline{L}_{i+1}) \cdots c_1(\overline{L}_{n+1}).$$

If X is integral, this is exactly the local intersection number

$$\widehat{\mathrm{div}}(s_1) \cdot \widehat{\mathrm{div}}(s_2) \cdots \widehat{\mathrm{div}}(s_{n+1}).$$

See [CT, §2] or [YZ1, Appendix 1] for basic properties of the local intersection number.

4.2.3 Relation to integral schemes

The following result converts the Deligne pairing of non-integral schemes to those of its irreducible components. It can serve as a substitute of the above treatment of non-integral schemes, and will also be used later.

Lemma 4.2.2. *Let Y be either the spectrum of a field or an integral Dedekind scheme. Let $f : X \to Y$ be a projective and flat morphism of pure relative dimension n. Denote by $X_1, \cdots X_r$ the irreducible components of X, endowed with the reduced structures. Assume that for each $i = 1, \cdots, r$, the morphism $X_i \to Y$ is smooth at the generic point of X_i. Let $L_1, \cdots, L_{n+1}$ be line bundles on X. Then there is a canonical isomorphism*

$$\langle L_1, \cdots, L_{n+1}\rangle \longrightarrow \otimes_{i=1}^{r} \langle L_1|_{X_i}, \cdots, L_{n+1}|_{X_i}\rangle^{\otimes \delta(X_i)}.$$

Moreover, if $Y = \mathrm{Spec}\,\mathbb{C}$, $L_1, \cdots, L_{n+1}$ are endowed with integrable metrics on X, and both sides are endowed with the induced metrics, then the isomorphism is an isometry.

Proof. We will only prove the first statement, as the second statement can be checked through the same process.

Denote by η_i the generic point of X_i, and denote by $\widetilde{X}_i$ the schematic closure of η_i in X. Then we have a birational morphism

$$\coprod_{i=1}^{r} \widetilde{X}_i \longrightarrow X.$$

Apply [MG, Thm. 5.3.1] to this morphism, we have a canonical isomorphism

$$\langle L_1, \cdots, L_{n+1}\rangle \longrightarrow \otimes_{i=1}^{r} \langle L_1|_{\widetilde{X}_i}, \cdots, L_{n+1}|_{\widetilde{X}_i}\rangle.$$

Therefore, it suffices to establish for each i a canonical isomorphism

$$\langle L_1|_{\widetilde{X}_i}, \cdots, L_{n+1}|_{\widetilde{X}_i}\rangle \longrightarrow \langle L_1|_{X_i}, \cdots, L_{n+1}|_{X_i}\rangle^{\otimes \delta(X_i)}.$$

Let $\widetilde{U}$ be an affine open subscheme of $\widetilde{X}_i$ such that the reduced structure $U = (\widetilde{U})_{\mathrm{red}}$ is smooth over Y. By the infinitesimal lifting theorem (cf. [BLR, §2.2, Prop. 6]), the identity morphism $U \to U$ can be lifted to a morphism $\phi : \widetilde{U} \to U$ over Y. Replacing $\widetilde{U}$ by an open subscheme if necessary, we can further assume that $\phi : \widetilde{U} \to U$ is flat. Note that $\phi : \widetilde{U} \to U$ is finite automatically. By the morphism ϕ, a little argument gives

$$\delta(X_i) = \delta(U, \widetilde{U}) = \deg(\phi).$$

The morphism $\widetilde{U} \to U$ gives a rational map $\widetilde{X}_i \dashrightarrow X_i$. By blowing-up $\widetilde{X}_i$, the rational map becomes a morphism $\widetilde{X}_i' \dashrightarrow X_i$. Apply the Raynaud–Gruson flattening theorem in [RG, Thm. 5.2.2]. We can further blow-up $\widetilde{X}_i'$ and X_i to change the rational map into a flat morphism $\psi : \widetilde{Z} \to Z$. Here $\widetilde{U}$ and U are respectively open subschemes of $\widetilde{Z}$ and Z, and $\widetilde{Z} \to Z$ extends the morphism $\widetilde{U} \to U$. We can assume that Z is normal by taking the base change of $\widetilde{Z} \to Z$ by the normalization of Z. The morphism $(\widetilde{Z})_{\mathrm{red}} \to Z$ is finite, birational, and equi-dimensional, so it must be an isomorphism. Note that the blowing-up does not affect the Deligne pairings by [MG, Thm. 5.3.1].

Now it suffices to establish a canonical isomorphism

$$\tau : \langle \psi^* M_1, \cdots, \psi^* M_{n+1}\rangle \longrightarrow \langle M_1, \cdots, M_{n+1}\rangle^{\otimes \deg(\psi)}$$

for line bundles $M_1, \cdots, M_{n+1}$ on Z. Here $\psi : \widetilde{Z} \to Z$ is the finite and flat morphism. Write the isomorphism in the form

$$\langle \psi^* M_1, \cdots, \psi^* M_{n+1} \rangle \longrightarrow \langle M_1, \cdots, M_n, N_{\widetilde{Z}/Z}(\psi^* M_{n+1}) \rangle$$

This isomorphism follows from the projection formula of [MG, Prop. 5.2.3.b].

Now we have established the desired isomorphism. We can also check that the isomorphism is independent of the choice of $\widetilde{U} \to U$. In fact, the morphism τ sends the section $\langle s_1, \cdots, s_{n+1} \rangle$ to the section $\langle s_1|_Z, \cdots, s_{n+1}|_Z \rangle^{\otimes \delta(Z)}$, for any strongly regular sequence $(s_1, \cdots, s_{n+1})$ of sections of $(\psi^* M_1, \cdots, \psi^* M_{n+1})$ on $\widetilde{Z}$ which can be descended to a strongly regular sequence of sections of $(M_1, \cdots, M_{n+1})$ on Z. Then we can check the independence by comparing different strongly regular sequences. $\square$

4.2.4 Deligne Pairing: metrics in a family

Let $f : X \to Y$ be a projective morphism of quasi-projective varieties over $\mathbb{C}$ of finite Tor-dimension and pure relative dimension n. Let $\overline{L}_1, \cdots, \overline{L}_{n+1}$ be line bundles on X, endowed with integrable metrics. The goal is to endow a natural metric of $\langle L_1, \cdots, L_{n+1} \rangle$ on Y in this general setting.

For any closed point $y \in Y$, we have a canonical metric $\| \cdot \|_{X_y}$ of $\langle L_{1,y}, \cdots, L_{n+1,y} \rangle$ at y. This is just the above construction applied to $f_y : X_y \to y$. By the canonical isomorphism

$$\langle L_1, \cdots, L_{n+1} \rangle_y \longrightarrow \langle L_{1,y}, \cdots, L_{n+1,y} \rangle,$$

we get a natural metric of the left-hand side. Varying y, this gives a "metric" of $\langle L_1, \cdots, L_{n+1} \rangle$ on Y. Denote this metric by $\| \cdot \|_{X/Y,\mathrm{fibral}}$, to indicate that it is fiberwise defined. The metric is not a priori continuous. The main result of this subsection asserts that it is indeed continuous if f is flat, and can be "modified" to a continuous one if Y is normal.

Theorem 4.2.3. *Let $f : X \to Y$ be a projective morphism of quasi-projective varieties over $\mathbb{C}$ of finite Tor-dimension and pure relative dimension n. Assume that either f is flat or Y is normal. Let $\overline{L}_1, \cdots, \overline{L}_{n+1}$ be line bundles on X, endowed with integrable metrics. Then there is a continuous integrable metric $\| \cdot \|_{X/Y}$ of $\langle L_1, \cdots, L_{n+1} \rangle$ on Y satisfying the following properties.*

(1) Let V be the maximal open subscheme of Y such that X_V is flat over V. Then the metric $\| \cdot \|_{X/Y}$ is equal to the metric $\| \cdot \|_{X/Y,\mathrm{fibral}}$ at all fibers of $\langle L_1, \cdots, L_{n+1} \rangle$ above V.

(2) *The metric $\|\cdot\|_{X/Y}$ is compatible with base changes by morphisms $Y' \to Y$ of quasi-projective varieties such that the image of Y' intersects V and that $X \times Y' \to Y'$ has finite Tor-dimension.*

(3) *The metric $\|\cdot\|_{X/Y}$ is symmetric and multi-linear in the components $L_1, \cdots, L_{n+1}$.*

(4) *The Chern current*

$$c_1(\langle L_1, \cdots, L_{n+1}\rangle, \|\cdot\|_{X/Y}) = f_*(c_1(\overline{L}_1)\cdots c_1(\overline{L}_{n+1}))$$

as $(1,1)$-currents on Y.

(5) *If the metrics of $\overline{L}_1, \cdots, \overline{L}_{n+1}$ are semipositive, then the metric $\|\cdot\|_{X/Y}$ of $\langle L_1, \cdots, L_{n+1}\rangle$ is also semipositive.*

If f is flat, then we have $\|\cdot\|_{X/Y} = \|\cdot\|_{X/Y,\mathrm{fibral}}$ everywhere, so $\|\cdot\|_{X/Y}$ is continuous. In this case, part (2) holds for any base change $Y' \to Y$.

In general, $\|\cdot\|_{X/Y}$ is determined by $\|\cdot\|_{X/Y,\mathrm{fibral}}$ by continuity, but it may happen that they are not equal.

In the theorem, by continuity, (1) determines the metric uniquely and implies (2) and (3). It is also easy to see that (4) implies (5). Thus the task is to prove that the metric $\|\cdot\|_{X/Y}$ determined by (1) exists and also satisfies (4). The proof of these two parts will be given in the next subsection.

4.3 Metrics of the Deligne pairing: proofs

The goal of this subsection is to prove Theorem 4.2.3. The idea is to apply the classical analytic results of Stoll and King to treat continuity of relative integrals.

4.3.1 Continuity of relative integral

As a preparation to prove Theorem 4.2.3, we first convert classical results of Stoll [Sto1, Sto2] and King [Kin] into the following statement.

Theorem 4.3.1. *Let $f : X \to Y$ be a projective morphism of quasi-projective varieties over $\mathbb{C}$ of pure relative dimension n. Let α be a continuous differential (n,n)-form on X. Denote by $I_{X/Y} : Y(\mathbb{C}) \to \mathbb{R}$ the function defined by*

$$I_{X/Y}(y) = \int_{X_y} \alpha, \qquad y \in Y(\mathbb{C}).$$

The following are true:

(1) If f is flat, then $I_{X/Y}(y)$ is continuous for all $y \in Y(\mathbb{C})$.

(2) If Y is normal, there is a unique continuous function $\tilde{I}_{X/Y} : Y(\mathbb{C}) \to \mathbb{R}$ such that $\tilde{I}_{X/Y}(y) = I_{X/Y}(y)$ for all $y \in Y(\mathbb{C})$ over which X is flat.

For a singular complex variety, there are notions of continuous differential forms and smooth differential forms in [Kin, §1.1]. Some of these are recalled in §2.1.

Recall that by definition, the integration

$$I_{X/Y}(y) = \int_{X_y} \alpha = \sum_{i=1}^{r} \delta(W_i) \int_{W_i} \alpha|_{W_i},$$

where $W_1, \cdots, W_r$ are irreducible components of X_y endowed with reduced structures, and $\delta(W_i)$ is the multiplicity of W_i in X_y introduced in last subsection.

We first recall some results of Stoll [Sto1, Sto2]. Let $f : X \to Y$ and α be as in the above proposition. Assume furthermore that X is regular and Y is normal. Then [Sto2, Thm. 3.9] asserts that the integral

$$I^*_{X/Y}(y) = \int_{(X_y)_{\mathrm{red}}} \nu_f \alpha$$

defines a continuous function of $y \in Y(\mathbb{C})$. Here the multiplicity function $\nu_f : (X_y)_{\mathrm{red}}(\mathbb{C}) \to \mathbb{Z}$ is defined in [Sto1, p. 17, p. 48]. Instead of reviewing the definitions of the multiplicity function, we first state the following result, which is sufficient for our application, and then we review some details on the multiplicity function in the proof.

Lemma 4.3.2. *Let $f : X \to Y$ be a projective morphism of smooth varieties over $\mathbb{C}$ of pure relative dimension n. Let $y \in Y(\mathbb{C})$ be a closed point. Then the following holds:*

(1) For any smooth (closed) point x of $(X_y)_{\mathrm{red}}$, we have $\nu_f(x) = \delta(W(x), X_y)$. Here $W(x)$ is the irreducible component of $(X_y)_{\mathrm{red}}$ containing x.

(2) For any continuous differential (n,n)-form on X, we have

$$\int_{(X_y)_{\mathrm{red}}} \nu_f \alpha = \int_{X_y} \alpha.$$

Proof. Note that (1) implies (2), since it implies $\nu_f(x) = \delta(W(x), X_y)$ for x outside a subset of $(X_y)_{\text{red}}$ of measure 0.

Now we prove the case $n = 0$ of (1). For the purpose later, we will prove the following slightly more general statement:

Let $f : X \to Y$ be a morphism of varieties over $\mathbb{C}$ with $\dim X = \dim Y$. Let $x \in X$ and $y \in Y$ be closed points with $f(x) = y$. Assume that X is smooth at x and that Y is smooth at y. Assume that x is an isolated point of X_y, i.e., $\{x\}$ is a connected component of X_y. Then

$$\nu_f(x) = \dim_{\mathbb{C}}(\mathcal{O}_{X,x}/m_y\mathcal{O}_{X,x}).$$

Here m_x (resp. m_y) denotes the maximal ideal of $\mathcal{O}_{X,x}$ (resp. $\mathcal{O}_{Y,y}$).

For a brief definition of $\nu_f(x)$, recall that there is an open neighborhood U of x under the analytic topology such that $U \to f(U)$ is proper and $f^{-1}(y) \cap U = \{x\}$. Then $\nu_f(x)$ is the degree of $U \to f(U)$, i.e., the common order of $f^{-1}(f(z))$ for any $z \in U \setminus R$, where R is an analytic subset of U of positive codimension. See also [Mum2, Chap. 3, Def. 3.12].

Denote by $\mathcal{O}^{\text{an}}_{X,x}$ (resp. $\mathcal{O}^{\text{an}}_{Y,y}$) the local ring of germs of analytic functions at a point $x \in X(\mathbb{C})$ (resp. $y \in Y(\mathbb{C})$). By [Mum2, Appendix to Chap. 6, Thm. A.8], the formula of Weil gives

$$\nu_f(x) = \operatorname{rank}_{\mathcal{O}^{\text{an}}_{Y,y}} \mathcal{O}^{\text{an}}_{X,x}.$$

Here by convention, the rank of an R-module M for an integral domain R means the dimension of the base change of M to the fraction field of R.

Note that $\mathcal{O}^{\text{an}}_{X,x}$ is a finite module over $\mathcal{O}^{\text{an}}_{Y,y}$ by [Mum2, Appendix to Chap. 6, Prop. A.7]. As a consequence,

$$\nu_f(x) = \operatorname{rank}_{\widehat{\mathcal{O}}_{Y,y}} \widehat{\mathcal{O}}_{X,x}.$$

Here $\widehat{\mathcal{O}}_{X,x}$ (resp. $\widehat{\mathcal{O}}_{Y,y}$) is the completion of $\mathcal{O}^{\text{an}}_{X,x}$ (resp. $\mathcal{O}^{\text{an}}_{Y,y}$), which is canonically isomorphic to the completion of $\mathcal{O}_{X,x}$ (resp. $\mathcal{O}_{Y,y}$).

Note that $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$ by the miracle flatness (cf. [Mat, Thm. 23.1]). It follows that $\widehat{\mathcal{O}}_{X,x}$ is flat (and finite) over $\widehat{\mathcal{O}}_{Y,y}$. It follows that

$$\nu_f(x) = \dim_{\mathbb{C}}(\widehat{\mathcal{O}}_{X,x}/m_y\widehat{\mathcal{O}}_{X,x}) = \dim_{\mathbb{C}}(\mathcal{O}_{X,x}/m_y\mathcal{O}_{X,x}).$$

This proves the case $n = 0$.

Now we prove (1) for $n > 0$. The idea is to reduce it to the case $n = 0$. Assume $n > 0$. Fix an irreducible component W of $(X_y)_{\text{red}}$. Denote by

U an affine open subscheme of W which is smooth over $\mathrm{Spec}(\mathbb{C})$. Denote by $\widetilde{U}$ the unique open subscheme of X_y supported on U. Then we have the reduced structure $U = (\widetilde{U})_{\mathrm{red}}$. By the infinitesimal lifting theorem (cf. [BLR, §2.2, Prop. 6]), the identity morphism $U \to U$ can be lifted to a morphism $\phi : \widetilde{U} \to U$. Replacing $\widetilde{U}$ by an open subscheme if necessary, we can further assume that $\phi : \widetilde{U} \to U$ is flat. Note that $\phi : \widetilde{U} \to U$ is finite automatically. By the morphism ϕ, a little argument gives

$$\delta(W, X_y) = \delta(U, \widetilde{U}) = \deg(\phi).$$

This technique to treat the multiplicity is also used in the proof of Lemma 4.2.2.

We are going to prove $\nu_f(x) = \delta(W, X_y)$ for any closed point $x \in U$. This extends to all closed points of W that are smooth in $(X_y)_{\mathrm{red}}$ by [Sto1, Thm. 5.6] about the global multiplicity function.

Let $x \in U$ be any closed point. Let $t_1, \cdots, t_n \in \mathcal{O}_{U,x}$ be a coordinate system; i.e., a minimal set of generators of the maximal ideal of the regular local ring $\mathcal{O}_{U,x}$. For $i = 1, \cdots, n$, denote by $\widetilde{t}_i = \phi^* t_i \in \mathcal{O}_{\widetilde{U},x}$ the pull-back via the morphism $\phi : \widetilde{U} \to U$. Denote by t_i^* a lifting of $\widetilde{t}_i$ in $\mathcal{O}_{X,x}$. Then $t_1^*, \cdots, t_n^*$ are defined on an open neighborhood W of x in X. Finally, denote by Z the closed subscheme of W defined by the equations $t_1^*, \cdots, t_n^*$. The base change of $\phi : \widetilde{U} \to U$ gives a finite and flat morphism $\mathrm{Spec}(\mathcal{O}_{Z \cap X_y, x}) \to x$ of the same degree. It follows that

$$\delta(x, Z \cap X_y) = \deg(\phi) = \delta(W, X_y).$$

On the other hand, by [Sto1, Thm. 5.5], $\nu_f(x) = \nu_{f|_Z}(x)$. By the case $n = 0$ we have just proved, we further have $\nu_{f|_Z}(x) = \delta(x, Z \cap X_y)$. Thus $\nu_f(x) = \delta(W, X_y)$. This finishes the proof. $\square$

Now we can prove Theorem 4.3.1.

Proof of Theorem 4.3.1. In (1), by Lemma 3.6.5, we can take a normalization and take the base change, so we will assume that Y is also normal in (1).

We will start the proof with (2) and then move to (1). Let $f : X \to Y$ be as in (2), so that Y is normal. By [Kin, Thm. 3.3.2], there is a continuous function $\widetilde{I}_{X/Y} : Y(\mathbb{C}) \to \mathbb{R}$ representing the current $f_*\alpha$. Recall that we also have functions $I_{X/Y} : Y(\mathbb{C}) \to \mathbb{R}$ and $I^*_{X/Y} : Y(\mathbb{C}) \to \mathbb{R}$ defined by

$$I_{X/Y}(y) = \int_{X_y} \alpha, \qquad I^*_{X/Y}(y) = \int_{(X_y)_{\mathrm{red}}} \nu_f \alpha.$$

Here we will only need $I^*_{X/Y}(y)$ for the case that X and Y are smooth. We are going to compare $\tilde{I}_{X/Y}$, $I_{X/Y}$ and $I^*_{X/Y}$.

Denote by $\psi : X' \to X$ a generic desingularization of X. Then there is a Zariski open and dense subset V_0 of Y such that V_0 is regular, X is flat over V_0, and X' is smooth over V_0. Shrinking V_0 if necessary, we can assume that for any point $y \in V_0$, the morphism $X'_y \to X_y$ is a birational morphism of reduced schemes. Then for any $y \in V_0(\mathbb{C})$,

$$I_{X/Y}(y) = \int_{X_y} \alpha = \int_{X'_y} \psi^*\alpha = \int_{X'_y} \nu_{f'}\, \psi^*\alpha = I^*_{X'/Y}(y).$$

Here the third equality inequality follows easily from Lemma 4.3.2. By [Sto2, Thm. 3.9], $I^*_{X/Y}(y) = I^*_{X'/Y}(y)$ is continuous in $y \in V_0(\mathbb{C})$. This is also easy to prove directly, since X'_{V_0} is diffeomorphic to a constant family over V_0 by Ehresmann's fibration theorem. By continuity, we have $I_{X/Y}(y) = \widetilde{I}_{X/Y}(y)$ for any $y \in V_0(\mathbb{C})$.

Now let V be the maximal open subscheme of Y such that X_V is flat over V. We need to prove $I_{X/Y}(y) = \widetilde{I}_{X/Y}(y)$ for any $y \in V$. Note that V contains V_0. It suffices to treat the case $Y = V$; i.e. $f : X \to Y$ is flat. This is actually case (1).

By taking a desingularization of Y and taking the base change of f accordingly, we can assume that Y is smooth over $\mathbb{C}$. This uses Lemma 3.6.5 again. Fix a point $y \in Y(\mathbb{C})$. Take a smooth curve $C \subset Y$ passing through y and intersecting V_0. This can be done by successively applying Bertini's theorem. Consider the base change $g : Z \to C$ of $f : X \to Y$ by $C \to Y$. Then g is projective and flat. Since C intersects V_0, the generic fiber of g is integral. Then the flatness of g implies that Z is integral.

Note that we need to prove $I_{X/Y}(y) = \widetilde{I}_{X/Y}(y)$ for all $y \in C(\mathbb{C})$. As they are equal for $y \in C(\mathbb{C}) \cap V_0(\mathbb{C})$, it suffices to prove that $I_{X/Y}(y)$ is continuous in $y \in C(\mathbb{C})$. Since $I_{X/Y}(y) = I_{Z/C}(y)$ for all $y \in C(\mathbb{C})$, we only need to consider everything for the fibration $g : Z \to C$.

If Z is smooth over $\mathbb{C}$, this is a consequence of [Sto2, Thm. 3.9] and Lemma 4.3.2. Otherwise, we need to take a resolution of singularity and check that $I_{Z/C}(y)$ does not change in this process. The advantage of $\dim C = 1$ is that the resolution of singularity does not violate the flatness of Z over C.

By Hironaka's theorem, there is a birational and projective morphism $Z' \to Z$ from a projective and smooth variety Z' over $\mathbb{C}$. We need to check that $I_{Z/C}(y) = I_{Z'/C}(y)$ for any $y \in C(\mathbb{C})$. Let W be an irreducible component of $(Z_y)_{\mathrm{red}}$. Denote by $W_1', \cdots W_a'$ the irreducible components of $(Z_y')_{\mathrm{red}}$ mapping surjectively to W. To prove $I_{Z/C}(y) = I_{Z'/C}(y)$, by pull-back of integrals, it suffices to prove

$$\delta(W, Z_y) = \sum_{i=1}^{a} \delta(W_i', Z_y') \deg(W_i'/W).$$

Take a finite morphism $Z \to \mathbb{P}^n_C$ over C, which exists by replacing C by a Zariski open cover. The construction is similar to the construction of the morphism $\mathcal{U}_{O_{F_\wp}} \to \mathbb{P}^d_{O_{F_\wp}}$ in the proof of Lemma 3.1.1, so we will not repeat it here. Denote $Z_0 = \mathbb{P}^n_C$ in the following.

Denote by η_0 (resp. η, η_i') the generic point of $Z_{0,y}$ (resp. W and W_i'). Denote by $\mathcal{O}_{C,y}$, $\mathcal{O}_{Z_0,\eta_0}$, $\mathcal{O}_{Z,\eta}$ the local rings. Denote by $\mathcal{O}_{Z',\eta}$ the base change $\mathcal{O}_{Z'} \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z,\eta}$, which is the semi-local ring of Z' at the points $\eta_1', \cdots, \eta_a'$. All these rings are integral domains of dimension 1. Moreover, $\mathcal{O}_{C,y}$ and $\mathcal{O}_{Z_0,\eta_0}$ are discrete valuation rings. Then $\mathcal{O}_{Z',\eta}$ and $\mathcal{O}_{Z,\eta}$ are finite and flat over $\mathcal{O}_{Z_0,\eta_0}$.

The inclusion $\mathcal{O}_{Z,\eta} \to \mathcal{O}_{Z',\eta}$ gives the same fraction fields, since it comes from the birational morphism $Z' \to Z$. As a consequence, $\mathcal{O}_{Z,\eta}$ and $\mathcal{O}_{Z',\eta}$ have the same rank over $\mathcal{O}_{Z_0,\eta_0}$. Computing the degrees between the fibers above y, we have

$$\deg(\operatorname{Spec} \mathcal{O}_{Z,\eta}/\operatorname{Spec} \mathcal{O}_{Z_0,\eta_0}) = \delta(W) \deg(\eta/\eta_0)$$

and

$$\deg(\operatorname{Spec} \mathcal{O}_{Z',\eta}/\operatorname{Spec} \mathcal{O}_{Z_0,\eta_0}) = \sum_{i=1}^{a} \delta(W_i') \deg(\eta_i'/\eta_0).$$

The equality of these two degrees gives the desired result. The proof of Theorem 4.3.1 is complete. □

4.3.2 Deligne Pairing: patching metrics

Now we prove Theorem 4.2.3. The major task is to prove part (1) of the theorem. Note that we have two cases: f is flat, or Y is normal. These correspond to the two cases of Theorem 4.3.1.

For convenience, denote

$$\langle \overline{L}_1, \cdots, \overline{L}_{n+1} \rangle_{\mathrm{fibral}} = \langle \overline{L}_1, \cdots, \overline{L}_{n+1} \rangle_{X/Y,\mathrm{fibral}} = (\langle L_1, \cdots, L_{n+1} \rangle, \| \cdot \|_{X/Y,\mathrm{fibral}})$$

and

$$\langle \overline{L}_1, \cdots, \overline{L}_{n+1} \rangle = \langle \overline{L}_1, \cdots, \overline{L}_{n+1} \rangle_{X/Y} = (\langle L_1, \cdots, L_{n+1} \rangle, \| \cdot \|_{X/Y})$$

in the following, where the second metric is the continuous one to be constructed.

Find a smooth metric. By multi-linearity, we can assume that $L_1, \cdots, L_{n+1}$ are all isomorphic to the same f-ample line bundle L on X; see [Mor2, §1, Step 2] for the argument for this reduction process. Of course, the metrics of $\overline{L}_i$ are allowed to be very different.

We first claim that, up to passing to a Zariski open cover of Y, there exists a smooth metric $\| \cdot \|$ of L, such that the induced metric $\| \cdot \|_{X/Y,\mathrm{fibral}}$ of $\langle L, \cdots, L \rangle$ is also smooth.

In fact, replacing Y by a Zariski open cover and replacing L by a tensor power if necessary, we can assume that there is a finite morphism $\psi : X \to \mathbb{P}^n_Y$ over Y such that $\psi^* \mathcal{O}_{\mathbb{P}^n_Y}(1) \simeq L$. The construction is similar to the construction of the morphism $\mathcal{U}_{O_{F_\wp}} \to \mathbb{P}^d_{O_{F_\wp}}$ in the proof of Lemma 3.1.1, so we will not repeat here.

Denote $\overline{M}_0 = (\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1), \| \cdot \|_{\mathrm{FS}})$ with the Fubini-Study metric $\| \cdot \|_{\mathrm{FS}}$ on $\mathbb{P}^n_{\mathbb{C}}$. Denote $\overline{M} = p^* \overline{M}_0$, where $p : \mathbb{P}^n_Y \to \mathbb{P}^n_{\mathbb{C}}$ is the projection. Denote $\overline{L} = \psi^* \overline{M}$, or equivalently $\overline{L} = (L, \| \cdot \|)$ with $\| \cdot \| = (p \circ \psi)^* \| \cdot \|_{\mathrm{FS}}$.

By the base change $q : Y \to \operatorname{Spec} \mathbb{C}$, we have a canonical isometry

$$q^* \langle \overline{M}_0, \cdots, \overline{M}_0 \rangle_{\mathbb{P}^n_{\mathbb{C}}/\mathbb{C},\mathrm{fibral}} \longrightarrow \langle \overline{M}, \cdots, \overline{M} \rangle_{\mathbb{P}^n_Y/Y,\mathrm{fibral}}.$$

As a consequence, the right-hand side is isomorphic to the trivial bundle $\mathcal{O}_Y$ with a constant metric.

There is also a natural isometry

$$\psi^* \langle \overline{M}, \cdots, \overline{M} \rangle_{\mathbb{P}^n_Y/Y,\mathrm{fibral}} \longrightarrow \langle \overline{L}, \cdots, \overline{L} \rangle_{X/Y,\mathrm{fibral}}.$$

In fact, the functoriality gives a natural isomorphism of the underlying line bundles, and a natural isometry on the fibers. The isomorphism and the isometries are compatible.

As a consequence, the metric $\| \cdot \|_{X/Y,\mathrm{fibral}}$ of $\langle L, \cdots, L \rangle_{X/Y}$ is smooth. This gives the requirement.

Compare the metrics. Consider the identity map

$$\gamma : \langle L, \cdots, L \rangle \longrightarrow \langle L_1, \cdots, L_{n+1} \rangle.$$

We first prove Theorem 4.2.3(1) in the case that f is flat. Then it suffices to prove that the norm $\|\gamma\|$ of γ under the fibral metrics is continuous on Y in this case.

For $i > 1$, denote $f_i = -\log(\|\cdot\|_i/\|\cdot\|)$, which is a continuous function on X. Write γ as the composition of

$$\gamma_i : \langle L_1, \cdots L_{i-1}, L, \cdots, L\rangle \longrightarrow \langle L_1, \cdots L_i, L, \cdots, L\rangle.$$

for $i = 1, \cdots, n+1$. The norm of γ at any $y \in Y(\mathbb{C})$ is given by

$$-\log\|\gamma\|(y) = \sum_{i=1}^{n+1} \int_{X_y} f_i c_1(\overline{L}_1)c_1(\overline{L}_2)\cdots c_1(\overline{L}_{i-1})c_1(\overline{L})^{n+1-i}.$$

Denote $d = \partial + \bar\partial$ and $d^c = (\partial - \bar\partial)/(2\pi i)$. Note that some literature normalizes d^c by a denominator $4\pi i$ instead of $2\pi i$. By $c_1(\overline{L}_j) = c_1(\overline{L}) + dd^c f_j$, we see that $-\log\|\gamma\|(y)$ is a linear combination of

$$\int_{X_y} f_i(\wedge^{j\in J} dd^c f_j) \wedge c_1(\overline{L})^{n-|J|}.$$

Here $i \in \{1, \cdots, n+1\}$ and $J \subset \{1, \cdots, i-1, i+1, \cdots, n+1\}$.

We are going to prove that for any $i = 1, \cdots, n+1$, and for any integrable functions $f_1, \cdots, f_i$ on X, the function

$$y \longmapsto \int_{X_y} f_1(dd^c f_2) \wedge \cdots \wedge (dd^c f_i) \wedge c_1(\overline{L})^{n+1-i}$$

is continuous in $y \in Y(\mathbb{C})$. Here an integrable function f on X is a continuous function such that the trivial bundle $\mathcal{O}_X$ with the metric defined by $\|1\| = e^{-f}$ is integrable.

If $f_2, \cdots, f_i$ are all smooth, the continuity is given by Theorem 4.3.1. In general, the strategy is to approximate them by smooth functions. For any $j = 2, \cdots, i$, by the Stokes formula,

$$\begin{aligned} &\int_{X_y} f_1(dd^c f_2) \wedge \cdots \wedge (dd^c f_i) \wedge c_1(\overline{L})^{n+1-i} \\ = &\int_{X_y} f_j(dd^c f_1) \wedge \cdots \wedge (dd^c f_{j-1}) \wedge (dd^c f_{j+1}) \wedge \cdots \wedge (dd^c f_i) \wedge c_1(\overline{L})^{n+1-i}. \end{aligned}$$

This is an easier version of Lemma 4.2.1. Over any compact subset of Y, f_j is a uniform limit of smooth functions on X. Looking the second integral, it suffices to prove the same statement assuming that f_j is smooth. By this method, we can assume that all $f_2, \cdots, f_i$ are all smooth. This proves the continuity for flat f.

In the case that Y is normal (but f is not necessarily smooth), let V be the maximal open subscheme of Y over which X is flat. Then we have already proved that all the relative integrals above are continuous on V, and it suffices to prove that they can extended to continuous functions on Y. This is proved in the same way by Theorem 4.3.1(2).

The Chern current. Once we have part (1) of Theorem 4.2.3, it is easy to obtain part (4) of the theorem. The goal is to prove

$$c_1(\langle L_1, \cdots, L_{n+1}\rangle, \|\cdot\|_{X/Y}) = f_*(c_1(\overline{L}_1)\cdots c_1(\overline{L}_{n+1}))$$

as $(1,1)$-currents on Y. Recall that for any metrized line bundle $(M, \|\cdot\|)$ on Y, the Chern current

$$c_1(M, \|\cdot\|) = dd^c(-\log\|s\|) + \delta_{\mathrm{div}(s)}$$

for any rational section s of M.

Similar to the above, it suffices to prove the formula when all L_i are isomorphic to a single L. In the above, we have the identity map

$$\gamma : \langle L, \cdots, L\rangle \longrightarrow \langle L_1, \cdots, L_{n+1}\rangle.$$

Then

$$c_1(\langle \overline{L}_1, \cdots, \overline{L}_{n+1}\rangle) = c_1(\langle \overline{L}, \cdots, \overline{L}\rangle) + dd^c(-\log\|\gamma\|).$$

Here if f is not flat, then $\|\cdot\|_{X/Y}$ is not necessarily equal to $\|\cdot\|_{X/Y,\mathrm{fibral}}$ at a subvariety of Y of positive codimension, but the ambiguity can be ignored in the sense of currents.

Note that the identity

$$c_1(\langle \overline{L}, \cdots, \overline{L}\rangle) = f_*(c_1(\overline{L})^n)$$

holds as both sides are 0 since $\overline{L}$ is constructed from a constant family. Considering the expression of $\log\|\gamma\|$ above in terms of the function

$$F(y) = \int_{X_y} f_1(dd^c f_2)\wedge\cdots\wedge(dd^c f_i)\wedge c_1(\overline{L})^{n+1-i}, \qquad y \in Y(\mathbb{C}).$$

It suffices to prove that

$$dd^c F = f_*((dd^c f_1) \wedge \cdots \wedge (dd^c f_i) \wedge c_1(\overline{L})^{n+1-i}).$$

Denote $d = \dim Y$. For any smooth and compactly supported $(d-1, d-1)$-form α on Y, we have by definition

$$\langle dd^c F,\ \alpha\rangle = \int_Y F dd^c \alpha.$$

By the expression of F, the right-hand side is equal to

$$\int_X f_1(dd^c f_2) \wedge \cdots \wedge (dd^c f_i) \wedge c_1(\overline{L})^{n+1-i} \wedge f^* dd^c \alpha.$$

By the Stokes formula, this becomes

$$\int_X (dd^c f_1) \wedge (dd^c f_2) \wedge \cdots \wedge (dd^c f_i) \wedge c_1(\overline{L})^{n+1-i} \wedge f^* \alpha,$$

which is exactly

$$\langle f_*((dd^c f_1) \wedge \cdots \wedge (dd^c f_i) \wedge c_1(\overline{L})^{n+1-i}),\ \alpha\rangle.$$

As α is an arbitrary test form, this finishes the proof.

4.4 Positivity of the Deligne pairing

In this subsection, we consider the Deligne pairing for projective varieties in both the geometric case and the arithmetic case. We will focus on some positivity results for the purpose later. For simplicity, we will only focus on the flat case. For clarity, we do not take the uniform terminology here.

4.4.1 Geometric case

The following easy result asserts that Deligne pairing sends nef (resp. ample) line bundles to nef (resp. ample) line bundles. It was proved by Nakayama [Nak, Cor. 4.6], but we provide a more direct proof here.

Lemma 4.4.1. *Let $f : X \to Y$ be a flat morphism of relative dimension n of projective varieties of over a field k. Let $L_1, \cdots, L_{n+1}$ be line bundles on X. Then the following are true:*

(1) If $\dim Y = 1$*, then*

$$\deg(\langle L_1, \cdots, L_{n+1}\rangle) = L_1 \cdot L_2 \cdots L_{n+1}.$$

(2) If $L_1, \cdots, L_{n+1}$ *are nef, then* $\langle L_1, \cdots, L_{n+1}\rangle$ *is nef.*

(3) If $L_1, \cdots, L_{n+1}$ *are ample, then* $\langle L_1, \cdots, L_{n+1}\rangle$ *is ample.*

Proof. For (1), we can assume that Y is regular by taking its normalization (and taking the corresponding base change of $X \to Y$). We can assume that X is normal by taking its normalization and applying [MG, Thm. 5.3.1]. By linearity, we can assume that $L_1, \cdots, L_{n+1}$ are very ample on X. The intersection number of $(L_1, \cdots, L_{n+1})$ on X is equal to $\deg(\mathrm{div}(s_1) \cap \cdots \cap \mathrm{div}(s_{n+1}))$ for a strongly regular sequence $(s_1, \cdots, s_{n+1})$ of sections of $(L_1, \cdots, L_{n+1})$ on X. There are many Bertini-type of results to guarantee the existence of strongly regular sequences in the current situation. The quickest one is the Bertini-type of theorem of Seidenberg [Sei] for normal varieties. Then (1) holds essentially by definition.

To prove (2), it suffices to prove that $\langle L_1, \cdots, L_{n+1}\rangle$ has a non-negative degree on any closed integral curves C in Y. Take the base change of $X \to Y$ by $C \to Y$. It suffices to compute the degree of the Deligne pairing for $X_C \to C$. If X_C is integral, this is just (1). If X_C is not integral, take a finite and flat base change $C' \to C$ for some regular projective curve C' so that the reduced structure of $X_{C'}$ is smooth over C' at all the generic points of $X_{C'}$. Then we can apply Lemma 4.2.2 to convert to the integral case in (1).

To prove (3), assume that $L_1, \cdots, L_{n+1}$ are ample on X. Let L be an ample line bundle on Y. By [MG, Prop. 5.2.1], $\langle f^*L, L_2, \cdots, L_{n+1}\rangle$ is a positive multiple of L and thus is ample. Replacing L_1 by a multiple if necessary, we can assume that $L_1 - f^*L$ is ample on X. Now we have

$$\langle L_1, L_2, \cdots, L_{n+1}\rangle \simeq \langle f^*L, L_2, \cdots, L_{n+1}\rangle + \langle L_1 - f^*L, L_2, \cdots, L_{n+1}\rangle.$$

The two terms on the right-hand sides are respectively ample and nef, so the left-hand side is ample. This finishes the proof. $\square$

Next we introduce a mixed pairing between Cartier divisors and line bundles in a suitable situation, and consider the effectivity in this situation.

Let $f : X \to Y$ be a projective and flat morphism of integral noetherian schemes of pure relative dimension n. Let $L_1, \cdots, L_n$ be line bundles on X.

Let D be a Cartier divisor on X, and $\mathcal{O}(D)$ be the line bundle associated to D. Let V be a dense and open subvariety of Y, and denote by $U \to V$ the base change of $X \to Y$ by $V \to Y$. Assume that $D|_U$ is the trivial divisor on U, which gives a canonical isomorphism $\mathcal{O}_U \to \mathcal{O}(D)|_U$. There is a canonical isomorphism

$$\langle \mathcal{O}(D), L_1, \cdots, L_n \rangle|_V \longrightarrow \langle \mathcal{O}(D)|_U, L_1|_U, \cdots, L_n|_U \rangle,$$

and canonical isomorphisms

$$\langle \mathcal{O}(D)|_U, L_1|_U, \cdots, L_n|_U \rangle \longrightarrow \langle \mathcal{O}_U, L_1|_U, \cdots, L_n|_U \rangle \longrightarrow \mathcal{O}_V.$$

Here the last map is a special case of [MG, Prop. 5.2.1.a]. Thus we have a canonical isomorphism

$$\mathcal{O}_V \longrightarrow \langle \mathcal{O}(D), L_1, \cdots, L_n \rangle|_V.$$

This defines a rational map

$$\mathcal{O}_Y \dashrightarrow \langle \mathcal{O}(D), L_1, \cdots, L_n \rangle$$

and thus a rational section s of $\langle \mathcal{O}(D), L_1, \cdots, L_n \rangle$. Define our *mixed Deligne pairing* by

$$\langle D, L_1, \cdots, L_n \rangle := \mathrm{div}(s),$$

which is a Cartier divisor on Y, supported on $Y \backslash V$. Note that $\langle D, L_1, \cdots, L_n \rangle$ is multi-linear in $L_1, \cdots, L_n$.

The following result concerns the effectivity of the pairing, which is compatible with the general fact that the intersection number of an effective divisor with nef divisors is non-negative.

Lemma 4.4.2. *Let $f : X \to Y$ be a flat morphism of relative dimension n of projective varieties over a field k. Let $L_1, \cdots, L_n$ be line bundles on X. Let D be a Cartier divisor on X. Let V be a dense and open subvariety of Y, and denote by $U \to V$ the base change of $X \to Y$ by $V \to Y$. Assume that $D|_U$ is the trivial divisor on U. Then the following are true:*

(1) If $D = f^*D_0$ for a Cartier divisor D_0 on Y, then

$$\langle D, L_1, \cdots, L_n \rangle = (L_{1,\eta} \cdot L_{2,\eta} \cdots L_{n,\eta})\, D_0,$$

where $(L_{1,\eta} \cdot L_{2,\eta} \cdots L_{n,\eta})$ is the intersection numbers of $L_1, \cdots, L_n$ on the generic fiber of $f : X \to Y$.

(2) *If Y is normal, D is effective, and $L_1, \cdots, L_n$ are nef, then $\langle D, L_1, \cdots, L_n\rangle$ is effective on Y.*

Proof. Note that (1) is a consequence of [MG, Prop. 5.2.1.a]. For (2), we first assume that $L_1, \cdots, L_n$ are ample on X. As Y is normal, by passing to Weil divisors, $\langle D, L_1, \cdots, L_n\rangle$ is effective on Y if and only if some positive multiple of it is effective. Thus we can replace $L_1, \cdots, L_n$ by positive multiples if necessary. Therefore, passing to a Zariski open cover of Y, we can find a strongly regular sequence $(s_1, \cdots, s_n)$ of sections of $(L_1, \cdots, L_n)$ on X. By the induction formula, this reduces the problem to $Z_n = \mathrm{div}(s_1)\cap\cdots\cap\mathrm{div}(s_n)$. Then the effectivity follows since the norm map from Z_n to Y sends global sections to global sections by [MG, Prop. 1.2.4(4)]. This proves the ample case.

Now we consider the case that $L_1, \cdots, L_n$ are nef on X. Let A be an ample line bundle on X. Then we have proved that

$$D_m = \langle D, mL_1 + A, \cdots, mL_n + A\rangle$$

is effective for all positive integers m. Note that D_m is a linear combination of the finitely many prime divisors of Y supported on $Y \setminus V$. Then

$$D = \lim_{m\to\infty} m^{-n} D_m$$

is effective. □

4.4.2 Arithmetic case

Now we consider the arithmetic analogues of the above results. Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of relative dimension n of projective arithmetic varieties (over $\mathbb{Z}$). Let $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}$ be hermitian line bundles with integrable metrics on $\mathcal{X}$. Define their Deligne pairing

$$\langle \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}\rangle := (\langle \mathcal{L}_1, \cdots, \mathcal{L}_{n+1}\rangle, \|\cdot\|_{X/Y})$$

Here the metric on the right-hand side is given by Theorem 4.2.3. This defines a functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})^{n+1}_{\mathrm{int}} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{Y})_{\mathrm{int}}.$$

Following [YZ1, Appendix 1, Def. 5.3], we say that a hermitian line bundle $\overline{\mathcal{L}}$ on a projective variety $\pi : \mathcal{X} \to \mathrm{Spec}\,\mathbb{Z}$ is *arithmetically positive* if the following holds:

(1) the generic fiber $\mathcal{L}_{\mathbb{Q}}$ is ample on $\mathcal{X}_{\mathbb{Q}}$;

(2) there exist a hermitian line bundle $\overline{\mathcal{N}}$ on $\operatorname{Spec}\mathbb{Z}$ with $\widehat{\deg}(\overline{\mathcal{N}}) > 0$ such that $\overline{\mathcal{L}} - \pi^*\overline{\mathcal{N}}$ is nef on $\mathcal{X}$.

Zhang's arithmetic Nakai–Moishezon theorem holds for arithmetically positive hermitian line bundles, see [Zha1, Cor. 4.8] and [Mor7, Cor. 5.1].

Lemma 4.4.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of relative dimension n of projective arithmetic varieties (over $\mathbb{Z}$). Let $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}$ be hermitian line bundles with integrable metrics on $\mathcal{X}$. Then the following are true:*

(1) If $\dim\mathcal{Y} = 1$, then

$$\deg(\langle\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}\rangle) = \overline{\mathcal{L}}_1 \cdot \overline{\mathcal{L}}_2 \cdots \overline{\mathcal{L}}_{n+1}.$$

(2) If $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}$ are nef, then $\langle\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}\rangle$ is nef.

(3) If $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}$ are arithmetically positive, then $\langle\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}\rangle$ is arithmetically positive.

Proof. By Theorem 4.2.3, the Deligne pairing of semipositive metrics is semipositive. The other parts of the proof are similar to that of Lemma 4.4.1. We omit it here. □

Now we introduce the arithmetic counterpart of Lemma 4.4.2. The situation is more or less included in the geometric case, except that there is an extra metric involved.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a projective and flat morphism of projective arithmetic varieties of pure relative dimension n. Let $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n$ be hermitian line bundles on $\mathcal{X}$ with integrable metrics. Let $\overline{\mathcal{D}}$ be an arithmetic divisor on $\mathcal{X}$, with an integrable Green's function, and $\mathcal{O}(\overline{\mathcal{D}})$ be the hermitian line bundle associated to $\overline{\mathcal{D}}$. Let $\mathcal{V}$ be a dense and open subvariety of $\mathcal{Y}$, and denote by $\mathcal{U} \to \mathcal{V}$ the base change of $\mathcal{X} \to \mathcal{Y}$ by $\mathcal{V} \to \mathcal{Y}$. Assume that $\mathcal{D}|_{\mathcal{U}}$ is the trivial divisor on $\mathcal{U}$. As in the geometric case, we have a rational map

$$\mathcal{O}_{\mathcal{Y}} \dashrightarrow \langle\mathcal{O}(\mathcal{D}), \mathcal{L}_1, \cdots, \mathcal{L}_n\rangle$$

and thus a rational section s of $\langle\mathcal{O}(\mathcal{D}), \mathcal{L}_1, \cdots, \mathcal{L}_n\rangle$. Define our *mixed Deligne pairing* by

$$\langle\overline{\mathcal{D}}, \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n\rangle := \widehat{\operatorname{div}}(s) = (\operatorname{div}(s), -\log\|s\|),$$

which is an arithmetic divisor on $\mathcal{Y}$. The Green's function uses the canonical metric of the Deligne pairing, which is simply given by

$$-\log \|s\| = \int_{\mathcal{X}(\mathbb{C})} g_{\overline{\mathcal{D}}}\, c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_n).$$

As in the geometric case, $\langle \overline{\mathcal{D}}, \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n \rangle$ is multi-linear in $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n$.

The following effectivity result is the arithmetic version of Lemma 4.4.2.

Lemma 4.4.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a flat morphism of relative dimension n of projective arithmetic varieties (over $\mathbb{Z}$). Let $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n$ be hermitian line bundles with integrable metrics on $\mathcal{X}$. Let $\overline{\mathcal{D}}$ be an arithmetic divisor on $\mathcal{X}$ with an integrable Green's function. Let $\mathcal{V}$ be a dense and open subvariety of $\mathcal{Y}$, and denote by $\mathcal{U} \to \mathcal{V}$ the base change of $\mathcal{X} \to \mathcal{Y}$ by $\mathcal{V} \to \mathcal{Y}$. Assume that $\mathcal{D}|_{\mathcal{U}}$ is the trivial divisor on $\mathcal{U}$. Then the following are true:*

(1) If $\overline{\mathcal{D}} = f^* \overline{\mathcal{D}}_0$ for an arithmetic divisor $\overline{\mathcal{D}}_0$ on $\mathcal{Y}$, then

$$\langle \overline{\mathcal{D}}, \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n \rangle = (\mathcal{L}_{1,\eta} \cdot \mathcal{L}_{2,\eta} \cdots \mathcal{L}_{n,\eta})\, \overline{\mathcal{D}}_0,$$

where $(\mathcal{L}_{1,\eta} \cdot \mathcal{L}_{2,\eta} \cdots \mathcal{L}_{n,\eta})$ is the intersection numbers of $\mathcal{L}_1, \cdots, \mathcal{L}_n$ on the generic fiber of $f : \mathcal{X} \to \mathcal{Y}$.

(2) If $\mathcal{Y}$ is normal, $\overline{\mathcal{D}}$ is effective, and $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n$ are nef, then $\langle \overline{\mathcal{D}}, \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n \rangle$ is effective on $\mathcal{Y}$.

Proof. This is similar to Lemma 4.4.2. In (2), the Green's function

$$-\log \|s\| = \int_{\mathcal{X}(\mathbb{C})} g_{\overline{\mathcal{D}}}\, c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_n)$$

is positive, since the current $c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_n)$ is positive by the nefness of $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_n$. □

4.5 Deligne pairing of adelic line bundles

Now we are ready to prove Theorem 4.1.3. With the preparation in the previous subsections, the proof here is similar to that of Proposition 4.1.1.

Proof of Theorem 4.1.3. Note that

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{int}} = \varinjlim_{\mathcal{U}\to\mathcal{V}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{int}},$$

where the direct limit is over all quasi-projective models $\mathcal{U} \to \mathcal{V}$ of $X \to Y$, i.e., projective and flat morphisms $\mathcal{U} \to \mathcal{V}$ extending $X \to Y$, where $\mathcal{U}$ and $\mathcal{V}$ are quasi-projective models of X and Y over k. In fact, similar to Lemma 2.3.3, for any quasi-projective models $\mathcal{U}$ and $\mathcal{V}$ of X and Y, the rational map $\mathcal{U} \dashrightarrow \mathcal{V}$ can be turned into a projective and flat morphism by shrinking $\mathcal{U}$ and $\mathcal{V}$ suitably. We can further assume that $\mathcal{V}$ is normal.

Therefore, it suffices to prove the results for projective and flat morphisms $f : \mathcal{U} \to \mathcal{V}$ of quasi-projective varieties $\mathcal{U}, \mathcal{V}$ over k, where $\mathcal{V}$ is assumed to be normal. We only need to define the functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)^{n+1}_{\mathrm{snef}} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{V}/k)_{\mathrm{snef}}.$$

In fact, the functor is extended to integrable adelic line bundles by linearity. To extend it to nef adelic line bundles, it suffices to check that if $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}$ are nef on $\mathcal{U}/k$, then $\overline{\mathcal{M}} = \langle \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1} \rangle \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{V}/k)_{\mathrm{int}}$ is nef over $\mathcal{V}/k$. In fact, there is a strongly nef adelic line bundle $\overline{\mathcal{N}}$ on $\mathcal{U}/k$ such that $\overline{\mathcal{L}}_1 + a\overline{\mathcal{N}}, \cdots, \overline{\mathcal{L}}_{n+1} + a\overline{\mathcal{N}}$ are strongly nef for all positive rational numbers a. It follows that

$$\overline{\mathcal{M}}_a = \langle \overline{\mathcal{L}}_1 + a\overline{\mathcal{N}}, \cdots, \overline{\mathcal{L}}_{n+1} + a\overline{\mathcal{N}} \rangle$$

is strongly nef all positive rational numbers a. Expanding it in terms of powers of a, we see that

$$\overline{\mathcal{M}}_a = \overline{\mathcal{M}} + a\overline{\mathcal{N}}_1 + \cdots + a^{n+1}\overline{\mathcal{N}}_{n+1}$$

is strongly nef for integrable adelic line bundles $\overline{\mathcal{N}}_1, \cdots, \overline{\mathcal{N}}_{n+1}$ on $\mathcal{V}/k$. By integrability, there is a strongly nef adelic line bundle $\overline{\mathcal{K}}$ such that $\overline{\mathcal{K}} - \overline{\mathcal{N}}_1, \cdots, \overline{\mathcal{K}} - \overline{\mathcal{N}}_{n+1}$ are strongly nef. As a consequence, $\overline{\mathcal{M}} + (a + \cdots + a^{n+1})\overline{\mathcal{K}}$ is strongly nef. This implies that $\overline{\mathcal{M}}$ is nef.

Now we construct the functor for strongly nef adelic line bundles. For the sake of the boundary topology, let $(\mathcal{Y}_0, \overline{\mathcal{E}}_0)$ be a boundary divisor of $\mathcal{V}$ over k. Assume that there is a projective model $\mathcal{X}_0$ of $\mathcal{U}$ with a morphism $f_0 : \mathcal{X}_0 \to \mathcal{Y}_0$ extending $f : \mathcal{U} \to \mathcal{V}$. Then $(\mathcal{X}_0, f^*\overline{\mathcal{E}}_0)$ is a boundary divisor of $\mathcal{U}$ over k.

Let $\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}$ be objects of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{snef}}$. For each $j = 1, \cdots, n+1$, suppose that $\overline{\mathcal{L}}_j$ is represented by a Cauchy sequence $(\mathcal{L}_j, (\mathcal{X}_i, \overline{\mathcal{L}}_{j,i}, \ell_{j,i})_{i\geq 1})$ with each $\overline{\mathcal{L}}_{j,i}$ nef on a projective model $\mathcal{X}_i$ of $\mathcal{U}$ over k. Here we assume that the integral model $\mathcal{X}_i$ is independent of j, which is always possible. For any $i \geq 1$, assume that there is a projective model $\mathcal{Y}_i$ of $\mathcal{V}$ with a morphism $f_i : \mathcal{X}_i \to \mathcal{Y}_i$ extending $f : \mathcal{U} \to \mathcal{V}$. We assume that for each $i' > i \geq 0$, we have morphisms $\mathcal{X}_{i'} \to \mathcal{X}_i$ and $\mathcal{Y}_{i'} \to \mathcal{Y}_i$ extending the identity maps of $\mathcal{U}$ and $\mathcal{V}$.

Apply the Raynaud–Gruson flattening theorem in [RG, Thm. 5.2.2]. After blowing up $\mathcal{Y}_i$ and replacing $\mathcal{X}_i$ by its pure transform, we can assume that $f_i : \mathcal{X}_i \to \mathcal{Y}_i$ is flat for any $i \geq 0$. By the Deligne pairing, we have a line bundle

$$\mathcal{M} = \langle \mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_{n+1} \rangle$$

over $\mathcal{V}$, and a hermitian $\mathbb{Q}$-line bundle

$$\overline{\mathcal{M}}_i = \langle \overline{\mathcal{L}}_{1,i}, \overline{\mathcal{L}}_{2,i}, \cdots, \overline{\mathcal{L}}_{n+1,i} \rangle$$

over $\mathcal{Y}_i$ for any $i \geq 1$. The isomorphism $\ell_{j,i} : \mathcal{L}_j \to \mathcal{L}_{j,i}|_{\mathcal{U}}$ induces an isomorphism $m_i : \mathcal{M} \to \mathcal{M}_i|_{\mathcal{V}}$ of $\mathbb{Q}$-line bundles on $\mathcal{V}$. By Lemma 4.4.1 and Lemma 4.4.3, each $\overline{\mathcal{M}}_i$ is nef on $\mathcal{Y}_i$.

To prove the theorem, we will define the Deligne pairing $\langle \overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2, \cdots, \overline{\mathcal{L}}_{n+1} \rangle$ to be

$$\overline{\mathcal{M}} = (\mathcal{M}, (\mathcal{Y}_i, \overline{\mathcal{M}}_i, m_i)_{i\geq 1}).$$

For that, we need to check that $(\mathcal{M}, (\mathcal{Y}_i, \overline{\mathcal{M}}_i, m_i)_{i\geq 1})$ is indeed a Cauchy sequence in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{V})_{\mathrm{mod}}$. Then it suffices to prove that $\{\widehat{\mathrm{div}}(m_i m_1^{-1})\}_i$ is a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{V})_{\mathrm{mod}}$.

For any $j = 1, \cdots, n+1$, by the Cauchy condition,

$$-\epsilon_i f_0^* \overline{\mathcal{E}}_0 \leq \widehat{\mathrm{div}}(\ell_{j,i'} \ell_{j,i}^{-1}) \leq \epsilon_i f_0^* \overline{\mathcal{E}}_0, \qquad 1 \leq i \leq i'.$$

Here $\{\epsilon_i\}_{i\geq 1}$ is a sequence of rational numbers converging to zero.

We claim that for any $i < i'$,

$$-\epsilon_i \deg(\mathcal{U}_\eta) \overline{\mathcal{E}}_0 \leq \widehat{\mathrm{div}}(m_{i'} m_i^{-1}) \leq \epsilon_i \deg(\mathcal{U}_\eta) \overline{\mathcal{E}}_0$$

in $\widehat{\mathrm{Div}}(\mathcal{V})_{\mathrm{mod}}$. Here

$$\deg(\mathcal{U}_\eta) = \sum_{j=1}^{n+1} \deg(\mathcal{U}_\eta)_j,$$

with
$$\deg(\mathcal{U}_\eta)_j = \deg(\mathcal{L}_{1,\eta}\cdot\mathcal{L}_{2,\eta}\cdots\mathcal{L}_{j-1,\eta}\cdot\mathcal{L}_{j+1,\eta}\cdots\mathcal{L}_{n+1,\eta}),$$
where $\mathcal{U}_\eta \to \eta$ is the generic fiber of $f:\mathcal{U}\to\mathcal{V}$, and $\mathcal{L}_{j,\eta}$ is the restriction of $\mathcal{L}_j$ to $\mathcal{U}_\eta$.

The situation is similar to the proof of Proposition 4.1.1. Note that the isomorphism $m_{i'}\circ m_i^{-1}:\mathcal{M}_i|_\mathcal{V}\to\mathcal{M}_{i'}|_\mathcal{V}$ is induced by the isomorphism $\ell_{j,i'}\circ\ell_{j,i}^{-1}:\mathcal{L}_{j,i}|_\mathcal{U}\to\mathcal{L}_{j,i'}|_\mathcal{U}$ for $j=1,\cdots,n+1$ via the construction of the Deligne pairing.

In the following, for simplicity of notations, view line bundles on $\mathcal{X}_i$ as line bundles on $\mathcal{X}_{i'}$ via pull-back by abuse of notations. Apply similar conventions to $\mathcal{Y}_i$ and $\mathcal{Y}_{i'}$.

Write the rational map $m_{i'}\circ m_i^{-1}:\mathcal{M}_i\dashrightarrow\mathcal{M}_{i'}$ as a composition of the rational maps
$$t_j:\langle\overline{\mathcal{L}}_{1,i'},\cdots,\overline{\mathcal{L}}_{j-1,i'},\overline{\mathcal{L}}_{j,i},\cdots,\overline{\mathcal{L}}_{n+1,i}\rangle\dashrightarrow\langle\overline{\mathcal{L}}_{1,i'},\cdots,\overline{\mathcal{L}}_{j,i'},\overline{\mathcal{L}}_{j+1,i},\cdots,\overline{\mathcal{L}}_{n+1,i}\rangle$$
for $j=1,\cdots,n+1$, which are induced by the natural isomorphisms on $\mathcal{U}$. View t_j as a rational section of
$$\langle\overline{\mathcal{L}}_{1,i'},\cdots,\overline{\mathcal{L}}_{j,i'},\overline{\mathcal{L}}_{j+1,i},\cdots,\overline{\mathcal{L}}_{n+1,i}\rangle-\langle\overline{\mathcal{L}}_{1,i'},\cdots,\overline{\mathcal{L}}_{j-1,i'},\overline{\mathcal{L}}_{j,i},\cdots,\overline{\mathcal{L}}_{n+1,i}\rangle,$$
which is canonically isomorphic to
$$\overline{\mathcal{N}}_j=\langle\overline{\mathcal{L}}_{1,i'},\cdots,\overline{\mathcal{L}}_{j-1,i'},\overline{\mathcal{L}}_{j,i'}-\overline{\mathcal{L}}_{j,i},\overline{\mathcal{L}}_{j+1,i},\cdots,\overline{\mathcal{L}}_{n+1,i}\rangle$$
over $\mathcal{Y}_{i'}$. It suffices to prove
$$-\epsilon_i\deg(\mathcal{U}_\eta)_j\overline{\mathcal{E}}_0\le\widehat{\mathrm{div}}(t_j)\le\epsilon_i\deg(\mathcal{U}_\eta)_j\overline{\mathcal{E}}_0$$
in $\widehat{\mathrm{Div}}(\mathcal{V})_{\mathrm{mod}}$.

The line bundle $\overline{\mathcal{N}}_j$ fits the framework of Lemma 4.4.2 and Lemma 4.4.4. In terms of the mixed Deligne pairing, we exactly have
$$\widehat{\mathrm{div}}(t_j)=\langle\widehat{\mathrm{div}}(\ell_{j,i'}\ell_{j,i}^{-1}),\,\overline{\mathcal{L}}_{1,i'},\cdots,\overline{\mathcal{L}}_{j-1,i'},\overline{\mathcal{L}}_{j+1,i},\cdots,\overline{\mathcal{L}}_{n+1,i}\rangle.$$
Apply Lemma 4.4.2 and Lemma 4.4.4. We get
$$\widehat{\mathrm{div}}(t_j)\le\langle\epsilon_i f_0^*\overline{\mathcal{E}}_0,\,\overline{\mathcal{L}}_{1,i'},\cdots,\overline{\mathcal{L}}_{j-1,i'},\overline{\mathcal{L}}_{j+1,i},\cdots,\overline{\mathcal{L}}_{n+1,i}\rangle=\epsilon_i\deg(\mathcal{U}_\eta)_j\overline{\mathcal{E}}_0$$
by the Cauchy condition
$$-\epsilon_i f_0^*\overline{\mathcal{E}}_0\le\widehat{\mathrm{div}}(\ell_{j,i'}\ell_{j,i}^{-1})\le\epsilon_i f_0^*\overline{\mathcal{E}}_0.$$
Similarly, we have
$$\widehat{\mathrm{div}}(t_j)\ge-\epsilon_i\deg(\mathcal{U}_\eta)_j\overline{\mathcal{E}}_0.$$
It finishes the proof. □

4.6 More functorialities of the pairing

In Theorem 4.1.3, we have listed that the Deligne pairing is compatible with base change. In this subsection, we list two more natural properties. The first one is the behavior of the pairing in some situation under compositions, and the second one is a non-archimedean local version of the pairing.

4.6.1 Functoriality properties

We first present various projection formulas on Deligne pairings. To avoid confusion, for a morphism $\psi : X \to Y$, we use write $\psi_*\langle \cdots \rangle$ for the Deligne pairing with respect to this morphism.

Lemma 4.6.1. *Let k be either $\mathbb{Z}$ or a field. Let $\psi : X \to Y$ be a projective morphism of relative dimension r over k, and $\pi : Y \to S$ be a projective morphism of relative dimension m over k. Here X, Y, S are quasi-projective and flat integral schemes over k. Assume that $\pi : Y \to S$ and $\pi \circ \psi : X \to S$ are flat, and assume that S is normal. Let $\overline{L}_1, \cdots, \overline{L}_{r+1}$ be integrable adelic line bundles on X, $\overline{M}_1, \cdots, \overline{M}_{m+1}$ be integrable adelic line bundles on Y, and $\overline{N}_1$ be an integrable adelic line bundles on S.*

(1) Assume that $\psi : X \to Y$ is flat, and assume that Y is normal. Then there is a canonical isomorphism

$$\begin{aligned}&(\pi \circ \psi)_*\langle \overline{L}_1, \cdots, \overline{L}_{r+1}, \psi^*\overline{M}_1, \cdots, \psi^*\overline{M}_m\rangle \\ \longrightarrow &\pi_*\langle \psi_*\langle \overline{L}_1, \cdots, \overline{L}_{r+1}\rangle, \overline{M}_1, \cdots, \overline{M}_m\rangle.\end{aligned}$$

(2) There is a canonical isomorphism

$$\pi_*\langle \overline{M}_1, \cdots, \overline{M}_m, \pi^*\overline{N}_1\rangle \longrightarrow e\,\overline{N}_1.$$

Here e is the intersection number of the underlying line bundles of $\overline{M}_1, \cdots, \overline{M}_m$ on the generic fiber of $\pi : Y \to S$.

(3) There is a canonical isomorphism

$$(\pi \circ \psi)_*\langle \overline{L}_1, \cdots, \overline{L}_r, \psi^*\overline{M}_1, \cdots, \psi^*\overline{M}_{m+1}\rangle \longrightarrow d\,\pi_*\langle \overline{M}_1, \cdots, \overline{M}_{m+1}\rangle.$$

Here d is the intersection number of the underlying line bundles of $\overline{L}_1, \cdots, \overline{L}_r$ on the generic fiber of $\psi : X \to Y$ if ψ is surjective; set $d = 0$ if ψ is not surjective.

(4) If $r = 0$, then there is a canonical isomorphism

$$(\pi \circ \psi)_*\langle \psi^*\overline{M}_1, \cdots, \psi^*\overline{M}_{m+1}\rangle \longrightarrow \deg(\psi)\, \pi_*\langle \overline{M}_1, \cdots, \overline{M}_{m+1}\rangle.$$

Here $\deg(\psi)$ is degree of the extension between the function fields of X and Y induced by $\psi : X \to Y$ if ψ is surjective; set $\deg(\psi) = 0$ if ψ is not surjective.

Proof. We first prove (1). See [MG, Prop. 5.2.3.b] for the isomorphism of the underlying line bundles. By taking limit, this already implies the result for the geometric case that k is a field.

If $k = \mathbb{Z}$, we need an extra argument to check the compatibility of the hermitian metrics. By Theorem 4.2.3, metrics of Deligne pairings are fiber-wise defined, so it suffices to check the equality of the metrics assuming that $S = \operatorname{Spec}\mathbb{C}$, and $\overline{L}_i, \overline{M}_j$ are metrized line bundles on the complex varieties X, Y. Induct on $m = \dim Y$. By linearity, assume that M_m is very ample on Y, and take a section $s \in \Gamma(Y, M_m)$ such that $Y' = \operatorname{div}(s)$ is integral and $X' = X \times_Y Y'$ is also integral. Denote by $\psi' : X' \to Y'$ and $\pi' : Y' \to S$ the morphisms. Denote $\overline{L}_i = \overline{L}_i|_{X'}$ and $\overline{M}_j' = \overline{M}_j|_{Y'}$. By induction, we have an isometry

$$\begin{aligned}&(\pi' \circ \psi')_*\langle \overline{L}_1', \cdots, \overline{L}_{r+1}', \psi^*\overline{M}_1', \cdots, \psi^*\overline{M}_{m-1}'\rangle \\ \longrightarrow\, &\pi_*'\langle \psi_*'\langle \overline{L}_1', \cdots, \overline{L}_{r+1}'\rangle, \overline{M}_1', \cdots, \overline{M}_{m-1}'\rangle.\end{aligned}$$

It suffice to check that the changes of the metrics of both sides are equal. By §4.2.2, this amounts to check

$$\begin{aligned}&\int_X \log \|\psi^* s\| c_1(\overline{L}_1)\cdots c_1(\overline{L}_{r+1})c_1(\psi^*\overline{M}_1)\cdots c_1(\psi^*\overline{M}_{m-1}) \\ =&\int_Y \log \|s\| c_1(\psi_*\langle \overline{L}_1, \cdots, \overline{L}_{r+1}\rangle)c_1(\overline{M}_1)\cdots c_1(\overline{M}_{m-1}).\end{aligned}$$

This follows from Theorem 4.2.3(4). It proves (1).

Note that (2) is actually the special case of (3) when π is the identity map on Y, and (4) is also a special case of (3). We also have two quick proofs of (2). First, [MG, Prop. 5.2.1.a] gives an isomorphism of the underlying line bundles in (2), and then we can extend it to the adelic case as in the proof of (1). Alternatively, write $\overline{N}_1 = \mathcal{O}(\overline{D})$ for some adelic divisor $\overline{D}$ on S. Then the result follows from limit versions of Lemma 4.4.2(1) and Lemma 4.4.4(1).

To prove (3), the key is to establish a canonical isomorphism

$$(\pi\circ\psi)_*\langle L_1,\cdots,L_r,\psi^*M_1,\cdots,\psi^*M_{m+1}\rangle \longrightarrow d\,\pi_*\langle M_1,\cdots,M_{m+1}\rangle$$

of the underlying line bundles, since the extension of this to the adelic case is similar to that in (1). For the isomorphism of the underlying line bundles, we can assume that all M_i, L_j are very ample by linearity. By passing to a finite Zariski open cover of S, we can find a section t_{m+1} of M_{m+1} such that $\mathrm{div}(t_{m+1})$ is integral and flat over S. This reduces the problem from (X,Y,S) to $(X_{\mathrm{div}(t)},\mathrm{div}(t),S)$, and thus eventually we can assume that $m=0$. Then $\pi: Y\to S$ is finite and flat, and by passing to a Zariski open cover of S, we can take a section of L_r to reduce $\dim X$, and eventually we can also assume that $r=0$. The case $m=r=0$ can be checked by an easy relation of the norm maps. This finishes the proof. □

4.6.2 Local theory

In this subsection, we are going to consider the Deligne paring over a non-archimedean field, and write the metrics in this setting.

Let K be a non-archimedean field with a discrete valuation, and let O_K be the valuation ring. Let X be a *projective* variety of dimension n over K. Recall that in §2.7, we have defined $\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)$ as the completion of

$$\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)_{\mathrm{mod}} = \varinjlim_{\mathcal{X}} \mathcal{P}\mathrm{ic}(\mathcal{X},X)$$

along the boundary topology. Here the limit is over projective models $\mathcal{X}$ of X over O_K.

Similar to the global case, we can introduce the category $\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)_{\mathrm{snef}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)_{\mathrm{nef}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)_{\mathrm{int}}$) of *strongly nef* (resp. *nef*, *integrable*) objects of $\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)$. In fact, a line bundle on a projective model $\mathcal{X}$ of X over O_K is *nef* if it has a non-negative degree on every projective and integral curve in the special fiber of $\mathcal{X}\to\mathrm{Spec}\,O_K$. An adelic line bundle on X is *strongly nef* if it is the limit under the boundary topology of model adelic line bundles induced by nef line bundles on projective models of X. An adelic line bundle $\overline{L}$ on X is *nef* if there exists a strongly nef adelic line bundle $\overline{M}$ on X such that $a\overline{L}+\overline{M}$ is strongly nef for all positive integers a. An adelic line bundle on X is *integrable* if it is isomorphic to the difference of two strongly nef adelic line bundles on X.

By continuity, the Deligne pairing

$$\mathcal{P}\mathrm{ic}(\mathcal{X})^{n+1} \longrightarrow \mathcal{P}\mathrm{ic}(O_K)$$

extends to a canonical pairing

$$\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(K/O_K).$$

Note that $\widehat{\mathcal{P}\mathrm{ic}}(K/O_K) = \widehat{\mathcal{P}\mathrm{ic}}(\mathrm{Spec}\, K/O_K)$ is equivalent to the category of triples $(L, \mathcal{L}, \ell)$ with $L \in \mathcal{P}\mathrm{ic}(K)$, $\mathcal{L} \in \mathcal{P}\mathrm{ic}(O_K)_{\mathbb{Q}}$, and $\ell : L \to \mathcal{L}|_{\mathrm{Spec}\, K}$ an isomorphism in $\mathcal{P}\mathrm{ic}(K)_{\mathbb{Q}}$. If O_K is a discrete valuation ring, the proof of this extension is similar to Theorem 4.1.3, and it is actually easier without the archimedean metrics. If O_K is not a discrete valuation ring, an extra ingredient of the extension is from Xia [Xia], who extends the Deligne pairing of [Del, Elk1, MG, Duc] to non-noetherian schemes. In fact, the idea of [Xia, Prop. 3.7] is that a projective and flat morphism of (possibly non-noetherian) schemes can be Zariski locally descended to a projective and flat morphism of noetherian schemes.

On the other hand, in Proposition 3.6.1, we have a canonical fully faithful functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X/O_K) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}}).$$

We have essential images

$$\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{cptf}} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(X/O_K) \to \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{snef}} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)_{\mathrm{snef}} \to \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{nef}} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)_{\mathrm{nef}} \to \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{int}} = \mathrm{Im}(\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)_{\mathrm{int}} \to \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})).$$

Note that in §3.6, we have also given direct descriptions of compactified (resp. strongly nef, nef, integrable) metrized line bundles on X^{an}.

Parallel to the archimedean setting in §4.2.2, we use integration to define a Deligne pairing

$$\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(K^{\mathrm{an}}).$$

In fact, let $\overline{L}_1, \cdots, \overline{L}_{n+1}$ be integrable metrized line bundles on X^{an}, we endow a metric of the 1-dimension K-space $\langle L_1, \cdots, L_{n+1} \rangle$ as follows.

We assume that all L_i are very ample by linearity. Let s_1 be a nonzero section of L_1 on X. We have a natural isomorphism

$$[s_1] : \langle L_1, \cdots, L_{n+1} \rangle \longrightarrow \langle L_2, \cdots, L_{n+1} \rangle_{Z_1/K}.$$

Here $Z_1 = \mathrm{div}(s_1)$ and the right-hand side is the Deligne pairing with respect to the morphism $Z_1 \to \operatorname{Spec} K$. Define the norm of the map $[s_1]$ by

$$\log \|[s_1]\| = -\int_{X^{\mathrm{an}}} \log \|s_1\| c_1(\overline{L}_2) \cdots c_1(\overline{L}_{n+1}).$$

Here the right-hand side uses the Chambert-Loir measure. This defines the metric of $\langle L_1, \cdots, L_{n+1} \rangle$ by induction on $\dim X$. Note that if Z_1 is not integral, we can use Lemma 4.2.2 to convert it to the pairings from its irreducible components. As in the archimedean case, the definition is independent of the choice of s_1 by [CT, Thm. 4.1], and the pairing is symmetric and multi-linear.

Similar to the archimedean case, in a single formula, if $(s_1, \cdots, s_{n+1})$ is a strongly regular sequence of sections of $(L_1, \cdots, L_{n+1})$ on X, then the metric of $\langle L_1, \cdots, L_{n+1} \rangle$ is given by

$$-\log \|\langle s_1, \cdots, s_{n+1} \rangle\| = -\sum_{i=1}^{n+1} \int_{Z_{i-1}^{\mathrm{an}}} \log \|s_i\| c_1(\overline{L}_{i+1}) \cdots c_1(\overline{L}_{n+1}).$$

This is exactly the local intersection number

$$\widehat{\mathrm{div}}(s_1) \cdot \widehat{\mathrm{div}}(s_2) \cdots \widehat{\mathrm{div}}(s_{n+1}).$$

See [CT, §2] or [YZ1, Appendix 1] for basic properties of the local intersection number.

Finally, we have the following result, which asserts that the two pairings are compatible.

Theorem 4.6.2. *Let K be a non-archimedean field, and let O_K be its valuation ring. Let X be a projective variety of dimension n over K. The Deligne pairings*

$$\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(K/O_K)$$

and

$$\widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(K^{\mathrm{an}})$$

are compatible with the analytification functors

$$\widehat{\mathcal{P}\mathrm{ic}}(X/O_K)_{\mathrm{int}} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}})_{\mathrm{int}}$$

and

$$\widehat{\mathcal{P}\mathrm{ic}}(K/O_K) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(K^{\mathrm{an}}).$$

Proof. It suffices to prove the model case. Namely, let $\mathcal{L}_1, \cdots, \mathcal{L}_{n+1}$ be line bundles on a projective model $\mathcal{X}$ of X over O_K, with generic fibers $L_1, \cdots, L_{n+1}$ on X. Then the metric of $\langle L_1, \cdots, L_{n+1}\rangle$ on X^{an} induced by $\langle \mathcal{L}_1, \cdots, \mathcal{L}_{n+1}\rangle$ is equal to the one defined by the integrals.

To prove the model case, assume that $\mathcal{X}$ is normal by taking a normalization. Assume that all $\mathcal{L}_i$ are very ample by linearity. Let s_1 be a nonzero section of $\mathcal{L}_1$ on $\mathcal{X}$ such that $\mathcal{Z}_1 = \mathrm{div}(s_1)$ is flat over O_K. Then we have a natural isomorphism

$$[s_1] : \langle \mathcal{L}_1, \cdots, \mathcal{L}_{n+1}\rangle \longrightarrow \langle \mathcal{L}_2, \cdots, \mathcal{L}_{n+1}\rangle_{\mathcal{Z}_1/O_K}.$$

Thus $\langle \mathcal{L}_1, \cdots, \mathcal{L}_{n+1}\rangle$ and $\langle \mathcal{L}_2, \cdots, \mathcal{L}_{n+1}\rangle_{\mathcal{Z}_1/O_K}$ induce compatible metrics on the line bundles $\langle L_1, \cdots, L_{n+1}\rangle$ and $\langle L_2, \cdots, L_{n+1}\rangle_{\mathcal{Z}_{1,K}/K}$. By induction, it suffices to prove that the analytic term

$$\log \|[s_1]\| = -\int_{X^{\mathrm{an}}} \log \|s_1\| c_1(\overline{L}_2) \cdots c_1(\overline{L}_{n+1})$$

vanishes. By definition, the Chambert-Loir measure on the right-hand side is supported on the divisorial points of X^{an} corresponding to the irreducible components of the special fiber of $\mathcal{X}$. On the other hand, $\|s_1\| = 1$ at these divisorial points by the assumption that $\mathrm{div}(s_1)$ is flat over O_K. Then the integral vanishes. $\square$

5 Volumes and heights

In this section, we are going to study effective section of adelic line bundles, volumes of adelic line bundles, heights of algebraic points and subvarieties, and equidistribution of small points. It turns out that many definitions and results for hermitian line bundles can be extended to the current situation.

As before, we will treat the geometric case and the arithmetic case uniformly, taking the uniform terminology in §1.6.

5.1 Effective sections of adelic line bundles

The goal of this subsection is to introduce effective sections of adelic line bundles and derive some basic finiteness properties.

5.1.1 Effective adelic divisors

Effective sections of adelic line bundles are defined in terms of effective adelic divisors, so we start with the following definition.

Definition 5.1.1. Let k be either $\mathbb{Z}$ or a field.

(1) Let $\mathcal{U}$ be a quasi-projective variety over k. An adelic divisor $\overline{\mathcal{D}}$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ is called *effective* if it can be represented by a Cauchy sequence of effective divisors in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$.

(2) Let X be a flat and essentially quasi-projective integral scheme over k. An adelic divisor $\overline{\mathcal{D}}$ in $\widehat{\mathrm{Div}}(X/k)$ is called *effective* if it is the image of an effective adelic divisor of $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ for some quasi-projective model $\mathcal{U}$ of X.

As before, we will use $\geq$ and $\leq$ to denote the partial order on $\widehat{\mathrm{Div}}(X/k)$ induced by effectivity.

Note that the above definition is very similar to the definition of strong nefness. The following is the justification of this definition.

Lemma 5.1.2. *Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. Then an adelic divisor $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(X/k)$ is effective if and only if its image $\overline{\mathcal{D}}^{\mathrm{an}}$ in $\widehat{\mathrm{Div}}(X^{\mathrm{an}})$ is effective.*

If furthermore X is normal, then $\overline{\mathcal{D}}$ is effective if and only the total Green's function $\widetilde{g}_{\overline{\mathcal{D}}}$ induced by $\overline{\mathcal{D}}$ is non-negative on $X^{\mathrm{an}} \setminus |\mathcal{D}|^{\mathrm{an}}$.

Proof. It suffices to prove the case that $X = \mathcal{U}$ is a quasi-projective variety over k. For the first statement, assume that $\overline{\mathcal{D}}^{\mathrm{an}}$ is effective, and we need to prove that $\overline{\mathcal{D}}$ is effective. Assume that $\overline{\mathcal{D}}$ is represented by a sequence $\{\overline{\mathcal{D}}_i\}_{i\geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$. By definition, there is a sequence $\{\epsilon_j\}_{j\geq 1}$ of positive rational numbers converging to 0 such that

$$-\epsilon_j\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}}_i - \overline{\mathcal{D}}_j \leq \epsilon_j\overline{\mathcal{E}}_0, \quad i \geq j \geq 1.$$

This implies
$$-\epsilon_j \tilde{g}_{\mathcal{E}_0} \leq \tilde{g}_{\mathcal{D}_i} - \tilde{g}_{\mathcal{D}_j} \leq \epsilon_j \widetilde{g}_{\mathcal{E}_0}, \qquad i \geq j \geq 1.$$
Here $\widetilde{g}_\bullet$ denotes the corresponding Green's function on $\mathcal{U}^{\mathrm{an}}$. Set $i \to \infty$. It gives
$$\tilde{g}_{\mathcal{D}_j} + \epsilon_j \tilde{g}_{\mathcal{E}_0} \geq \tilde{g}_{\mathcal{D}} \geq 0, \qquad j \geq 1.$$
By Lemma 3.3.3, $\overline{\mathcal{D}}_j + \epsilon_j \overline{\mathcal{E}}_0$ is effective in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$. Note that $\overline{\mathcal{D}}$ is also represented by the Cauchy sequence $\{\overline{\mathcal{D}}_i + \epsilon_i \overline{\mathcal{E}}_0\}_{i \geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$. Then it is effective.

For the second statement, it suffices to prove that $\widetilde{g}_{\overline{\mathcal{D}}} \geq 0$ implies $\mathcal{D} \geq 0$. This can be proved as in the proof of Corollary 3.4.2. □

5.1.2 Effective sections of adelic line bundles

Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k, and let $\overline{L}$ be an adelic line bundle on X. For any nonzero rational section s of L on X, there is an arithmetic divisor $\widehat{\mathrm{div}}(s)$, defined as an element of $\widehat{\mathrm{Div}}(X/k)$. In fact, it suffices to define this for any quasi-projective model $\mathcal{U}$ of X. This is in Lemma 2.5.1. Namely, if $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i \geq 1})$ is an adelic line bundle on $\mathcal{U}$, and s is a nonzero rational section of $\mathcal{L}$ on $\mathcal{U}$, then
$$\widehat{\mathrm{div}}(s) = \widehat{\mathrm{div}}_{(\mathcal{X}_1, \overline{\mathcal{L}}_1)}(s) + \lim_{i \to \infty} \widehat{\mathrm{div}}(\ell_i \ell_1^{-1})$$
in $\widehat{\mathrm{Div}}(\mathcal{U}/k)$.

Now we are ready to introduce the key definitions.

Definition 5.1.3. Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k.

(1) Let $\overline{L}$ be an adelic line bundle on X with underlying line bundle L on X. Define
$$\widehat{H}^0(X, \overline{L}) := \{s \in H^0(X, L) : \widehat{\mathrm{div}}(s) \geq 0\}.$$
Here the partial order is in $\widehat{\mathrm{Div}}(X/k)$. Elements of $\widehat{H}^0(X, \overline{L})$ are called *effective sections* of $\overline{L}$ on X. If $k = \mathbb{Z}$, denote
$$\widehat{h}^0(X, \overline{L}) := \log \# \widehat{H}^0(X, \overline{L});$$

if k is a field, denote

$$\widehat{h}^0(X, \overline{L}) := \dim_k \widehat{H}^0(X, \overline{L}).$$

We say that $\overline{L}$ if *effective* if $\widehat{h}^0(X, \overline{L}) > 0$.

(2) Let $\overline{L}$ be a metrized line bundle on X^{an} with underlying line bundle L. For any $s \in H^0(X, L)$ and any $v \in \mathcal{M}(k)$, define the supremum norms

$$\|s\|_{\sup} := \sup_{x \in X^{\mathrm{an}}} \|s(x)\|,$$

$$\|s\|_{v,\sup} := \sup_{x \in X_v^{\mathrm{an}}} \|s(x)\|.$$

Both "sup" are allowed to be infinity. Define

$$\widehat{H}^0(X, \overline{L}) := \{s \in H^0(X, L) : \|s\|_{\sup} \leq 1\}.$$

Elements of $\widehat{H}^0(X, \overline{L})$ are called *effective sections* of $\overline{L}$ on X. If $k = \mathbb{Z}$, denote

$$\widehat{h}^0(X, \overline{L}) := \log \# \widehat{H}^0(X, \overline{L});$$

if k is a field, denote

$$\widehat{h}^0(X, \overline{L}) := \dim_k \widehat{H}^0(X, \overline{L}).$$

The definitions in (1) and (2) are compatible. Namely, for any adelic line bundle $\overline{L}$ on X, which induces a metrized line bundle $\overline{L}^{\mathrm{an}}$ on X^{an}, the canonical map

$$\widehat{H}^0(X, \overline{L}) \longrightarrow \widehat{H}^0(X, \overline{L}^{\mathrm{an}})$$

is bijective. This follows from Lemma 5.1.2.

If k is a field, it is easy to see that $\widehat{H}^0(X, \overline{L})$ is a vector space over k in the setting of (2), and thus the same holds in the setting of (1). So the dimension $\widehat{h}^0(X, \overline{L})$ is well-defined (as a finite number or infinity).

In both the arithmetic case and the geometric case, we will prove that the number $\widehat{h}^0(X, \overline{L})$ in (1) is finite. The proof is not hard, but we will postpone it till Lemma 5.1.6 to set up a framework to bound sections of adelic line bundles.

In terms of arithmetic divisors, the definitions are written more easily. For example, if $\overline{D}$ is an adelic divisor on X, then

$$\widehat{H}^0(X, \mathcal{O}(\overline{D})) = \{f \in k(X)^\times : \widehat{\mathrm{div}}(f) + \overline{D} \geq 0\} \cup \{0\}.$$

Because of this, we may work on adelic divisors instead of adelic line bundles. For simplicity, we will denote

$$\widehat{H}^0(X, \overline{D}) = \widehat{H}^0(X, \mathcal{O}(\overline{D})), \qquad \widehat{h}^0(X, \overline{D}) = \widehat{h}^0(X, \mathcal{O}(\overline{D})).$$

5.1.3 Effective sections of arithmetic $\mathbb{Q}$-divisors

For the purpose later, we generalize the definition of effective sections to arithmetic $\mathbb{Q}$-divisors.

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{X}$ be a projective variety over k of absolute dimension d. Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{Q}$-divisor on $\mathcal{X}$. Denote

$$\widehat{H}^0(\mathcal{X}, \overline{\mathcal{D}})' = \{f \in k(\mathcal{X})^\times : \widehat{\mathrm{div}}(f) + \overline{\mathcal{D}} \geq 0 \text{ in } \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}\} \cup \{0\}.$$

If $k = \mathbb{Z}$, denote

$$\widehat{h}^0(\mathcal{X}, \overline{\mathcal{D}})' := \log \# \widehat{H}^0(\mathcal{X}, \mathcal{O}(\overline{\mathcal{D}}))';$$

if k is a field, denote

$$\widehat{h}^0(\mathcal{X}, \overline{\mathcal{D}})' := \dim_k H^0(\mathcal{X}, \mathcal{O}(\overline{\mathcal{D}}))'.$$

If $\overline{\mathcal{D}}$ is integral, recall the usual set of effective sections defined by

$$\widehat{H}^0(\mathcal{X}, \overline{\mathcal{D}}) := \{f \in k(\mathcal{X})^\times : \widehat{\mathrm{div}}(f) + \overline{\mathcal{D}} \geq 0 \text{ in } \widehat{\mathrm{Div}}(\mathcal{X})\} \cup \{0\}.$$

There is a canonical injection

$$\widehat{H}^0(\mathcal{X}, \overline{\mathcal{D}}) \to \widehat{H}^0(\mathcal{X}, \overline{\mathcal{D}})',$$

which might fail to be bijective due to the difference of the effectivity relations in $\widehat{\mathrm{Div}}(\mathcal{X})$ and $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$. However, if $\mathcal{X}$ is normal, then it is bijective by Lemma 2.3.5.

In general, $\widehat{h}^0(\mathcal{X}, \overline{\mathcal{D}})'$ is always finite. In fact, the finiteness holds if $\mathcal{X}$ is normal and $\overline{\mathcal{D}}$ is integral by $\widehat{h}^0(\mathcal{X}, \overline{\mathcal{D}})' = \widehat{h}^0(\mathcal{X}, \overline{\mathcal{D}})$. The normality condition can be removed by passing to the normalization, and the integrality condition can be removed by bounding $\overline{\mathcal{D}}$ by an integral arithmetic divisor on $\mathcal{X}$ under the relation "$\leq$".

5.1.4 Model case

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{X}$ be a projective variety over k of absolute dimension d. Let $\mathcal{U}$ be an open subscheme of $\mathcal{X}$. Let $\overline{\mathcal{D}}$ be an arithmetic $(\mathbb{Q},\mathbb{Z})$-divisor on $(\mathcal{X},\mathcal{U})$. Definition 5.1.3 gives

$$\widehat{H}^0(\mathcal{U},\overline{\mathcal{D}}) = \{f \in k(\mathcal{U})^\times : \widehat{\mathrm{div}}(f) + \overline{\mathcal{D}} \geq 0 \text{ in } \widehat{\mathrm{Div}}(\mathcal{U}/k)\} \cup \{0\}.$$

Note that the partial order is taken in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$. Here $\widehat{\mathrm{div}}(f) \in \widehat{\mathrm{Div}}(\mathcal{X})$ is viewed as an element of $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ via the canonical map $\widehat{\mathrm{Div}}(\mathcal{X}) \to \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$.

Using the rational part of $\overline{\mathcal{D}}$, we have a well-defined $\widehat{H}^0(\mathcal{X},\overline{\mathcal{D}})'$ in the above, which might be different from $\widehat{H}^0(\mathcal{U},\overline{\mathcal{D}})$, as they use different effectivity relations. The following result give some inequalities between these different notions.

Lemma 5.1.4. *Let X be a flat and essentially quasi-projective integral scheme over k. Let $\mathcal{U}$ be a quasi-projective model of X, and let $\mathcal{X}$ be a projective model of $\mathcal{U}$. Let $\overline{\mathcal{D}}$ be an arithmetic $(\mathbb{Q},\mathbb{Z})$-divisor on $(\mathcal{X},\mathcal{U})$.*

(1) There is a canonical injection

$$\widehat{H}^0(X,\overline{\mathcal{D}}) \longrightarrow \widehat{H}^0(\mathcal{X}',\pi^*\overline{\mathcal{D}})'.$$

Here $\pi : \mathcal{X}' \to \mathcal{X}$ is the normalization of $\mathcal{X}$.

(2) If $\overline{\mathcal{D}}$ is the image of an integral arithmetic divisor $\overline{\mathcal{D}}^* \in \widehat{\mathrm{Div}}(\mathcal{X})$ in $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$, then there is a canonical injection

$$\widehat{H}^0(\mathcal{X},\overline{\mathcal{D}}^*) \longrightarrow \widehat{H}^0(X,\overline{\mathcal{D}}).$$

(3) $\widehat{h}^0(X,\overline{\mathcal{D}})$ is always finite.

Proof. Part (2) is trivial. Part (3) is a direct consequence of (1). For (1), denote by X' the generic point of $\mathcal{X}'$. The canonical map

$$\widehat{H}^0(X,\overline{\mathcal{D}}) \longrightarrow \widehat{H}^0(X',\pi^*\overline{\mathcal{D}})$$

is injective, so it suffices to prove that the canonical injection

$$\widehat{H}^0(\mathcal{X}',\pi^*\overline{\mathcal{D}})' \longrightarrow \widehat{H}^0(X',\pi^*\overline{\mathcal{D}})$$

is bijective. It suffices to note that for any $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{X}',\mathcal{U})$, the relations $\overline{\mathcal{D}} \geq 0$, viewed in

$$\widehat{\mathrm{Div}}(\mathcal{X}',\mathcal{U}'), \quad \widehat{\mathrm{Div}}(\mathcal{X}')_{\mathbb{Q}}, \quad \widehat{\mathrm{Div}}(X'/k),$$

are all equivalent. This is a consequence of Lemma 2.3.5, Lemma 3.3.3 and Lemma 5.1.2 by converting effectivity to positivity of Green's functions. $\square$

Remark 5.1.5. If X is normal, then the injection in (1) is actually an isomorphism.

5.1.5 Adelic case

Now we can easily obtain the finiteness of $\widehat{h}^0$ in Definition 5.1.3(1).

Lemma 5.1.6. *Let k be either $\mathbb{Z}$ or a field. Let $\overline{D}$ be an adelic divisor on a flat and essentially quasi-projective integral scheme X over k. Then the following are true:*

(1) There is a model adelic divisor $\overline{D}'$ on $\mathcal{U}$, induced by an effective and nef arithmetic divisor on a projective model of $\mathcal{U}/k$, such that

$$-\overline{D}' \leq \overline{D} \leq \overline{D}'$$

in $\widehat{\mathrm{Div}}(\mathcal{U}/k)$.

(2) $\widehat{h}^0(X,\overline{\mathcal{D}})$ is always finite.

Proof. Part (1) implies part (2) by Lemma 5.1.4(3). For part (1), assume that $\overline{D}$ is represented by a Cauchy sequence $\{\overline{\mathcal{D}}_i\}_{i\geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ for a quasi-projective model $\mathcal{U}$ of X. The Cauchy condition implies that for some rational number $\epsilon_1 > 0$,

$$-\epsilon_1\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}}_i - \overline{\mathcal{D}}_1 \leq \epsilon_1\overline{\mathcal{E}}_0, \quad \forall i > 1.$$

The limit gives

$$-\epsilon_1\overline{\mathcal{E}}_0 \leq \overline{D} - \overline{\mathcal{D}}_1 \leq \epsilon_1\overline{\mathcal{E}}_0.$$

This gives a model adelic divisors $\overline{D}'$ on $\mathcal{U}$ such that $-\overline{D}' \leq \overline{D} \leq \overline{D}'$. Assume that $\overline{D}'$ is defined on a projective model $\mathcal{X}$ of X. We can find a nef and effective arithmetic divisor $\overline{D}'$ on $\mathcal{X}$ such that $-\overline{D}'' \leq \overline{D}' \leq \overline{D}''$. This finishes the proof. $\square$

5.2 Volumes of adelic line bundles

The goal of this subsection is to extend many fundamental properties on volumes of hermitian line bundles to adelic line bundles, including the arithmetic Hilbert–Samuel formula, the arithmetic bigness theorems, the Fujita approximation theorem, the log-concavity theorem, and continuity of the volume function. The key to these extensions is that volumes of adelic line bundles are naturally approximated by volumes of hermitian line bundles.

5.2.1 Volumes on arithmetic varieties

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{X}$ be a projective variety over k of absolute dimension d. For any hermitian line bundles $\overline{\mathcal{L}}$ on $\mathcal{X}$ (with continuous metrics), denote the volume

$$\widehat{\mathrm{vol}}(\mathcal{X}, \overline{\mathcal{L}}) := \lim_{m\to\infty} \frac{d!}{m^d} \widehat{h}^0(\mathcal{X}, m\overline{\mathcal{L}}).$$

The limit defining the volume always exists. In the geometric case, this is a result of Fujita (cf. [Laz2, 11.4.7]). In the arithmetic case, this is independently proved by Yuan [Yua2] and Chen [Che1]. We need the following basic properties of the volume function:

(1) In the arithmetic case, if there is a sequence $\{\overline{\mathcal{L}}_i\}_{i\geq 1}$ of hermitian line bundles on $\mathcal{X}$ with underlying line bundles $\mathcal{L}_i = \mathcal{L}$ such that the metrics of $\overline{\mathcal{L}}_i$ converges to the metric of $\overline{\mathcal{L}}$ uniformly, then $\widehat{\mathrm{vol}}(\mathcal{X}, \overline{\mathcal{L}}_i)$ converges to $\widehat{\mathrm{vol}}(\mathcal{X}, \overline{\mathcal{L}})$. This is a direct consequence of [YZt, Prop. 2.1]. As a consequence, in many situations, we can easily extend the results from smooth metrics to continuous metrics.

(2) In both cases, the volume function is homogeneous in that $\widehat{\mathrm{vol}}(\mathcal{X}, m\overline{\mathcal{L}}) = m^d \widehat{\mathrm{vol}}(\mathcal{X}, \overline{\mathcal{L}})$ for any positive integer m. Therefore, the definition of $\widehat{\mathrm{vol}}$ extends to all hermitian $\mathbb{Q}$-line bundles by homogeneity.

(3) In both cases, the volume function is a birational invariant. Namely, if $\pi : \mathcal{X}' \to \mathcal{X}$ is a birational morphism of projective varieties over k, then $\widehat{\mathrm{vol}}(\mathcal{X}', \pi^*\overline{\mathcal{L}}) = \widehat{\mathrm{vol}}(\mathcal{X}, \overline{\mathcal{L}})$. The geometric case is proved in [Laz1, Prop. 2.2.43], and the arithmetic case is proved by Moriwaki [Mor5, Thm. 4.3].

The arithmetic Hilbert–Samuel formula asserts that, for any nef hermitian line bundles $\overline{\mathcal{L}}$ on $\mathcal{X}$,

$$\widehat{\mathrm{vol}}(\mathcal{X}, \overline{\mathcal{L}}) = \overline{\mathcal{L}}^d.$$

Here the right-hand side denotes the arithmetic self-intersection number. In the geometric case, this is the classical Hilbert–Samuel formula in algebraic geometry (cf. [Laz1, Cor. 1.4.41]). Now we briefly describe the history of the formula for the arithmetic case. If $\overline{\mathcal{L}}$ is ample in the sense of Zhang [Zha1], then the formula is a consequence of the arithmetic Riemann–Roch theorem of Gillet–Soulé [GS2], an estimate of analytic torsions of Bismut–Vasserot [BV], and the arithmetic Nakai–Moishezon theorem of Zhang [Zha1]. See [Yua1, Corollary 2.7] for more details of the implications. The formula was further extended to the nef case with continuous metrics by Moriwaki [Mor5, Mor6].

We will also need a bigness theorem, which asserts that for hermitian line bundles $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ on $\mathcal{X}$ such that $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ are ample,

$$\widehat{\mathrm{vol}}(\mathcal{X}, \overline{\mathcal{L}} - \overline{\mathcal{M}}) \geq \overline{\mathcal{L}}^d - d\,\overline{\mathcal{L}}^{d-1}\overline{\mathcal{M}}.$$

In the geometric case, this is a theorem of Siu [Siu]. In the arithmetic case, this is the main theorem of Yuan [Yua1]. This extends to the case that $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ are nef. In fact, fix an ample hermitian line bundle $\overline{\mathcal{A}}$ on $\mathcal{X}$. For any positive rational number $\epsilon > 0$, apply the result to $(\overline{\mathcal{L}} + \epsilon\overline{\mathcal{A}},\ \overline{\mathcal{M}} + \epsilon\overline{\mathcal{A}})$. Then set $\epsilon \to 0$.

5.2.2 Main theorems on volumes

Now we are ready to state our generalization of the theorems to adelic line bundles.

Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. Let $\overline{L}$ be an adelic line bundles on X. Define

$$\widehat{\mathrm{vol}}(X, \overline{L}) := \lim_{m\to\infty} \frac{d!}{m^d}\widehat{h}^0(X, m\overline{L}).$$

Here d is the absolute dimension of a quasi-projective model of X over k.

An adelic line bundle $\overline{L}$ on X is said to be *big* if $\widehat{\mathrm{vol}}(X, \overline{L}) > 0$. Many results on big hermitian line bundle can be generalized to the current setting.

Our first result asserts that the limit defining $\widehat{\mathrm{vol}}(X, \overline{L})$ exists.

Theorem 5.2.1. *Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. Let $\overline{L}$ be an adelic line bundle on X.*

(1) The limit

$$\widehat{\mathrm{vol}}(X, \overline{L}) = \lim_{m\to\infty} \frac{d!}{m^d} \widehat{h}^0(X, m\overline{L})$$

exists. Here d is the absolute dimension of a quasi-projective model of X over k.

(2) If $\overline{L}$ is represented by an adelic line bundle $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ on $\mathcal{U}$ for a quasi-projective model $\mathcal{U}$ of X over k, then

$$\widehat{\mathrm{vol}}(X, \overline{L}) = \lim_{i\to\infty} \widehat{\mathrm{vol}}(\mathcal{X}_i, \overline{\mathcal{L}}_i).$$

On the right-hand side, $\widehat{\mathrm{vol}}(\mathcal{X}_i, \overline{\mathcal{L}}_i)$ is the volume of $\overline{\mathcal{L}}_i$ as a hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}_i$, defined by homogeneity. By the theorem, the definition of $\widehat{\mathrm{vol}}(X, \overline{L})$ extends to adelic $\mathbb{Q}$-line bundles on X by homogeneity.

The proof of Theorem 5.2.1 will take up most of the rest of this subsection. Let us first note that by the theorem, the arithmetic Hilbert–Samuel formula and the arithmetic bigness theorem can be generalized to the following theorem.

Theorem 5.2.2. *Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. Denote by d the absolute dimension of a quasi-projective model of X over k.*

(1) Let $\overline{L}$ be a nef adelic line bundles on X. Then

$$\widehat{\mathrm{vol}}(X, \overline{L}) = \overline{L}^d.$$

(2) Let $\overline{L}, \overline{M}$ be nef adelic line bundles on X. Then

$$\widehat{\mathrm{vol}}(X, \overline{L} - \overline{M}) \geq \overline{L}^d - d\,\overline{L}^{d-1}\overline{M}.$$

It is immediate that Theorem 5.2.2 holds for strongly nef adelic line bundles, as a limit version of its counterpart on projective (arithmetic) varieties by Theorem 5.2.1. The theorem will be further extended to nef adelic line bundles by the continuity of the volume function in Theorem 5.2.9 below.

In application, the above theorem is usually combined with the following basic result.

Proposition 5.2.3. *Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. Denote by d the absolute dimension of a quasi-projective model of X over k. Let $\overline{L}$ be an adelic line bundle on X.*

(1) If $k=\mathbb{Z}$, let $\overline{N}\in\widehat{\mathrm{Pic}}(\mathbb{Z})$ be a hermitian line bundle with $\widehat{\deg}(\overline{N})>0$. Then for any positive rational number c,

$$\widehat{\mathrm{vol}}(X,\overline{L}-c\overline{N})\geq\widehat{\mathrm{vol}}(X,\overline{L})-d\,c\,\widehat{\deg}(\overline{N})\,\widehat{\mathrm{vol}}(X_{\mathbb{Q}},\widetilde{L}).$$

Here $\overline{N}$ is viewed as an adelic line bundle on X via pull-back, and $\widetilde{L}$ is the image of $\overline{L}$ under the canonical map $\widehat{\mathrm{Pic}}(X/\mathbb{Z})\to\widehat{\mathrm{Pic}}(X_{\mathbb{Q}}/\mathbb{Q})$.

(2) If k is a field, assume that there is a projective and regular curve B over k together with a flat k-morphism $X\to B$. Let $N\in\mathrm{Pic}(B)$ be a line bundle with $\deg(N)>0$. Then for any positive rational number c,

$$\widehat{\mathrm{vol}}(X,\overline{L}-cN)\geq\widehat{\mathrm{vol}}(X,\overline{L})-d\,c\,\deg(N)\,\widehat{\mathrm{vol}}(X_K,\widetilde{L}).$$

Here N is viewed as an adelic line bundle on X via pull-back, K is the function field of B, and $\widetilde{L}$ is the image of $\overline{L}$ under the canonical composition

$$\widehat{\mathrm{Pic}}(X/k)\longrightarrow\widehat{\mathrm{Pic}}(X_K/k)\longrightarrow\widehat{\mathrm{Pic}}(X_K/K).$$

Proof. By Theorem 5.2.1, the problem is reduced to the case that X is projective over k, and $\overline{L}$ is a hermitian line bundle on X. Then the result is more or less well-known, and one easily checks that the result depends only on $c\,\widehat{\deg}(\overline{N})$ (or $c\deg(N)$). In fact, the arithmetic case is implied by [Mor5, Prop. 4.2(2)]. The geometric case can be proved by assuming that N is linearly equivalent to a closed point $P\in B$, and applying the exact sequence

$$0\longrightarrow H^0(X,aL-bN)\longrightarrow H^0(X,aL-(b-1)N)\longrightarrow H^0(X_P,(aL-(b-1)N)|_{X_P})$$

to count the dimensions for $a\geq b\geq 1$. ∎

5.2.3 Volumes of model adelic divisors

For the proof of Theorem 5.2.1, we need a slightly generalized limiting expression about volumes of arithmetic $\mathbb{Q}$-divisors.

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{X}$ be a projective variety over k of absolute dimension d. Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{Q}$-divisor on $\mathcal{X}$. Recall that in §5.1.3 we have introduced

$$\widehat{H}^0(\mathcal{X},\overline{\mathcal{D}})' = \{f \in k(\mathcal{X})^\times : \widehat{\mathrm{div}}(f) + \overline{\mathcal{D}} \geq 0 \text{ in } \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}\} \cup \{0\}.$$

If $k = \mathbb{Z}$, denote

$$\widehat{h}^0(\mathcal{X},\overline{\mathcal{D}})' := \log \#\widehat{H}^0(\mathcal{X},\mathcal{O}(\overline{\mathcal{D}}))';$$

if k is a field, denote

$$\widehat{h}^0(\mathcal{X},\overline{\mathcal{D}})' := \dim_k H^0(\mathcal{X},\mathcal{O}(\overline{\mathcal{D}}))'.$$

On the other hand, we have an extended definition of

$$\widehat{\mathrm{vol}}(\mathcal{X},\overline{\mathcal{D}}) = \widehat{\mathrm{vol}}(\mathcal{X},\mathcal{O}(\overline{\mathcal{D}}))$$

from integral divisors to $\mathbb{Q}$-divisors by homogeneity. Namely, let a be a positive integer such that $a\overline{\mathcal{D}}$ can be realized as an integral arithmetic divisor $\overline{\mathcal{D}}^*$ on $\mathcal{X}$. Then

$$\widehat{\mathrm{vol}}(\mathcal{X},\overline{\mathcal{D}}) := \frac{1}{a^d}\widehat{\mathrm{vol}}(\mathcal{X},\overline{\mathcal{D}}^*) = \frac{1}{a^d}\lim_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(\mathcal{X},m\overline{\mathcal{D}}^*).$$

It turns out that we have the following compatibility.

Lemma 5.2.4. *Let $\overline{\mathcal{D}},\overline{\mathcal{E}}$ be arithmetic $\mathbb{Q}$-divisors on $\mathcal{X}$. Then*

$$\widehat{\mathrm{vol}}(\mathcal{X},\overline{\mathcal{D}}) = \lim_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(\mathcal{X},m\overline{\mathcal{D}}+\overline{\mathcal{E}})'.$$

Proof. This actually holds for arithmetic $\mathbb{R}$-divisors by Moriwaki [Mor6, Thm. 5.2.2(1)] (in the arithmetic case). For our purpose of arithmetic $\mathbb{Q}$-divisors, the situation is much easier. We sketch a proof, which will be used later.

Let $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ be integral arithmetic divisors on $\mathcal{X}$ with $\overline{\mathcal{D}}_1 \leq \overline{\mathcal{D}} \leq \overline{\mathcal{D}}_2$ in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$. Let a be a positive integer such that $a\overline{\mathcal{D}}$ can be realized as an integral arithmetic divisor on $\mathcal{X}$. Let $\mathcal{X}' \to \mathcal{X}$ be the normalization, and let $(\overline{\mathcal{D}}',\overline{\mathcal{D}}_i',\overline{\mathcal{E}}')$ be the pull-back of $(\overline{\mathcal{D}},\overline{\mathcal{D}}_i,\overline{\mathcal{E}})$ to $\mathcal{X}'$. For any $r = 0,\cdots,a-1$, we have

$$\widehat{h}^0(\mathcal{X},ma\overline{\mathcal{D}}+r\overline{\mathcal{D}}_1+\overline{\mathcal{E}}) \leq \widehat{h}^0(\mathcal{X},(am+r)\overline{\mathcal{D}}+\overline{\mathcal{E}}) \leq \widehat{h}^0(\mathcal{X}',ma\overline{\mathcal{D}}'+r\overline{\mathcal{D}}_2'+\overline{\mathcal{E}}').$$

On the other hand, we have

$$\lim_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(\mathcal{X}, ma\overline{\mathcal{D}}+r\overline{\mathcal{D}}_1+\overline{\mathcal{E}})=\widehat{\mathrm{vol}}(\mathcal{X}, a\overline{\mathcal{D}})$$

and

$$\lim_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(\mathcal{X}', ma\overline{\mathcal{D}}'+r\overline{\mathcal{D}}_2'+\overline{\mathcal{E}}')=\widehat{\mathrm{vol}}(\mathcal{X}, a\overline{\mathcal{D}}).$$

In the arithmetic case, the extra terms $r\overline{\mathcal{D}}_1$ and $r\overline{\mathcal{D}}_2'$ and the normalization do not change the limit by [Mor5, Thm, 4.3, Thm. 4.4]. The corresponding results also hold in the geometric case, and we omit them. □

Now we introduce a compatibility result on volumes of model adelic divisors, which will be used in the proof of Theorem 5.2.1. The setting is similar to that of Lemma 5.1.4.

Lemma 5.2.5. *Let X be a flat and essentially quasi-projective integral scheme over k. Let $\mathcal{U}$ be a quasi-projective model of X, and let $\mathcal{X}$ be a projective model of $\mathcal{U}$. Let $\overline{\mathcal{D}}$ be an arithmetic $(\mathbb{Q},\mathbb{Z})$-divisor on $(\mathcal{X},\mathcal{U})$. Then the limit*

$$\widehat{\mathrm{vol}}(X,\overline{\mathcal{D}})=\lim_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(X, m\overline{\mathcal{D}})$$

exists and equals

$$\widehat{\mathrm{vol}}(\mathcal{X},\overline{\mathcal{D}})=\lim_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(\mathcal{X}, m\overline{\mathcal{D}})'.$$

Proof. Denote by

$$\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})^*=\mathrm{Im}(\widehat{\mathrm{Div}}(\mathcal{X})\to\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})).$$

We claim that there are $\overline{\mathcal{D}}_1,\overline{\mathcal{D}}_2\in\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})^*$ with $\overline{\mathcal{D}}_1\leq\overline{\mathcal{D}}\leq\overline{\mathcal{D}}_2$ in $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$. In fact, it suffices to construct $\overline{\mathcal{D}}_2$, since $\overline{\mathcal{D}}_1$ can be obtained by considering $-\overline{\mathcal{D}}$. We will find effective divisors $\overline{\mathcal{D}}_2',\overline{\mathcal{D}}_2''\in\widehat{\mathrm{Div}}(\mathcal{X})$ with $\overline{\mathcal{D}}\leq\overline{\mathcal{D}}_2'$ in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ and $\mathcal{D}|_{\mathcal{U}}\leq\mathcal{D}_2''|_{\mathcal{U}}$ in $\widehat{\mathrm{Div}}(\mathcal{U})$. Then we can just set $\overline{\mathcal{D}}_2$ to be the image of $\overline{\mathcal{D}}_2'+\overline{\mathcal{D}}_2''$ in $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$.

The existence of $\overline{\mathcal{D}}_2'$ is easy. For the existence of $\overline{\mathcal{D}}_2''$, we can assume that $\mathcal{D}|_{\mathcal{U}}$ is effective on $\mathcal{U}$, and then extend it to a closed subscheme $\widetilde{\mathcal{D}}$ of $\mathcal{X}$ by taking Zariski closure. Take a very ample line bundle $\mathcal{A}$ on $\mathcal{X}$ with a nonzero global section s vanishing along $\widetilde{\mathcal{D}}$; i.e., s lies in the kernel of

$H^0(\mathcal{X},\mathcal{A})\to H^0(\widetilde{\mathcal{D}},\mathcal{A}|_{\widetilde{\mathcal{D}}})$. Then we can set $\overline{\mathcal{D}}_2''$ to be $\mathrm{div}(s)$ (with a positive Green's function in the arithmetic case).

With the claim, by the method of the proof of Lemma 5.2.4, it suffices to prove that for any $\overline{\mathcal{D}},\overline{\mathcal{E}}\in\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})^*$,

$$\lim_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(X,m\overline{\mathcal{D}}+\overline{\mathcal{E}})=\widehat{\mathrm{vol}}(\mathcal{X},\overline{\mathcal{D}}).$$

Assume $\overline{\mathcal{E}}=0$ for simplicity of notations, since the general case is similar.

By Lemma 5.1.4(1),

$$\limsup_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(X,m\overline{\mathcal{D}})\le\limsup_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(\mathcal{X}',m\pi^*\overline{\mathcal{D}})'=\widehat{\mathrm{vol}}(\mathcal{X}',\pi^*\overline{\mathcal{D}}).$$

The right-hand side is further equal to $\widehat{\mathrm{vol}}(\mathcal{X},\overline{\mathcal{D}})$ by the birational invariance of the volume function. It remains to prove

$$\liminf_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(X,m\overline{\mathcal{D}})\ge\widehat{\mathrm{vol}}(\mathcal{X},\overline{\mathcal{D}}).$$

Assume that $\overline{\mathcal{D}}$ is the image of $\overline{\mathcal{D}}^*\in\widehat{\mathrm{Div}}(\mathcal{X})$ in $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$. Lemma 5.1.4(2) implies

$$\liminf_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(X,m\overline{\mathcal{D}})\ge\liminf_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(\mathcal{X},m\overline{\mathcal{D}}^*)=\widehat{\mathrm{vol}}(\mathcal{X},\overline{\mathcal{D}}^*)=\widehat{\mathrm{vol}}(\mathcal{X},\overline{\mathcal{D}}).$$

This finishes the proof. $\square$

5.2.4 Proof of Theorem 5.2.1

Now we are ready to prove Theorem 5.2.1. It is easier to write the proof in terms of divisors, so we reformulate the problem as follows.

Let $\overline{D}$ be an adelic divisor on X. Assume that $\overline{D}$ is represented by $\overline{\mathcal{D}}\in\widehat{\mathrm{Div}}(\mathcal{U}/k)$ for a quasi-projective model $\mathcal{U}$ of X, and that $\overline{\mathcal{D}}$ is a Cauchy sequence $\{\overline{\mathcal{D}}_i\}_{i\ge1}$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$. The goal is to prove that the limit

$$\lim_{m\to\infty}\frac{d!}{m^d}\widehat{h}^0(X,m\overline{D})$$

and the limit

$$\lim_{i\to\infty}\widehat{\mathrm{vol}}(\mathcal{X}_i,\overline{\mathcal{D}}_i)$$

exist and are equal. Here we write $\overline{D}$ instead of $\mathcal{O}(\overline{D})$ in the notations for $\widehat{h}^0$ and $\widehat{\mathrm{vol}}$, and take similar conventions in the following.

For convenience, denote

$$\widehat{\mathrm{vol}}(X,\overline{D})_- = \liminf_{m\to\infty} \frac{d!}{m^d}\widehat{h}^0(X, m\overline{D}),$$

$$\widehat{\mathrm{vol}}(X,\overline{D})_+ = \limsup_{m\to\infty} \frac{d!}{m^d}\widehat{h}^0(X, m\overline{D}).$$

The Cauchy condition implies that there is a sequence $\{\epsilon_j\}_{j\geq 1}$ of positive rational numbers converging to 0 such that

$$\overline{\mathcal{D}}_j - \epsilon_j\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}}_i \leq \overline{\mathcal{D}}_j + \epsilon_j\overline{\mathcal{E}}_0, \qquad i \geq j \geq 1.$$

Here $(\mathcal{X}_0, \overline{\mathcal{E}}_0)$ is a boundary divisor, and we have assumed that there is a morphism $\mathcal{X}_i \to \mathcal{X}_0$ extending the identity map on $\mathcal{U}$ for any $i \geq 1$. The effectivity relations hold in $\widehat{\mathrm{Div}}(\mathcal{U}/k)$, but we can actually assume that it holds in $\widehat{\mathrm{Div}}(\mathcal{X}_j)_{\mathbb{Q}}$ by replacing each $\mathcal{X}_i$ by a projective model of $\mathcal{U}$ dominating $\mathcal{X}_i$. Set $i \to \infty$. This gives

$$\overline{\mathcal{D}}_j - \epsilon_j\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}} \leq \overline{\mathcal{D}}_j + \epsilon_j\overline{\mathcal{E}}_0, \qquad j \geq 1.$$

The effectivity relations hold in $\widehat{\mathrm{Div}}(\mathcal{U}/k)$.

Take volumes in the above inequality. By Lemma 5.2.5, this implies

$$\widehat{\mathrm{vol}}(X,\overline{D})_- \geq \widehat{\mathrm{vol}}(\mathcal{X}_i, \overline{\mathcal{D}}_i - \epsilon_i\overline{\mathcal{E}}_0),$$

$$\widehat{\mathrm{vol}}(X,\overline{D})_+ \leq \widehat{\mathrm{vol}}(\mathcal{X}_i, \overline{\mathcal{D}}_i + \epsilon_i\overline{\mathcal{E}}_0).$$

Here $\overline{\mathcal{E}}_0$ is viewed as a divisor on $\mathcal{X}_i$ via pull-back. It suffices to prove

$$\lim_{i\to\infty}\left(\widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i + \epsilon_i\overline{\mathcal{E}}_0) - \widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i - \epsilon_i\overline{\mathcal{E}}_0)\right) = 0.$$

Here we also omit the dependence on $\mathcal{X}_i$ of the volume function, noting that the volume function on projective varieties is a birational invariant.

We first consider the case that

$$\liminf_{i\to\infty} \widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i + \epsilon_i\overline{\mathcal{E}}_0) > 0.$$

For convenience, we assume that $\widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i + \epsilon_i\overline{\mathcal{E}}_0) > 0$ for every i.

We are going to apply Fuijita's approximation theorem proved in [Fuj], and its arithmetic counterpart proved independently by Yuan [Yua2] and Chen [Che1]. As a consequence, for any $\delta > 0$, there is a birational morphism $\psi : \mathcal{X}_i' \to \mathcal{X}_i$ of projective varieties over k, together with an ample arithmetic $\mathbb{Q}$-divisor $\overline{\mathcal{F}}_i$ on $\mathcal{X}_i'$ such that

$$\widehat{\mathrm{vol}}(\overline{\mathcal{F}}_i) > \widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i + \epsilon_i \overline{\mathcal{E}}_0) - \delta$$

and such that $\psi^*(\overline{\mathcal{D}}_i + \epsilon_i \overline{\mathcal{E}}_0) - \overline{\mathcal{F}}_i$ is an effective arithmetic $\mathbb{Q}$-divisor on $\mathcal{X}_i$. Then we have

$$\widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i - \epsilon_i \overline{\mathcal{E}}_0) \geq \widehat{\mathrm{vol}}(\overline{\mathcal{F}}_i - 2\epsilon_i \overline{\mathcal{E}}_0).$$

Now we are going to apply Siu's theorem and Yuan's arithmetic version recalled before. Write $\overline{\mathcal{E}}_0 = \overline{\mathcal{A}} - \overline{\mathcal{B}}$ for nef divisors $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ on $\mathcal{X}_0$. Then

$$\widehat{\mathrm{vol}}(\overline{\mathcal{F}}_i - 2\epsilon_i \overline{\mathcal{E}}_0) = \widehat{\mathrm{vol}}(\overline{\mathcal{F}}_i + 2\epsilon_i \overline{\mathcal{B}} - 2\epsilon_i \overline{\mathcal{A}}) \geq (\overline{\mathcal{F}}_i + 2\epsilon_i \overline{\mathcal{B}})^d - 2d\epsilon_i (\overline{\mathcal{F}}_i + 2\epsilon_i \overline{\mathcal{B}})^{d-1} \overline{\mathcal{A}}.$$

We need a uniform upper bound on $\overline{\mathcal{F}}_i$. We claim that there is a nef arithmetic $\mathbb{Q}$-divisor $\overline{\mathcal{N}}$ on $\mathcal{X}_1$ such that $\overline{\mathcal{N}} \geq \overline{\mathcal{F}}_i$ in $\widehat{\mathrm{Div}}(\mathcal{X}_i')_{\mathbb{Q}}$ for any i. In fact, by the Cauchy condition,

$$\overline{\mathcal{F}}_i \leq \overline{\mathcal{D}}_i + \epsilon_i \overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}}_1 + \epsilon_i \overline{\mathcal{E}}_0 + \epsilon_1 \overline{\mathcal{E}}_0$$

for any i. Then it is easy to find $\overline{\mathcal{N}}$ to bound $\overline{\mathcal{D}}_1 + \epsilon_i \overline{\mathcal{E}}_0 + \epsilon_1 \overline{\mathcal{E}}_0$. See also Lemma 5.1.6(1).

With the uniform bound $\overline{\mathcal{N}}$, we have

$$(\overline{\mathcal{F}}_i + 2\epsilon_i \overline{\mathcal{B}})^{d-1} \overline{\mathcal{A}} \leq (\overline{\mathcal{F}}_i + 2\epsilon_i \overline{\mathcal{B}})^{d-2} (\overline{\mathcal{N}} + 2\epsilon_i \overline{\mathcal{B}})^{d-1} \overline{\mathcal{A}} \leq \cdots \leq (\overline{\mathcal{N}} + 2\epsilon_i \overline{\mathcal{B}})^{d-1} \overline{\mathcal{A}}.$$

It follows that

$$\widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i - \epsilon_i \overline{\mathcal{E}}_0) \geq \overline{\mathcal{F}}_i^d - 2\epsilon_i d (\overline{\mathcal{N}} + 2\epsilon_i \overline{\mathcal{B}})^{d-1} \overline{\mathcal{A}}.$$

Set $\delta \to 0$, so that $\overline{\mathcal{F}}_i^d \to \widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i + \epsilon_i \overline{\mathcal{E}}_0)$. The bound becomes

$$\widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i - \epsilon_i \overline{\mathcal{E}}_0) \geq \widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i + \epsilon_i \overline{\mathcal{E}}_0) - 2\epsilon_i d (\overline{\mathcal{N}} + 2\epsilon_i \overline{\mathcal{B}})^{d-1} \overline{\mathcal{A}}.$$

As a consequence, we have

$$\lim_{i \to \infty} \left(\widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i + \epsilon_i \overline{\mathcal{E}}_0) - \widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i - \epsilon_i \overline{\mathcal{E}}_0) \right) = 0.$$

This proves the theorem under the assumption $\liminf_{i\to\infty}\widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i+\epsilon_i\overline{\mathcal{E}}_0)>0$.

It is easy to extend the result to all cases. In fact, if

$$\liminf_{i\to\infty}\widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i+\epsilon_i\overline{\mathcal{E}}_0)=0,$$

then $\widehat{\mathrm{vol}}(X,\overline{\mathcal{D}})=0$. Moreover, $\lim_{i\to\infty}\widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i)=0$, since $\overline{\mathcal{D}}_i\leq\overline{\mathcal{D}}_j+\epsilon_j\overline{\mathcal{E}}_0$ for any $j>i$. It follows that the theorem also holds in this case. Combining these two cases, we have that if ν is a limit point of the sequence $\{\widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i+\epsilon_i\overline{\mathcal{E}}_0)\}_{i\geq 1}$, then $\widehat{\mathrm{vol}}(X,\overline{\mathcal{D}})$ is convergent and equal to ν. This proves the uniqueness of ν, and thus covers all cases.

5.2.5 More properties of the volume function

Here we generalize some other fundamental properties of volumes of (hermitian) line bundles to adelic line bundles. The first result is the log-concavity property.

Theorem 5.2.6 (log-convavity)**.** *Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. Let $\overline{L}_1,\overline{L}_2$ be two effective adelic line bundles on X. Then*

$$\widehat{\mathrm{vol}}(\overline{L}_1+\overline{L}_2)^{1/d}\geq\widehat{\mathrm{vol}}(\overline{L}_1)^{1/d}+\widehat{\mathrm{vol}}(\overline{L}_2)^{1/d}.$$

Here d is the absolute dimension of a quasi-projective model of X over k.

Proof. The result is easy if $\overline{L}_1$ or $\overline{L}_2$ is not big. Assume that both $\overline{L}_1$ and $\overline{L}_2$ are big. Apply Theorem 5.2.1. The problem is converted to the model case. Then the geometric case is the classical result in [Laz2, Thm. 11.4.9], and the arithmetic case is [Yua2, Thm. B]. □

A morphism between two flat and essentially quasi-projective integral schemes over k is called *birational* if it induces an isomorphism between the function fields. The next result says that the volume function is also a birational invariant.

Theorem 5.2.7 (birational invariant)**.** *Let k be either $\mathbb{Z}$ or a field. Let $\pi: X'\to X$ be a birational morphism of essentially quasi-projective integral schemes over k. Let $\overline{L}$ be an adelic line bundle on X. Then*

$$\widehat{\mathrm{vol}}(X',\pi^*\overline{L})=\widehat{\mathrm{vol}}(X,\overline{L}).$$

Proof. This is the adelic version of the geometric case in [Laz1, Prop. 2.2.43] and the arithmetic case in [Mor5, Thm. 4.3]. For the current case, it suffices to check the case that X' is the generic point of X. Then the result follows from Theorem 5.2.1(2). $\square$

We also have the Fujita approximation theorem for adelic line bundles. There are many slightly different notions of ampleness of hermitian line bundles, we take the notion of "arithmetically positive" introduced right before Lemma 4.4.3. By abuse of terminology, "arithmetically positive" on a projective variety over a field means "ample".

Theorem 5.2.8 (Fujita approximation)**.** *Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. Let $\overline{L}$ be a big adelic $\mathbb{Q}$-line bundle on X. Then for any $\epsilon > 0$, there exist a flat and essentially quasi-projective integral scheme X' over k, a birational projective morphism $\pi : X' \to X$ over k, a projective model $\mathcal{X}'$ of X' over k, and an arithmetically positive hermitian $\mathbb{Q}$-line bundle $\overline{\mathcal{A}}$ on $\mathcal{X}'$ such that $\widehat{\mathrm{vol}}(\mathcal{X}', \overline{\mathcal{A}}) \geq \widehat{\mathrm{vol}}(X, \overline{L}) - \epsilon$ and $\pi^*\overline{L} - \overline{\mathcal{A}}$ is effective in $\widehat{\mathrm{Pic}}(X'/k)_{\mathbb{Q}}$.*

Proof. Apply Theorem 5.2.1. The problem is converted to the original Fujita approximation theorem proved in [Fuj], and its arithmetic counterpart proved independently by Yuan [Yua2] and Chen [Che1]. $\square$

Now we consider a continuity property of the volume function $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q}} \to \mathbb{R}$. Recall that in the projective case, the volume function has very nice continuity properties by Lazarsfeld [Laz1, Thm. 2.2.44] for the geometric case and by Moriwaki [Mor5] for the arithmetic case. The following result generalizes these two, but our proof is different from theirs.

Theorem 5.2.9 (continuity)**.** *Let k be either $\mathbb{Z}$ or a field. Let X be a flat and essentially quasi-projective integral scheme over k. Let $\overline{L}, \overline{M}_1, \cdots, \overline{M}_r$ be adelic $\mathbb{Q}$-line bundles on X. Then*

$$\lim_{t_1,\cdots,t_r \to 0} \widehat{\mathrm{vol}}(\overline{L} + t_1\overline{M}_1 + \cdots + t_r\overline{M}_r) = \widehat{\mathrm{vol}}(\overline{L}).$$

Here $t_1, \cdots, t_r$ are rational numbers converging to 0.

Proof. For convenience, a model adelic line bundle is called *nef* if it is induced by a nef hermitian line bundle on a projective model. We will apply Theorem

5.2.2 for nef model adelic line bundles. In fact, we already know that Theorem 5.2.2 holds for strongly nef adelic line bundles by Theorem 5.2.1.

By Lemma 5.1.6(1), there is a nef and effective model adelic line bundle $\overline{M}_i'$ on X such that $\overline{M}_i' \pm \overline{M}_i$ are effective for any i. Denote $\overline{M} = \overline{M}_1' + \cdots + \overline{M}_r'$, which is still a nef and effective model adelic line bundle. It suffices to prove

$$\lim_{t\to 0} \widehat{\mathrm{vol}}(\overline{L} + t\overline{M}) = \widehat{\mathrm{vol}}(\overline{L}).$$

We first treat the case that $\overline{L}$ is big. If $t < 0$, denote $t' = -t$. Apply the Fujita approximation theorem in Theorem 5.2.8. By replacing X by some X' with a birational morphism $X' \to X$ if necessary, we can assume that $\overline{L} \geq \overline{A}$ for some nef model adelic line bundle $\overline{A}$ on X with $\widehat{\mathrm{vol}}(\overline{A}) \to \widehat{\mathrm{vol}}(\overline{L})$. Then the bigness result in Theorem 5.2.2(2) gives

$$\widehat{\mathrm{vol}}(\overline{L} - t'\overline{M}) \geq \widehat{\mathrm{vol}}(\overline{A} - t'\overline{M}) \geq \overline{A}^d - dt'\overline{A}^{d-1}\overline{M}.$$

Here d is the absolute dimension of a quasi-projective model of X over k.

We can bound $\overline{A}^{d-1}\overline{M}$ from above as in the proof of Theorem 5.2.1. In fact, by Lemma 5.1.6(1) again, we can find a nef model adelic line bundle $\overline{N}$ on X such that $\overline{L} \leq \overline{N}$. This implies $\overline{A} \leq \overline{N}$. It follows that

$$\overline{A}^{d-1}\overline{M} \leq \overline{N}^{d-1}\overline{M}$$

is bounded as $\overline{A}$ varies. Then the above lower bound of $\widehat{\mathrm{vol}}(\overline{L} - t'\overline{M})$ gives

$$\liminf_{t'\to 0+} \widehat{\mathrm{vol}}(\overline{L} - t'\overline{M}) \geq \widehat{\mathrm{vol}}(\overline{L}).$$

This proves the case $t < 0$.

If $t > 0$, by the log-concavity theorem in Theorem 5.2.6,

$$\widehat{\mathrm{vol}}(\overline{L} + t\overline{M})^{1/d} \leq 2\,\widehat{\mathrm{vol}}(\overline{L})^{1/d} - \widehat{\mathrm{vol}}(\overline{L} - t\overline{M})^{1/d}.$$

Then the result follows from the case $t < 0$. This idea of applying the log-concavity theorem is inspired by Chen [Che2].

Now we treat the case that $\overline{L}$ is not big. Then $\widehat{\mathrm{vol}}(\overline{L}) = 0$, and we need to prove $\widehat{\mathrm{vol}}(\overline{L} + t\overline{M})$ converges to 0. Assume that it is not true, and thus there is a constant $c > 0$ such that $\widehat{\mathrm{vol}}(\overline{L} + t\overline{M}) > c$ for all rational number $t > 0$. Apply the Fujita approximation theorem in Theorem 5.2.8 again. There is a birational morphism $X_t \to X$ such that $\overline{L} + t\overline{M} \geq \overline{A}_t$ on X_t for some nef model adelic line bundle $\overline{A}_t$ on X_t with $\widehat{\mathrm{vol}}(\overline{A}_t) > c/2$. Then the bigness result in Theorem 5.2.2(2) gives

$$\widehat{\mathrm{vol}}(\overline{L}) \geq \widehat{\mathrm{vol}}(\overline{A}_t - t\overline{M}) \geq \overline{A}_t^d - dt\overline{A}_t^{d-1}\overline{M}.$$

We can bound $\overline{A}_t^{d-1}\overline{M}$ by the above method. In fact, as above, we have a nef model adelic line bundle $\overline{N}$ on X such that $\overline{L} \leq \overline{N}$. This implies $\overline{A}_t \leq \overline{N} + t\overline{M}$. It follows that

$$\overline{A}_t^{d-1}\overline{M} \leq (\overline{N} + t\overline{M})^{d-1}\overline{M}$$

is bounded as $t \to 0$. As a consequence,

$$\widehat{\mathrm{vol}}(\overline{L}) \geq c/2 - O(t), \quad t \to 0.$$

This contradicts to $\widehat{\mathrm{vol}}(\overline{L}) = 0$. The proof is complete. $\square$

In the end, we present a basic result, which asserts that the bigness of the geometric part is not far from the bigness of the whole adelic line bundle.

Lemma 5.2.10. *Let k be either $\mathbb{Z}$ or a field. If $k = \mathbb{Z}$, let K be a number field; if k is a field, let K be a function field of one variable over k. Let X be a quasi-projective variety over K. Let $\overline{N} \in \widehat{\mathrm{Pic}}(K/k)$ be an adelic line bundle with $\widehat{\deg}(\overline{N}) > 0$, viewed as an adelic line bundle on X/k via pull-back. Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/k)$ be an adelic line bundle on X/k. Assume that the image $\widetilde{L}$ of $\overline{L}$ under the canonical map $\widehat{\mathrm{Pic}}(X/k) \to \widehat{\mathrm{Pic}}(X/K)$ is big on X/K. Then for sufficiently large rational number c, the adelic line bundle $\overline{L} + c\overline{N}$ is big on X/k.*

Proof. We only consider the arithmetic case $k = \mathbb{Z}$, since the geometric case is similar. Let $\mathcal{U}$ be a quasi-projective model of X over $\mathbb{Z}$ such that $\overline{L}$ actually lies in $\widehat{\mathrm{Pic}}(\mathcal{U}/\mathbb{Z})$. It is more convenience to work on adelic divisors, so we take an adelic divisor $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})$ linearly equivalent to $\overline{L}$. We first prove the general fact that $\overline{\mathcal{D}}$ is represented by an increasing Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$.

In fact, by definition, $\overline{\mathcal{D}}$ is represented by a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$, i.e., a sequence $\{\overline{\mathcal{D}}_i\}_{i\geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$ satisfying the property that there is a sequence $\{\epsilon_i\}_{i\geq 1}$ of positive rational numbers converging to 0 such that

$$-\epsilon_i\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}}_{i'} - \overline{\mathcal{D}}_i \leq \epsilon_i\overline{\mathcal{E}}_0, \quad i' \geq i \geq 1.$$

Replacing $\{\overline{\mathcal{D}}_i\}_{i\geq 1}$ by a subsequence if necessary, we can assume $\epsilon_{i+1} \leq \epsilon_i/2$ for every $i \geq 1$. Now $\overline{\mathcal{D}}$ is represented by the Cauchy sequence $\{\overline{\mathcal{D}}_i - 2\epsilon_i\overline{\mathcal{E}}_0\}_{i\geq 1}$, which is increasing by

$$(\overline{\mathcal{D}}_{i+1} - 2\epsilon_{i+1}\overline{\mathcal{E}}_0) - (\overline{\mathcal{D}}_i - 2\epsilon_i\overline{\mathcal{E}}_0) \geq -\epsilon_i\overline{\mathcal{E}}_0 - 2\epsilon_{i+1}\overline{\mathcal{E}}_0 + 2\epsilon_i\overline{\mathcal{E}}_0 \geq 0.$$

With the general fact, we assume that $\overline{\mathcal{D}}$ is the limit of an increasing sequence $\{\overline{\mathcal{D}}_i\}_{i\geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{U}/\mathbb{Z})_{\mathrm{mod}}$. Then the image $\widetilde{\mathcal{D}}$ of $\overline{\mathcal{D}}$ in $\widehat{\mathrm{Div}}(X/K)$ is the limit of the increasing sequence $\{\mathcal{D}_{i,\mathbb{Q}}\}_{i\geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{U}_{\mathbb{Q}}/\mathbb{Q})_{\mathrm{mod}}$. Note that $\widehat{\mathrm{vol}}(\mathcal{D}_{i,\mathbb{Q}})$ converges to $\widehat{\mathrm{vol}}(\widetilde{\mathcal{D}}) > 0$. Then there is an i such that $\widehat{\mathrm{vol}}(\mathcal{D}_{i,\mathbb{Q}}) > 0$. Note that

$$\widehat{\mathrm{vol}}(\overline{\mathcal{D}} + c\overline{N}) = \lim_{j\to\infty} \widehat{\mathrm{vol}}(\overline{\mathcal{D}}_j + c\overline{N}) \geq \widehat{\mathrm{vol}}(\overline{\mathcal{D}}_i + c\overline{N})$$

by the increasing property of the sequence. It suffices to prove that $\overline{\mathcal{D}}_i + c\overline{N}$ is big for sufficiently large rational number c, under the condition that $\mathcal{D}_{i,\mathbb{Q}}$ is big. This reduces the problem from the adelic divisor $\overline{\mathcal{D}}$ to the arithmetic divisor $\overline{\mathcal{D}}_i$.

The arithmetic case is well-known to experts. In fact, by linear equivalence, we can reduce the problem to the case that $\overline{N}$ is represented by (the pull-back of) the arithmetic divisor $(0, 1)$ on $\mathrm{Spec}\,\mathbb{Z}$ with underlying divisor $0 \in \mathrm{Div}(\mathbb{Z})$ and with Green's function 1 at the archimedean place. Then the arithmetic case of the result we need is an easy consequence of [Yua1, Cor. 2.4(1)(4)]. □

5.3 Heights on quasi-projective varieties

In this section, we introduce heights on quasi-projective varieties over finitely generated fields. In the projective case, we can define vector-valued heights, which refines the Moriwaki height in [Mor3, Mor4] in the arithmetic case. Note that vector-valued heights were introduced by Moret-Bailly [MB] for different purposes. The Northcott property in the projective case is deduced from that of Moriwaki, and the fundamental inequality is extended to the current case following an idea of Moriwaki.

5.3.1 Vector-valued heights

Let k be either $\mathbb{Z}$ or a field. Take the uniform terminology in §1.6. Let F be a finitely generated field over k and X be a quasi-projective variety over F. Let $\overline{L}$ be an element of $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}}$, i.e., an integrable adelic $\mathbb{Q}$-line bundle.

For any point $x \in X(\overline{F})$, define the *vector-valued height*

$$\mathfrak{h}_{\overline{L}}(x) := \mathfrak{h}_{\overline{L}}(x') := \frac{\langle \overline{L}|_{x'} \rangle}{\deg(x')} \ \in \widehat{\mathrm{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}.$$

Here x' denotes the closed point of X containing x, $\deg(x')$ is the degree of the residue field of x' over F, $\overline{L}|_{x'}$ denotes the *pull-back* of $\overline{L}$ in $\widehat{\mathrm{Pic}}(x'/k)_{\mathrm{int},\mathbb{Q}}$, and $\langle \overline{L}|_{x'} \rangle$ is the image of $\overline{L}|_{x'}$ under the norm map $\widehat{\mathrm{Pic}}(x'/k)_{\mathrm{int},\mathbb{Q}} \to \widehat{\mathrm{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}$, which is the Deligne pairing of relative dimension 0 in Theorem 4.1.3.

Therefore, we have a height function

$$\mathfrak{h}_{\overline{L}} : X(\overline{F}) \longrightarrow \widehat{\mathrm{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}.$$

Note that we do not require X to be projective here.

We can generalize the definition to high-dimensional projective subvarieties. Let Z be a closed projective $\overline{F}$-subvariety of X, i.e. a closed subvariety of $X_{\overline{F}}$ which is *projective* over $\overline{F}$. Define *the vector-valued height of Z with respect to $\overline{L}$* as

$$\mathfrak{h}_{\overline{L}}(Z) := \mathfrak{h}_{\overline{L}}(Z') := \frac{\left\langle (\overline{L}|_{Z'})^{\dim Z+1} \right\rangle}{(\dim Z + 1) \deg_L(Z')} \ \in \widehat{\mathrm{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}.$$

Here Z' denotes the image of $Z \to X$, $\overline{L}|_{Z'}$ denotes the *pull-back* in $\widehat{\mathrm{Pic}}(Z'/k)_{\mathrm{int}}$, and the self-intersection is as in Theorem 4.1.3. As we do not require X to be projective or L to be ample, the height $\mathfrak{h}_{\overline{L}}(Z)$ is only well-defined if Z is projective and $\deg_L(Z') \neq 0$.

The following are some special situations:

(1) If $\overline{L}$ is nef on X, then the height $\mathfrak{h}_{\overline{L}}(Z)$ is also nef if it is defined.

(2) If X is projective over F and L is ample on X, then the degree $\deg_L(Z')$ is well-defined and positive for all closed subvarieties of $X_{\overline{F}}$. This gives a function
$$\mathfrak{h}_{\overline{L}} : |X_{\overline{F}}| \longrightarrow \widehat{\mathrm{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}.$$
Here $|X_{\overline{F}}|$ denotes the set of closed subvarieties of $X_{\overline{F}}$.

(3) If $k = \mathbb{Z}$, let F be a number field; if k is a field, let F be a function field of one variable over k. There is a degree map
$$\widehat{\deg} : \widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q}} \longrightarrow \mathbb{R}.$$

This follows from Lemma 2.6.1 in the number field case, and limits of degrees of divisors on curves in the function field case. In both cases,

$$h_{\overline{L}}(Z) := \widehat{\deg}\, \mathfrak{h}_{\overline{L}}(Z)$$

generalizes the height function of Zhang [Zha2] (from the projective case to the quasi-projective case).

5.3.2 High-dimensional base

The above definition works well because any scheme over $\operatorname{Spec} F$ is automatically flat over $\operatorname{Spec} F$, which is required in the Deligne pairing. This still works well if we change $\operatorname{Spec} F$ to a Dedekind scheme, but if we change $\operatorname{Spec} F$ to a high-dimensional base, we easily lose this convenience and thus the definition only works in some special cases.

Let k be either $\mathbb{Z}$ or a field. Let B be a flat and essentially quasi-projective integral scheme over k, and X be a flat and quasi-projective integral scheme over B. Let $\overline{L}$ be an element of $\widehat{\operatorname{Pic}}(X/k)_{\text{int},\mathbb{Q}}$. Then we have a vector-valued height function

$$\mathfrak{h}_{\overline{L}} : X(B) \longrightarrow \widehat{\operatorname{Pic}}(B/k)_{\text{int},\mathbb{Q}}, \quad x \longmapsto x^*\overline{L}.$$

In general, for any integral subscheme Y of X which is projective and flat over B, we can still define a vector-valued height of Y with respect to $\overline{L}$ in terms of the Deligne pairing. We omit it here, since we will not use it in the current paper.

5.3.3 Moriwaki heights

Let $(k, F, X, \overline{L})$ be as above. Namely, k is either $\mathbb{Z}$ or a field, F is a field finitely generated over k, X is a quasi-projective variety over F, and $\overline{L}$ is an element of $\widehat{\operatorname{Pic}}(X/k)_{\text{int},\mathbb{Q}}$. Denote by d the absolute dimension of a quasi-projective model of F over k, and denote by n the dimension of X.

Let $\overline{H}_1, \cdots, \overline{H}_{d-1}$ be any $d-1$ elements in $\widehat{\operatorname{Pic}}(F/k)_{\text{int},\mathbb{Q}}$. For any point $x \in X(\overline{F})$, define the *Moriwaki height of x with respect to $\overline{L}$ and $(\overline{H}_1, \cdots, \overline{H}_{d-1})$* by

$$h_{\overline{L}}^{\overline{H}_1,\cdots,\overline{H}_{d-1}}(x) := h_{\overline{L}}^{\overline{H}_1,\cdots,\overline{H}_{d-1}}(x') := \frac{\overline{L}|_{x'} \cdot \overline{H}_1 \cdots \overline{H}_{d-1}}{\deg(x')} \in \mathbb{R}.$$

Here x' and $\overline{L}|_{x'}$ are as above, and the intersection number is taken in $\widehat{\mathrm{Pic}}(x'/k)_{\mathrm{int},\mathbb{Q}}$, as defined by Proposition 4.1.1, where $\overline{H}_1, \cdots, \overline{H}_{d-1}$ are viewed as elements of $\widehat{\mathrm{Pic}}(x'/k)_{\mathrm{int},\mathbb{Q}}$ via pull-back. This gives a height function

$$h_{\overline{L}}^{\overline{H}_1,\cdots,\overline{H}_{d-1}} : X(\overline{F}) \longrightarrow \mathbb{R}.$$

We can also generalize the definition to high-dimensions. For any closed $\overline{F}$-subvariety Z of X, *the Moriwaki height of Z with respect to $\overline{L}$ and $(\overline{H}_1, \cdots, \overline{H}_{d-1})$* is

$$h_{\overline{L}}^{\overline{H}_1,\cdots,\overline{H}_{d-1}}(Z) := h_{\overline{L}}^{\overline{H}_1,\cdots,\overline{H}_{d-1}}(Z') := \frac{(\overline{L}|_{Z'})^{\dim Z+1} \cdot \overline{H}_1 \cdots \overline{H}_{d-1}}{(\dim Z + 1)\deg_{\widetilde{L}}(Z'/F)}.$$

Here Z' and $\overline{L}|_{Z'}$ are as above, and the intersection number is taken in $\widehat{\mathrm{Pic}}(Z'/k)_{\mathrm{int},\mathbb{Q}}$, as defined by Proposition 4.1.1, where $\overline{H}_1, \cdots, \overline{H}_{d-1}$ are viewed as elements of $\widehat{\mathrm{Pic}}(Z'/k)_{\mathrm{int},\mathbb{Q}}$ via pull-back. The term $\widetilde{L}$ denotes the image of $\overline{L}$ under the canonical map

$$\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(X/F)_{\mathrm{int},\mathbb{Q}}$$

introduced at the end of §2.5.5, and

$$\deg_{\widetilde{L}}(Z'/F) := (\widetilde{L}|_{Z'})^{\dim Z} \in \mathbb{R}$$

is the self-intersection number of $\widetilde{L}|_{Z'}$ in $\widehat{\mathrm{Pic}}(Z'/F)_{\mathrm{int},\mathbb{Q}}$ defined in Proposition 4.1.1.

The height $h_{\overline{L}}^{\overline{H}_1,\cdots,\overline{H}_{d-1}}(Z)$ is only well-defined if $\deg_{\widetilde{L}}(Z'/F) \neq 0$. If Z is projective, then we have $\widetilde{L}|_{Z'} = L|_{Z'}$, and thus $\deg_{\widetilde{L}}(Z'/F) = \deg_L(Z')$.

The vector-valued height refines the Moriwaki height by the simple formula

$$h_{\overline{L}}^{\overline{H}_1,\cdots,\overline{H}_{d-1}}(Z) = \mathfrak{h}_{\overline{L}}(Z) \cdot \overline{H}_1 \cdots \overline{H}_{d-1}$$

as long as the right-hand side is well-defined. We introduce the following simplified notations.

(1) For any $\overline{H} \in \widehat{\mathrm{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}$, denote $h_{\overline{L}}^{\overline{H}} = h_{\overline{L}}^{\overline{H},\cdots,\overline{H}}$, where the right-hand has $d-1$ copies of $\overline{H}$.

(2) If F is a number field, then $d = 1$ and thus $h_{\overline{L}}^{\overline{H}}$ is independent of $\overline{H}$, so we just write $h_{\overline{L}} = h_{\overline{L}}^{\overline{H}}$. In this case, we simply have

$$h_{\overline{L}}(Z) = h_{\overline{L}}(Z') = \frac{(\overline{L}|_{Z'})^{\dim Z+1}}{(\dim Z + 1)\deg_{\widetilde{L}}(Z'/F)} \in \mathbb{R}.$$

A similar convention holds for function fields of one variable.

In the arithmetic case ($k = \mathbb{Z}$), if X is projective over F, and both $\overline{L}$ and $(\overline{H}_1, \cdots, \overline{H}_{d-1})$ are realized on some projective model $\mathcal{X} \to \mathcal{S}$ of $X \to \operatorname{Spec} F$, then $h_{\overline{L}}^{\overline{H}_1, \cdots, \overline{H}_{d-1}}$ is exactly the height function introduced in [Mor3]. In [Mor4], Moriwaki generalizes the definition to the case that $\overline{L}$ is given by an adelic sequence.

Let us briefly compare our adelic line bundles with the adelic sequence in [Mor4]. Roughly speaking, the adelic sequence in the loc. cit. are more numerical in nature since it uses intersection numbers to define its topology, while our adelic line bundles uses effectivity to define its topology. Then our notion includes more restrictive objects and allows coarser equivalence relations. These two notions are not too different in the definition of absolute intersection numbers, but our notion has the advantage of having Deligne pairings, effective sections and volumes.

5.3.4 Northcott property in the projective case

In the projective case, we have the following Northcott property of Moriwaki heights, which generalizes [Mor3, Prop. 3.3.7(4)].

Theorem 5.3.1 (Northcott property)**.** *Let k be either $\mathbb{Z}$ or a finite field. Let F be a finitely generated field over k, and let d be the absolute dimension of a quasi-projective model of F over k. Let X be a projective variety over F. Let $\overline{L}$ be an element in $\widehat{\operatorname{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}}$ with an ample generic fiber L. Let $\overline{H}_1, \cdots, \overline{H}_{d-1}$ be nef and big elements in $\widehat{\operatorname{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}$. Then for any $D \in \mathbb{R}$ and $A \in \mathbb{R}$, the set*

$$\{x \in X(\overline{F}) : \ \deg(x) < D, \ h_{\overline{L}}^{\overline{H}_1, \cdots, \overline{H}_{d-1}}(x) < A\}$$

is finite.

Proof. We only treat the arithmetic case $k = \mathbb{Z}$, since the geometric case is similar. If $\overline{L}, \overline{H}_1, \cdots, \overline{H}_{d-1}$ are model adelic divisors, this follows from [Mor3, Prop. 3.3.7(4)]. We will extend it to the current generality by replacing $\overline{L}, \overline{H}_1, \cdots, \overline{H}_{d-1}$ successively by more general adelic line bundles.

Let $\overline{L}'$ be any integrable adelic line bundle on X with underlying line bundle $L' = L$. To replace $\overline{L}$ by $\overline{L}'$, it suffices to check that $h_{\overline{L}}^{\overline{H}_1, \cdots, \overline{H}_{d-1}} - h_{\overline{L}'}^{\overline{H}_1, \cdots, \overline{H}_{d-1}}$ is a bounded function on $X(\overline{F})$. Assume that $\overline{L}$ and $\overline{L}'$ lie in

$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/\mathbb{Z})_{\mathbb{Q}}$ for a quasi-projective model $\mathcal{U}$ of X over $\mathbb{Z}$ with a projective and flat morphism $f : \mathcal{U} \to \mathcal{V}$ to a quasi-projective model $\mathcal{V}$ of F over $\mathbb{Z}$. Let $(\mathcal{Y}_0, \overline{\mathcal{E}}_0)$ be a boundary divisor for $\mathcal{V}$. By $L' = L$, the difference $\overline{L} - \overline{L}'$ is represented by a Cauchy sequence $\overline{\mathcal{D}} = \{\overline{\mathcal{D}}_i\}_{i \geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod},\mathbb{Q}}$ with $\mathcal{D}_i|_X = 0$. As in the proof of Lemma 5.1.6, the Cauchy condition implies

$$\overline{\mathcal{D}}_1 - \epsilon f^*\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}} \leq \overline{\mathcal{D}}_1 + \epsilon f^*\overline{\mathcal{E}}_0$$

for some positive rational number ϵ. Thus $-f^*\overline{\mathcal{D}}_1 \leq \overline{\mathcal{D}} \leq f^*\overline{\mathcal{D}}_1$ for some $\overline{\mathcal{D}}_1 \in \widehat{\mathrm{Div}}(\mathcal{V})_{\mathrm{mod},\mathbb{Q}}$. It follows that

$$|h_{\overline{L}}^{\overline{H}_1,\cdots,\overline{H}_{d-1}} - h_{\overline{L}'}^{\overline{H}_1,\cdots,\overline{H}_{d-1}}| \leq \overline{\mathcal{D}}_1 \cdot \overline{H}_1 \cdots \overline{H}_{d-1}.$$

Now we replace $\overline{H}_1, \cdots, \overline{H}_{d-1}$ successively by more general line bundles. By symmetry, it suffices to do that for $\overline{H}_1$. Let $\overline{H}_1'$ be a nef and big element in $\widehat{\mathrm{Pic}}(F/\mathbb{Z})_{\mathrm{int},\mathbb{Q}}$. To replace $\overline{H}_1$ by $\overline{H}_1'$, it suffices to have $h_{\overline{L}}^{\overline{H}_1',\cdots,\overline{H}_{d-1}} \geq c\, h_{\overline{L}}^{\overline{H}_1,\cdots,\overline{H}_{d-1}}$ for some positive rational number c. We can further assume that $\overline{L}$ is nef. Then it suffices to prove that $\overline{H}_1' - c\,\overline{H}_1$ is effective for some positive rational number c. By Theorem 5.2.2,

$$\widehat{\mathrm{vol}}(\overline{H}_1' - c\,\overline{H}_1) \geq \overline{H}_1^d - d\,c\,\overline{H}_1^{d-1}\overline{H}_1'$$

is positive if c is sufficiently small. This finishes the proof. □

5.3.5 The fundamental inequality

The fundamental inequality, a part of the theorem of successive minima of Zhang [Zha1, Zha2], is generalized to projective varieties over finitely generated fields by Moriwaki [Mor3]. Now we further generalize the result to quasi-projective varieties over finitely generated fields.

We first introduce the Moriwaki condition on polarizations of finitely generated fields. Let k be either $\mathbb{Z}$ or a field. Take the uniform terminology in §1.6. Let F be a finitely generated field over k. Denote by d the absolute dimension of a quasi-projective model of F over k. Let $\overline{H} \in \widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q}}$ be an adelic $\mathbb{Q}$-line bundle.

If $k = \mathbb{Z}$ and $d > 1$, we say that $\overline{H}$ satisfies *the Moriwaki condition* if $\overline{H}$ is nef on $F/\mathbb{Z}$, the arithmetic top self-intersection number $\widehat{\deg}_{\overline{H}}(F/\mathbb{Z}) = \overline{H}^d =$

0, and the geometric top self-intersection number $\deg_{\widetilde{H}}(F/\mathbb{Q}) = \widetilde{H}^{d-1} > 0$. Here $\widetilde{H}$ is the image of $\overline{H}$ under the canonical map

$$\widehat{\mathrm{Pic}}(F/\mathbb{Z})_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(F/\mathbb{Q})_{\mathbb{Q}}.$$

If k is a field and $d > 1$, we say that $\overline{H}$ satisfies *the Moriwaki condition* if $\overline{H}$ is nef on F/k, the geometric top self-intersection number $\deg_{\overline{H}}(F/k) = \overline{H}^d = 0$, and the geometric top self-intersection number $\deg_{\widetilde{H}}(F/K) = \widetilde{H}^{d-1} > 0$ for some extension K of k in F of transcendental degree 1, where $\widetilde{H}$ is the image of $\overline{H}$ under the canonical map

$$\widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(F/K)_{\mathbb{Q}}.$$

Note that the definition depends on the choice of K.

We will assume that $d \geq 1$, and take the convention that the Moriwaki condition is automatically satisfied if $d = 1$. Now we are ready to state the theorem.

Theorem 5.3.2 (fundamental inequality)**.** *Let k be either $\mathbb{Z}$ or a field. Let F be a finitely generated field over k. Assume that F is an infinite extension of k if k is a field. Let $\overline{H}$ be an element of $\widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q}}$ satisfying the Moriwaki condition. Let X be a quasi-projective variety over F. Let $\overline{L}$ be a nef element in $\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q}}$ such that $\deg_{\widetilde{L}}(X/F) > 0$. Then*

$$\lambda_1^{\overline{H}}(X, \overline{L}) \geq h_{\overline{L}}^{\overline{H}}(X).$$

Recall that the essential minimum

$$\lambda_1^{\overline{H}}(X, \overline{L}) = \sup_{U \subset X} \inf_{x \in U(\overline{F})} h_{\overline{L}}^{\overline{H}}(x),$$

where the supremum is taken over all Zariski open subschemes U of X. Recall that $\widetilde{L}$ denotes the image of $\overline{L}$ under the map

$$\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(X/F)_{\mathbb{Q}}.$$

The theorem is a part of [Mor3, Cor. 5.2] if $k = \mathbb{Z}$, X is projective over F, and $\overline{L}, \overline{H}$ are model adelic line bundles. The general case here is proved similarly, while the new ingredient is our results on volumes of adelic line bundles in Theorem 5.2.2.

Proof of Theorem 5.3.2. Let $\overline{N} \in \widehat{\mathrm{Pic}}(B/k)_{\mathbb{Q}}$ be an element of degree 1. Here if $k = \mathbb{Z}$, then $K = \mathbb{Q}$ and $B = \mathrm{Spec}\,\mathbb{Z}$; if k is a field, then K is the function field of one variable over k in F defining the Moriwaki condition, and B is the unique projective regular curve over k with function field K.

View $\overline{N}$ as elements of $\widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q}}$ and $\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q}}$ by pull-back. Denote $\overline{L}' = \overline{L} - c\overline{N}$ with $c \in \mathbb{Q}$. Note that

$$\lambda_1^{\overline{H}}(X, \overline{L}) - \lambda_1^{\overline{H}}(X, \overline{L}') = h_{\overline{L}}^{\overline{H}}(X) - h_{\overline{L}'}^{\overline{H}}(X) = c\, \widetilde{H}^{d-1}.$$

Thus it suffices to prove that for any $c \in \mathbb{Q}$ such that $h_{\overline{L}'}^{\overline{H}}(X) > 0$, we also have $\lambda_1^{\overline{H}}(X, \overline{L}') \geq 0$.

By the assumption $\overline{H}^d = 0$, we see that both $\lambda_1^{\overline{H}}(X, \overline{L}')$ and $h_{\overline{L}'}^{\overline{H}}(X)$ remain the same if we replace $\overline{L}'$ by $\overline{L}' + m\overline{H}$ for some positive rational number m. Note that we always have

$$\widehat{\mathrm{vol}}(\overline{L}' + m\overline{H}) \geq (\overline{L}' + m\overline{H})^{d+n}.$$

This follows from Theorem 5.2.2(1) if $c \leq 0$, and follows from Theorem 5.2.2(1) and Proposition 5.2.3 if $c \geq 0$.

By the assumption $\overline{H}^d = 0$,

$$(\overline{L}' + m\overline{H})^{d+n} = \binom{d+n}{d-1} \overline{L}'^{n+1} \overline{H}^{d-1} m^{d-1} + O(m^{d-2}), \quad m \to \infty.$$

Therefore, there is a positive integer m such that $\widehat{\mathrm{vol}}(\overline{L}' + m\overline{H}) > 0$. By definition, there is a positive integer $N > 0$ such that $N(\overline{L}' + m\overline{H})$ is an integral adelic line bundle with an effective section s on X. For any point $x \in X(\overline{F})$ outside the support $|\mathrm{div}(s)|$, we have

$$h_{\overline{L}'}^{\overline{H}}(x) = h_{\overline{L}' + m\overline{H}}^{\overline{H}}(x) = \frac{1}{N \deg(x)} \widehat{\mathrm{div}}(s)|_{x'} \cdot \pi^* \overline{H}^{d-1} \geq 0.$$

Here the intersection is on the closed point $x' \in X$ corresponding to x. This finishes the proof. $\square$

5.3.6 Fundamental inequality in the number field case

Here we provide a different proof for Theorem 5.3.2 for the special but rather important case that F is a number field. We can see more clearly the role of the small sections from this proof. The treatment also works for function fields of one variable. For convenience, we re-state the result as follows.

Theorem 5.3.3 (fundamental inequality: number field case)**.** *Let X be a quasi-projective variety over a number field K. Let $\overline{L}$ be a nef element in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ with $\deg_{\widetilde{L}}(X) > 0$. Then $\lambda_1(X, \overline{L}) \geq h_{\overline{L}}(X)$.*

The theorem is a consequence of the arithmetic Hilbert–Samuel formula in Theorem 5.2.2 by the following result.

Lemma 5.3.4. *Let X be a quasi-projective variety of dimension n over a number field K. Let $\overline{L}$ be an integrable adelic line bundle in $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{int}}$. Then for any positive integer m such that $\widehat{h}^0(X, m\overline{L}) > 0$,*

$$\lambda_1(X, \overline{L}) \geq \frac{\widehat{h}^0(X, m\overline{L})}{m\,\widehat{h}^0(X, m\widetilde{L})} - \frac{2}{m}[K : \mathbb{Q}]$$

if the right-hand side is strictly positive. As a consequence,

$$\lambda_1(X, \overline{L}) \geq \frac{\widehat{\mathrm{vol}}(\overline{L})}{(n+1)\widehat{\mathrm{vol}}(\widetilde{L})}$$

if both $\widehat{\mathrm{vol}}(\overline{L})$ and $\widehat{\mathrm{vol}}(\widetilde{L})$ are strictly positive.

Proof. It suffices to prove the first inequality. Recall that $\widehat{H}^0(X, m\widetilde{L})$ is a vector space of dimension $\widehat{h}^0(X, m\widetilde{L})$ over K, which contains the finite set $\widehat{H}^0(X, m\overline{L})$. By Definition 5.1.3, we have a v-adic norm $\|\cdot\|_{v,\sup}$ on $\widehat{H}^0(X, m\widetilde{L})$ for any place v of $\mathbb{Q}$. We claim that there is a nonzero element $s \in \widehat{H}^0(X, m\overline{L})$ such that

$$-\log\|s\|_\infty \geq \frac{\widehat{h}^0(X, m\overline{L})}{[K : \mathbb{Q}]\widehat{h}^0(X, m\widetilde{L})} - 2.$$

The claim follows from a basic result in the classical geometry of lattices. For example, we can apply [YZt, Prop. 2.1]. To match the notations, denote by M the $\mathbb{Z}$-submodule of $\widehat{H}^0(X, m\widetilde{L})$ generated by $\widehat{H}^0(X, m\overline{L})$. Then $\overline{M} = (M, \|\cdot\|_{\infty,\sup})$ is a normed $\mathbb{Z}$-module in the sense of the loc. cit.. Denote $r = \operatorname{rank} M$, which is at most $[K : \mathbb{Q}]\widehat{h}^0(X, m\widetilde{L})$. Note that

$$\alpha := \frac{1}{r}\widehat{h}^0(\overline{M}) - 2 \geq \frac{\widehat{h}^0(X, m\overline{L})}{[K : \mathbb{Q}]\widehat{h}^0(X, m\widetilde{L})} - 2 > 0.$$

By the first inequality of [YZt, Prop. 2.1], we have

$$\widehat{h}^0(\overline{M}(-\alpha)) > \widehat{h}^0(\overline{M}) - r\alpha - r\log 3 > 0.$$

Then there is a nonzero element $s \in \widehat{H}^0(\overline{M}(-\alpha))$ satisfying $-\log \|s\|_{\infty,\sup} \geq \alpha$. This proves the claim.

With the section $s \in \widehat{H}^0(X, m\overline{L})$, for any $x \in X(\overline{K})$ not contained in $\mathrm{div}_X(s)$, we have

$$m\, h_{\overline{L}}(x) = \frac{1}{\deg_K(x)} \widehat{\deg}(m\overline{L}|_{x'}) = \frac{1}{\deg_K(x)} \sum_v \sum_{y \in x'_v} (-\log \|s(y)\|_v^{\deg_{\mathbb{Q}_v}(y)}).$$

The first summation is over all places v of $\mathbb{Q}$, but the point $x \in X(\overline{K})$, so there are a lot of Galois orbits in the above. Namely, x' is the closed point of X corresponding to x, x'_v is the image of $x' \times_{\mathbb{Q}} \mathbb{Q}_v$ in $X_{\mathbb{Q}_v}$, which is a finite set of closed points of $X_{\mathbb{Q}_v}$. Any $y \in x'_v$ is also viewed as a classical point of X_v^{an}. Then we have

$$m\, h_{\overline{L}}(x) \geq [K : \mathbb{Q}]\alpha,$$

since $\|s\|_v \leq 1$ for any finite v and $-\log \|s\|_{\infty,\sup} \geq \alpha$. It follows that

$$\lambda_1(X, \overline{L}) \geq \frac{1}{m}[K : \mathbb{Q}]\alpha.$$

This finishes the proof. □

5.3.7 The height inequality

In the end, we present the following height inequality, which is a general form of the height inequality in Theorem 6.2.2. It holds over finitely generated fields in a suitable sense, but we restrict to global fields for simplicity.

Theorem 5.3.5 (height inequality)**.** *Let k be either $\mathbb{Z}$ or a field. If $k = \mathbb{Z}$, let K be a number field; if k is a field, let K be a function field of one variable over k. Let $\pi : X \to S$ be a morphism of quasi-projective varieties over K. Let $\overline{L} \in \widehat{\mathrm{Pic}}(X/k)$ and $\overline{M} \in \widehat{\mathrm{Pic}}(S/k)$ be adelic line bundles.*

(1) If $\overline{L}$ is big on X, then there exist $\epsilon > 0$ and a Zariski open and dense subvariety U of X such that

$$h_{\overline{L}}(x) \geq \epsilon\, h_{\overline{M}}(\pi(x)), \qquad \forall\, x \in U(\overline{K}).$$

(2) If $\overline{L}$ is nef on X/k, and the image $\widetilde{L}$ of $\overline{L}$ under the canonical map $\widehat{\mathrm{Pic}}(X/k) \to \widehat{\mathrm{Pic}}(X/K)$ is big on X/K, then for any $c > 0$, there exist $\epsilon > 0$ and a Zariski open and dense subvariety U of X such that

$$h_{\overline{L}}(x) \geq \epsilon\, h_{\overline{M}}(\pi(x)) - c, \qquad \forall\, x \in U(\overline{K}).$$

(3) If the image $\widetilde{L}$ of $\overline{L}$ under the canonical map $\widehat{\mathrm{Pic}}(X/k) \to \widehat{\mathrm{Pic}}(X/K)$ is big on X/K, then there exist $c > 0$, $\epsilon > 0$, and a Zariski open and dense subvariety U of X such that

$$h_{\overline{L}}(x) \geq \epsilon\, h_{\overline{M}}(\pi(x)) - c, \qquad \forall\, x \in U(\overline{K}).$$

Proof. We only write the proofs in the arithmetic case $k = \mathbb{Z}$, since the geometric case is similar.

We first see that (1) implies (2). In fact, if $(\overline{L}, \widetilde{L})$ is as in (2), let $\overline{N} \in \widehat{\mathrm{Pic}}(O_K)$ be a hermitian line bundle with $\widehat{\deg}(\overline{N}) = 1$, and view $\overline{N}$ as an adelic line bundle on X by pull-back. Denote $\overline{L}' = \overline{L} + c\overline{N}$ for a rational number $c > 0$. It follows that

$$\overline{L}'^d = (\overline{L} + c\overline{N})^d = \overline{L}^d + dc\widetilde{L}^{d-1} > 0.$$

Here $d = \dim X + 1$. Then $\overline{L}'$ is big and we can apply (1) to $(\overline{L}', \overline{M})$. This gives (2) by the simple relation

$$h_{\overline{L}'}(x) = h_{\overline{L}}(x) + c.$$

Now we see that (1) implies (3). In fact, if $(\overline{L}, \widetilde{L})$ is as in (3), we still denote $\overline{L}' = \overline{L} + c\overline{N}$ for a rational number $c > 0$. By Lemma 5.2.10, $\overline{L}'$ is big on $X/\mathbb{Z}$ for sufficiently large rational number $c > 0$. We still apply (1) to $(\overline{L}', \overline{M})$.

Now we prove (1). The key is that there exists a rational number $\epsilon > 0$ such that $\widehat{\mathrm{vol}}(\overline{L} - \epsilon\pi^*\overline{M}) > 0$. If $\overline{L}$ and $\overline{M}$ are nef, this is a consequence of Theorem 5.2.2, which asserts

$$\widehat{\mathrm{vol}}(\overline{L} - \epsilon\pi^*\overline{M}) \geq \overline{L}^d - d\epsilon\, \overline{L}^{d-1} \cdot \pi^*\overline{M}.$$

In general, by the continuity of the volume function in Theorem 5.2.9,

$$\lim_{\epsilon \to 0} \widehat{\mathrm{vol}}(\overline{L} - \epsilon\pi^*\overline{M}) = \widehat{\mathrm{vol}}(\overline{L}).$$

Then there still exists such an ϵ.

Consequently, there is a positive integer m and a nonzero effective section s of $m(\overline{L} - \epsilon\pi^*\overline{M})$ on X. We claim that this implies

$$h_{\overline{L}-\epsilon\pi^*\overline{M}}(x) \geq 0, \quad \forall x \in X(\overline{K}),\ s(x) \neq 0.$$

Then the result follows by the simple relation

$$h_{\overline{L}-\epsilon\pi^*\overline{M}}(x) = h_{\overline{M}}(x) - \epsilon h_{\overline{M}}(\pi(x)).$$

For the claim, the reason is already in the proof of Lemma 5.3.4. Alternatively, denote by x' the closed point of X corresponding to x, Then the pull-back of $\overline{L} - \epsilon\pi^*\overline{M}$ to x' gives an adelic line bundle on x' with a nonzero effective section, and thus is linearly equivalent to an effective adelic divisor on x'. This effective adelic divisor can be written as a limit of effective model divisors on a quasi-projective model of x', and thus the degree is non-negative. $\square$

5.4 Equidistribution: conjectures and theorems

In this subsection, we formulate an equidistribution conjecture and prove two equidistribution theorems for small points. More precisely, we have the following:

(1) Theorem 5.4.3, an equidistribution theorem for quasi-projective varieties over number fields or function fields of one variable;

(2) Conjecture 5.4.1, an equidistribution conjecture for quasi-projective varieties over finitely generated fields;

(3) Theorem 5.4.6, an equidistribution theorem for morphisms between quasi-projective varieties over number fields or function fields of one variable.

All these statements generalize the equidistribution theorems of Szpiro–Ullmo–Zhang [SUZ], Chambert-Loir [CL] and Yuan [Yua1] for projective varieties over number fields. Theorem 5.4.6 also generalizes an equidistribution theorem of Moriwaki [Mor3]. Conjecture 5.4.1 generalizes Theorem 5.4.3 by changing the base fields; Theorem 5.4.6 generalizes Theorem 5.4.3 by changing it to a relative version.

Our main ingredient is the extension of the arithmetic Hilbert–Samuel formula and Yuan's bigness theorem to quasi-projective varieties in Theorem 5.2.2, with which we are able to apply the variational principle of Szpiro–Ullmo–Zhang to the current quasi-projective situation.

5.4.1 Small points

We will first state the equidistribution conjecture (Conjecture 5.4.1), and then prove the two equidistribution theorems. We start with some definitions.

Let k be either $\mathbb{Z}$ or a field. Let X be a quasi-projective variety of dimension n over a finitely generated field F over k. Let d be the dimension of any quasi-projective model of F over k. Let $\overline{L}$ be a nef adelic line bundle on X. Recall that we have a Moriwaki height function

$$h_{\overline{L}}^{\overline{H}} : X(\overline{F}) \longrightarrow \mathbb{R}_{\geq 0}$$

for any polarization $\overline{H} \in \widehat{\mathrm{Pic}}(F/k)_{\mathrm{nef}}$.

Denote by $\widetilde{L}$ the image of $\overline{L}$ under the canonical map $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{nef}} \to \widehat{\mathrm{Pic}}(X/F)_{\mathrm{nef}}$ introduced in §2.5.5. Assume that the self-intersection number (defined in Proposition 4.1.1)

$$\deg_{\widetilde{L}}(X/F) = \widetilde{L}^{\dim X} > 0.$$

Then we have a well-defined Moriwaki height

$$h_{\overline{L}}^{\overline{H}}(X) = \frac{\overline{L}^{n+1} \cdot \overline{H}^{d-1}}{(n+1)\deg_{\widetilde{L}}(X/F)}.$$

A sequence $\{x_m\}_{m\geq 1}$ in $X(\overline{F})$ is said to be *generic* if any closed subvariety $Y \subsetneqq X$ contains only finitely many terms of the sequence.

Let $\overline{H} \in \mathrm{Pic}(F/k)_{\mathbb{Q},\mathrm{nef}}$ be a polarization. A sequence $\{x_m\}$ in $X(\overline{F})$ is said to be *directionally small with respect to* $\overline{H}$, or just $h_{\overline{L}}^{\overline{H}}$*-small*, if $h_{\overline{L}}^{\overline{H}}(x_m)$ converges to $h_{\overline{L}}^{\overline{H}}(X)$.

A sequence $\{x_m\}$ in $X(\overline{F})$ is said to be *small*, or just $\mathfrak{h}_{\overline{L}}$*-small*, if it is $h_{\overline{L}}^{\overline{H}}$-small for any polarization $\overline{H} \in \mathrm{Pic}(F/k)_{\mathbb{Q},\mathrm{nef}}$.

If X is projective, so that $\mathfrak{h}_{\overline{L}}(X)$ is well-defined. Then the sequence is $\mathfrak{h}_{\overline{L}}$-small if and only if $\mathfrak{h}_{\overline{L}}(x_m)$ converges to $\mathfrak{h}_{\overline{L}}(X)$ *numerically* in $\mathrm{Pic}(F/k)_{\mathrm{int},\mathbb{Q}}$ in the sense that

$$\lim_{m\to\infty} \mathfrak{h}_{\overline{L}}(x_m) \cdot \overline{H}_1 \cdot \overline{H}_2 \cdots \overline{H}_{d-1} = \mathfrak{h}_{\overline{L}}(X) \cdot \overline{H}_1 \cdot \overline{H}_2 \cdots \overline{H}_{d-1}$$

for any $\overline{H}_1, \cdots, \overline{H}_{d-1} \in \mathrm{Pic}(F/k)_{\mathrm{int},\mathbb{Q}}$.

If $k = \mathbb{Z}$ and F is a number field, both $\widehat{\deg}\, \mathfrak{h}_{\overline{L}}$ and $h_{\overline{L}}^{\overline{H}}$ are equal to the usual height function $h_{\overline{L}}$. Then both smallness are equivalent to the usual one given by $h_{\overline{L}}(x_m) \to h_{\overline{L}}(X)$.

5.4.2 Equilibrium measure

Resume the above notations for $(k, F, X, \overline{L})$.

Let $v \in \mathcal{M}(F/k)$ be a point corresponding to a non-trivial valuation of F; i.e., a non-trivial multiplicative norm $|\cdot|_v : F \to \mathbb{R}$. It could be either archimedean or non-archimedean. Denote by F_v the completion of F with respect to v. Then we have the Berkovich space X_v^{an} associated to the variety X_{F_v} over the complete field F_v.

There is an *equilibrium measure*

$$d\mu_{\overline{L},v} := \frac{1}{\deg_{\widetilde{L}}(X/F)} c_1(\overline{L})_v^n$$

over the analytic space X_v^{an}. If v is archimedean, this is classical. If v is non-archimedean, this is defined in terms of the Chambert-Loir measure developed by Chambert-Loir and Ducros in [CD]. We refer to §3.6.7 for the precise definition.

5.4.3 Equidistribution conjecture over finitely generated fields

For each point $x \in X(\overline{F})$, we have the measure

$$\mu_{x,v} := \frac{1}{\deg(x)} \delta_{x'_v}$$

on X_v^{an}. Here $\delta_{x'_v}$ is the Dirac measure for the Galois orbit x'_v of x in X_v^{an}. More precisely, x' is the closed point of X corresponding to x, and x'_v is the image of $x' \times_F F_v$ in X_{F_v}, viewed as a finite set of classical points of X_v^{an}.

We say the Galois orbit of a sequence $\{x_m\}_{m\geq 1}$ of points of $X(\overline{F})$ is *equidistributed in X_v^{an} with respect to $d\mu_{\overline{L},v}$* if the weak convergence

$$\mu_{x_m,v} \longrightarrow d\mu_{\overline{L},v}$$

holds on X_v^{an}. Namely,

$$\int_{X_v^{\mathrm{an}}} f\,\mu_{x_m,v} \longrightarrow \int_{X_v^{\mathrm{an}}} f\,d\mu_{\overline{L},v}$$

for any $f \in C_c(X_v^{\mathrm{an}})$. Here $C_c(X_v^{\mathrm{an}})$ is the space of real-valued continuous and compactly supported functions on X_v^{an}.

Finally, we are ready to state our equidistribution conjecture. Recall that $\widetilde{L}$ denotes the image of $\overline{L}$ under the map $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{nef}} \to \widehat{\mathrm{Pic}}(X/F)_{\mathrm{nef}}$.

Conjecture 5.4.1 (equidistribution over finitely generated fields)**.** *Let k be either $\mathbb{Z}$ or a field. Let F be a finitely generated field over k. Let v be a non-trivial valuation of F. Assume that the restriction of v to k is trivial if k is a field. Let X be a quasi-projective variety over F. Let $\overline{L}$ be a nef adelic line bundle on X/k such that $\deg_{\widetilde{L}}(X/F) > 0$. Let $\{x_m\}_m$ be a generic sequence of small points in $X(\overline{F})$. Then the Galois orbit of $\{x_m\}_m$ is equidistributed in X_v^{an} with respect to $d\mu_{\overline{L},v}$.*

In the arithmetic case ($k = \mathbb{Z}$), if F is a number field and X is projective, the conjecture is fully known previously. In fact, the pioneering work of Szpiro–Ullmo–Zhang [SUZ] proved the equidistribution for number fields F and archimedean places v assuming pointwise positivity of the Chern form $c_1(L, \|\cdot\|_v)$. Their work was extended to non-archimedean places v by Chambert-Loir [CL]. The full case of number fields with L ample was proved by Yuan [Yua1] by developing a bigness theorem for difference of ample hermitian line bundles. The proof of [Yua1] actually works by replacing the ampleness of L by the positivity $\deg_L(X) > 0$. For more history of this subject, we refer to [Yua3, §6.3].

The above arguments were also generalized to the geometric case. In that case, if X is projective over F, and the valuation v of F comes from a prime divisor of a projective model of F over k, the conjecture was proved independently by Faber [Fab] when the transcendental degree of F/k is 1 and by Gubler [Gub2] for general transcendental degrees.

In Theorem 5.4.3 below, we will prove the conjecture for any quasi-projective X and for any number field F or function field F of one variable. However, the conjecture seems widely open if F has a positive transcendental degree over $\mathbb{Z}$.

Remark 5.4.2. In the conjecture, we have assumed that v is a non-trivial valuation of F. Nonetheless, if v is the trivial valuation of F, a similar equidistribution theorem on X_v^{an} was proved by [Xie, Cor. 5.6]. Here the equilibrium measure on X_v^{an} is the Dirac measure supported at the point corresponding to the trivial valuation of the function field of X_{F_v}.

5.4.4 Equidistribution theorem over number fields

The goal here is prove the following theorem, which asserts that Conjecture 5.4.1 holds if F is a number field or a function field of one variable. It is a consequence of the variational principle of [SUZ, Yua1] and the bigness result in Theorem 5.2.2.

Theorem 5.4.3 (equidistribution over number fields)**.** *Let k be either $\mathbb{Z}$ or a field. Let K be a number field if $k = \mathbb{Z}$; let K be the function field of one variable over k if k is a field. Let X be a quasi-projective variety over K. Let $\overline{L}$ be a nef adelic line bundle on X/k such that $\deg_{\widetilde{L}}(X/K) > 0$. Let $\{x_m\}_m$ be a generic sequence in $X(\overline{K})$ such that $\{h_{\overline{L}}(x_m)\}_m$ converges to $h_{\overline{L}}(X)$. Then the Galois orbit of $\{x_m\}_m$ is equidistributed in X_v^{an} with respect to $d\mu_{\overline{L},v}$ for any place v of K.*

Proof. We only write the proof for the arithmetic case $k = \mathbb{Z}$, since the geometric case is similar. Apply the variational principle of [SUZ, Yua1] to Theorem 5.2.2. The process is standard at the beginning, and then there will be a new situation due to quasi-projectivity.

The conditions and the result do not change if replacing $\overline{L}$ by $\overline{L}+\pi^*\overline{N}$ for an element $\overline{N} \in \widehat{\mathrm{Pic}}(K)_{\mathrm{int}}$ with $\widehat{\deg}(\overline{N}) > 0$. Here $\pi^* : \widehat{\mathrm{Pic}}(K)_{\mathrm{int}} \to \widehat{\mathrm{Pic}}(X)_{\mathrm{int}}$ is the pull-back map. As a consequence, we can assume $\overline{L}^{n+1} > 0$. Here we denote $n = \dim X$.

Let $\overline{M}$ be an element in the kernel of the map $\widehat{\mathrm{Pic}}(X)_{\mathrm{int}} \to \widehat{\mathrm{Pic}}(X/K)_{\mathrm{int}}$. Let ϵ be a nonzero rational number. By Lemma 5.3.4,

$$\lambda_1(X, \overline{L} + \epsilon\overline{M}) \geq \frac{\widehat{\mathrm{vol}}(\overline{L} + \epsilon\overline{M})}{(n+1)\widehat{\mathrm{vol}}(\widetilde{L})},$$

if both $\widehat{\mathrm{vol}}(\overline{L} + \epsilon\overline{M})$ and $\widehat{\mathrm{vol}}(\widetilde{L})$ are strictly positive.

Now it is straight-forward to apply Theorem 5.2.2. In fact, by writing $\overline{M}$ as the difference of two nef adelic line bundles, Theorem 5.2.2 implies

$$\widehat{\mathrm{vol}}(\overline{L} + \epsilon\overline{M}) \geq \overline{L}^{n+1} + \epsilon(n+1)\overline{L}^n\overline{M} + O(\epsilon^2).$$

By the assumption $\overline{L}^{n+1} > 0$, the right-hand side is strictly positive if $|\epsilon|$ is sufficiently small. Theorem 5.2.2 also implies the geometric volume

$$\widehat{\mathrm{vol}}(\widetilde{L}) = \widetilde{L}^n = \deg_{\widetilde{L}}(X)$$

which is assumed to be strictly positive. It follows that

$$\lambda_1(X, \overline{L} + \epsilon\overline{M}) \geq \frac{\overline{L}^{n+1} + \epsilon(n+1)\overline{L}^n\overline{M}}{(n+1)\deg_{\widetilde{L}}(X)} + O(\epsilon^2).$$

Apply the inequality to the generic sequence $\{x_m\}_m$. We have

$$\liminf_{m\to\infty} h_{\overline{L}+\epsilon\overline{M}}(x_m) \geq \frac{\overline{L}^{n+1}+\epsilon(n+1)\overline{L}^n\overline{M}}{(n+1)\deg_{\widetilde{L}}(X)} + O(\epsilon^2).$$

By assumption,

$$\lim_{m\to\infty} h_{\overline{L}}(x_m) = h_{\overline{L}}(X) = \frac{\overline{L}^{n+1}}{(n+1)\deg_{\widetilde{L}}(X)}.$$

Then the inequality implies

$$\liminf_{m\to\infty} \epsilon h_{\overline{M}}(x_m) \geq \epsilon\frac{\overline{L}^n\overline{M}}{\deg_{\widetilde{L}}(X)} + O(\epsilon^2).$$

If $\epsilon > 0$, the above implies

$$\liminf_{m\to\infty} h_{\overline{M}}(x_m) \geq \frac{\overline{L}^n\overline{M}}{\deg_{\widetilde{L}}(X)} + O(\epsilon).$$

If $\epsilon < 0$, the above implies

$$\limsup_{m\to\infty} h_{\overline{M}}(x_m) \leq \frac{\overline{L}^n\overline{M}}{\deg_{\widetilde{L}}(X)} + O(|\epsilon|).$$

Set $\epsilon \to 0$ in each case. We obtain

$$\lim_{m\to\infty} h_{\overline{M}}(x_m) = \frac{\overline{L}^n\overline{M}}{\deg_{\widetilde{L}}(X)}.$$

We are going to deduce the equidistribution theorem on X_v^{an} from the above limit identity. Assume that $\overline{L} \in \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathbb{Q},\mathrm{nef}}$ for a quasi-projective model $\mathcal{U}$ of X over $\mathbb{Z}$, and assume that $\overline{L}$ is represented by a Cauchy sequence $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod},\mathbb{Q}}$. Here $\mathcal{X}_i$ is a projective model of $\mathcal{U}$, and $\overline{\mathcal{L}}_i$ is a hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}_i$. Assume that there is a morphism $\psi_i : \mathcal{X}_i \to \mathcal{X}_1$ extending the identity morphism of $\mathcal{U}$. Denote $X_i = \mathcal{X}_{i,\mathbb{Q}}$, which contains X as an open subvariety.

Let $\mathcal{X}_1'$ be another projective model of X_1 over $\mathbb{Z}$. Let $\overline{\mathcal{M}}$ be a hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}_1'$, with a fixed isomorphism $\mathcal{M}_{\mathbb{Q}} \to \mathcal{O}_{X_1}$. Then it induces a metric $\|\cdot\|_w$ of $\mathcal{O}_{X_1}$ on $X_{1,w}^{\mathrm{an}}$ for any place w of K. Assume that the metric $\|1\|_w = 1$ for any place $w \neq v$ of K. Denote $f = -\log \|1\|_v$, which is continuous on $X_{1,v}^{\mathrm{an}}$. By definition,

$$h_{\overline{\mathcal{M}}}(x_m) = \int_{X_v^{\mathrm{an}}} f \mu_{x_m,v},$$

and

$$\overline{L}^n \overline{\mathcal{M}} = \lim_{i\to\infty} \overline{\mathcal{L}}_i^n \overline{\mathcal{M}} = \lim_{i\to\infty} \int_{X_{i,v}^{\mathrm{an}}} f c_1(\overline{\mathcal{L}}_i)_v^n.$$

Then the above result gives a limit identity

$$\lim_{m\to\infty} \int_{X_v^{\mathrm{an}}} f \mu_{x_m,v} = \frac{1}{\deg_{\widetilde{L}}(X)} \lim_{i\to\infty} \int_{X_{i,v}^{\mathrm{an}}} f\, c_1(\overline{\mathcal{L}}_i)_v^n.$$

Here f is viewed as a function on $X_{i,v}^{\mathrm{an}}$ by the pull-back induced by $\psi_{i,\mathbb{Q}} : X_i \to X_1$.

Now we going to vary $f = -\log \|1\|_v$, which is a model function on $X_{1,v}^{\mathrm{an}}$ associated to $(\mathcal{X}_1', \overline{\mathcal{M}})$. By Gubler's density theorem (cf. [Gub3, Thm. 7.12] and [Yua1, Lem. 3.5]), the space of all such model functions f is dense in $C(X_{1,v}^{\mathrm{an}})$ under the topology of uniform convergence. Note that

$$\lim_{i\to\infty} \int_{X_{i,v}^{\mathrm{an}}} c_1(\overline{\mathcal{L}}_i)_v^n = \lim_{i\to\infty} (\mathcal{L}_{i,\mathbb{Q}})^n = \widetilde{L}^n = \deg_{\widetilde{L}}(X).$$

Therefore, the limit identity also holds for any $f \in C(X_{1,v}^{\mathrm{an}})$.

Finally, assume $f \in C_c(X_v^{\mathrm{an}})$, viewed as an element of $C(X_{i,v}^{\mathrm{an}})$ by the open immersion $X \to X_i$. Then

$$\lim_{i\to\infty} \int_{X_{i,v}^{\mathrm{an}}} f c_1(\overline{\mathcal{L}}_i)_v^n = \lim_{i\to\infty} \int_{X_v^{\mathrm{an}}} f\, c_1(\overline{\mathcal{L}}_i)_v^n|_{X_v^{\mathrm{an}}} = \int_{X_v^{\mathrm{an}}} f c_1(\overline{L})_v^n.$$

Here the last equality follows from the definition of $c_1(\overline{L})_v^n$ in §3.6, based on the theory of [CD] in the non-archimedean case. Therefore, the limit identity becomes

$$\lim_{m\to\infty} \int_{X_v^{\mathrm{an}}} f \mu_{x_m,v} = \frac{1}{\deg_{\widetilde{L}}(X)} \int_{X_v^{\mathrm{an}}} f c_1(\overline{L})_v^n.$$

This proves the equidistribution theorem. $\square$

5.4.5 Total volume

In the following, we prove that the equilibrium measure $d\mu_{\overline{L},v}$ in Theorem 5.4.3 is indeed a probability measure. Our proof uses the global intersection theory to bounded local integrals, so it only works over number fields and function fields of one variable. We refer to Gauthier–Vigny [GV, Thm. B] for a complex approach of such a result in the dynamical setting.

Lemma 5.4.4. *Let k be either $\mathbb{Z}$ or a field. Let K be a number field if $k=\mathbb{Z}$; let K be the function field of one variable over k if k is a field. Let X be a quasi-projective variety of dimension n over K. Let $\overline{L}_1,\cdots,\overline{L}_n$ be integrable adelic line bundles on X/k, and let $\widetilde{L}_1,\cdots,\widetilde{L}_n$ be their images under the map $\widehat{\mathrm{Pic}}(X/k)\to\widehat{\mathrm{Pic}}(X/K)$. Then for any place v of K,*

$$\int_{X_v^{\mathrm{an}}} c_1(\overline{L}_1)_v\cdots c_1(\overline{L}_n)_v=\widetilde{L}_1\cdot\widetilde{L}_2\cdots\widetilde{L}_n.$$

Proof. By multi-linearity, it suffices to assume that all $\overline{L}_1,\cdots,\overline{L}_n$ are strongly nef. By multi-linearity again, it suffices to assume that all $\overline{L}_1,\cdots,\overline{L}_n$ are isomorphic to the same adelic line bundle $\overline{L}$ on X.

Assume that $\overline{L}$ is represented by a Cauchy sequence $\overline{\mathcal{L}}=(\mathcal{L},(\mathcal{X}_i,\overline{\mathcal{L}}_i,\ell_i)_{i\geq 1})$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})_{\mathrm{mod}}$. Here $\mathcal{U}$ is a quasi-projective model of X over k, and each $\overline{\mathcal{L}}_i$ is nef on $\mathcal{X}_i$. We further assume that for each $i\geq 1$, there is a morphism $\phi_i:\mathcal{X}_i\to\mathcal{X}_0$ extending the identity morphism of $\mathcal{U}$. Here $(\mathcal{X}_0,\overline{\mathcal{E}}_0)$ is a boundary divisor. Denote $X_i=\mathcal{X}_{i,\mathbb{Q}}$, which is a projective model of X over K.

The weak convergence formula gives, for any $f\in C_c(X_v^{\mathrm{an}})$,

$$\int_{X_v^{\mathrm{an}}} fc_1(\overline{L})_v^n=\lim_{i\to\infty}\int_{X_{i,v}^{\mathrm{an}}} fc_1(\overline{\mathcal{L}}_i)_v^n.$$

See §3.6.6 for more details. As X_i is projective over K, the right-hand side is equal to the integration defined by global intersection numbers by [CL, Gub1]. It suffices to extend this formula to the case $f=1$.

Denote by $\widetilde{g}_v\geq 0$ the Green's function of $\overline{\mathcal{E}}_0$ on $X_{0,v}^{\mathrm{an}}$. For any $m\geq 1$, define a continuous and compactly supported function $f_m:X_{0,v}^{\mathrm{an}}\to\mathbb{R}$ by

(1) $f_m(x)=1$ if $\widetilde{g}_v(x)\leq m$;

(2) $f_m(x)=m+1-\widetilde{g}_v(x)$ if $m\leq\widetilde{g}_v(x)\leq m+1$;

(3) $f_m(x) = 0$ if $\widetilde{g}_v(x) \geq m+1$.

Then f_m increases to the constant function 1 on X_v^{an}. We have

$$\int_{X_v^{\mathrm{an}}} c_1(\overline{L})_v^n = \lim_{m\to\infty} \int_{X_v^{\mathrm{an}}} f_m c_1(\overline{L})_v^n = \lim_{m\to\infty} \lim_{i\to\infty} \int_{X_{i,v}^{\mathrm{an}}} f_m c_1(\overline{\mathcal{L}}_i)_v^n.$$

The first equality follows from Lebesgue's monotone convergence theorem, and second equality holds by viewing f_m as an element of $C_c(X_v^{\mathrm{an}})$. Then it suffices to prove

$$\lim_{m\to\infty} \lim_{i\to\infty} \int_{X_{i,v}^{\mathrm{an}}} (1-f_m) c_1(\overline{\mathcal{L}}_i)_v^n = 0.$$

By definition, $0 \leq 1 - f_m \leq \widetilde{g}_v/m$ everywhere on X_v^{an}. Therefore, it suffices to prove $\int_{X_{i,v}^{\mathrm{an}}} \phi_i^* \widetilde{g}_v \, c_1(\overline{\mathcal{L}}_i)_v^n$ is bounded above as i varies.

By the global intersection formula of Chambert-Loir and Thuillier in [CT, Thm. 1.4],

$$\overline{\mathcal{L}}_i^n \cdot \phi_i^* \overline{\mathcal{D}} = (\overline{\mathcal{L}}_i|_{\mathcal{H}_i})^n + \sum_v \int_{X_{i,v}^{\mathrm{an}}} \phi_i^* \widetilde{g}_v \, c_1(\overline{\mathcal{L}}_i)_v^n.$$

Here $\mathcal{H}_i$ is the horizontal part of $\phi_i^* \mathcal{D}$, as an effective divisor on $\mathcal{X}_i$. As $\overline{\mathcal{L}}_i$ is nef, $\overline{\mathcal{D}}$ is effective, and $\widetilde{g}_0 \geq 0$, every term on the right-hand side is non-negative. This gives

$$\int_{X_{i,v}^{\mathrm{an}}} \phi_i^* \widetilde{g}_v \, c_1(\overline{\mathcal{L}}_i)_v^n \leq \overline{\mathcal{L}}_i^n \cdot \phi_i^* \overline{\mathcal{D}}.$$

The right-hand sides converges to $\overline{L}^n \cdot \overline{\mathcal{D}}$ by Proposition 4.1.1. This finishes the proof. □

Remark 5.4.5. In Lemma 5.4.4, the result is local at the place v, but the condition assumes that the adelic line bundles come from a global field. This global assumption seems strange, but gives us convenience to bound local integrals by global intersection numbers. In a recent work, Guo [Guo] proves the volume formula by local methods, and thus removes our global assumption.

5.4.6 Equidistribution theorem in the relative situation

Inspired by an original idea of Moriwaki [Mor3], we generalize Theorem 5.4.3 to equidistribution of directionally small points in the relative situation. The statement is closely related to the fundamental inequality in Theorem 5.3.2. The key is still the variational principle of [SUZ, Yua1] and the bigness result in Theorem 5.2.2. The theorem is as follows.

Theorem 5.4.6 (equidistribution in the relative case)**.** *Let k be either $\mathbb{Z}$ or a field. Let K be a number field if $k=\mathbb{Z}$; let K be the function field of one variable over k if k is a field. Let $\pi: U \to V$ be a flat morphism of relative dimension n of quasi-projective varieties over K. Set d to be $\dim V+1$ if $k=\mathbb{Z}$; set d to be $\dim V$ if k is a field. Let $X \to \operatorname{Spec} F$ be the generic fiber of $U \to V$.*

Let $\overline{H}$ be an element of $\widehat{\operatorname{Pic}}(V/k)_{\rm nef}$ satisfying the Moriwaki condition that $\overline{H}$ is nef, $\overline{H}^d=0$ and $\widetilde{H}^{d-1}>0$. Here $\widetilde{H}$ is the image of $\overline{H}$ in $\widehat{\operatorname{Pic}}(V/K)_{\rm nef}$.

Let $\overline{L}$ be an element of $\widehat{\operatorname{Pic}}(U/k)_{\rm nef}$ such that $\deg_{\widetilde{L}}(X/F)>0$. Here $\widetilde{L}$ is the image of $\overline{L}$ under the canonical composition

$$\widehat{\operatorname{Pic}}(U/k)_{\rm nef} \longrightarrow \widehat{\operatorname{Pic}}(X/k)_{\rm nef} \longrightarrow \widehat{\operatorname{Pic}}(X/F)_{\rm nef}.$$

Let $\{x_m\}_m$ be a generic sequence in $X(\overline{F})$ such that $h_{\overline{L}}^{\overline{H}}(x_m)$ converges to $h_{\overline{L}}^{\overline{H}}(X)$. Then for any place v of K, there is a weak convergence

$$\frac{1}{\deg(x_m)}\delta_{\Delta(x_m),v}\, c_1(\pi^*\overline{H})_v^{d-1} \longrightarrow \frac{1}{\deg_{\widetilde{L}}(X/F)} c_1(\overline{L})_v^n c_1(\pi^*\overline{H})_v^{d-1}$$

of measures on $U_v^{\rm an}$. Here $\Delta(x_m)\subset U$ denotes the Zariski closure of the image of x_m in U, and $\delta_{\Delta(x_m),v}$ denotes the Dirac current of $\Delta(x_m)_{K_v}^{\rm an}$ in $U_v^{\rm an}$.

In the theorem, the weak convergence means that

$$\frac{1}{\deg(x_m)}\int_{\Delta(x_m)_{K_v}^{\rm an}} f\, c_1(\pi^*\overline{H})_v^{d-1} \longrightarrow \frac{1}{\deg_{\widetilde{L}}(X/F)}\int_{U_v^{\rm an}} f\, c_1(\overline{L})_v^n c_1(\pi^*\overline{H})_v^{d-1}$$

for any continuous and compactly supported function $f: U_v^{\rm an}\to\mathbb{R}$. Here the measures are defined in §3.6.7.

The prototype of the theorem is [Mor3, Thm. 6.1], which proves the equidistribution at archimedean places with the additional assumption that $U\to V$ is projective and the metric of $\overline{L}$ is smooth and strictly positive (at archimedean places).

Proof. The proof is a hybrid of the proofs of Theorem 5.4.3 and Theorem 5.3.2. As in the proof of Theorem 5.4.3, let $\overline{M}$ be an element in the kernel of the map $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int}} \to \widehat{\mathrm{Pic}}(X/K)_{\mathrm{int}}$. Let ϵ be a nonzero rational number. The key is the claim that

$$\lambda_1^{\overline{H}}(X, \overline{L} + \epsilon\overline{M}) \geq h^{\overline{H}}_{\overline{L}+\epsilon\overline{M}}(X) + O(\epsilon^2), \qquad \epsilon \to 0.$$

Let us first see how the claim implies the equidistribution theorem, following the proof of Theorem 5.4.3. In fact, the claim gives

$$\liminf_{m\to\infty} \epsilon\, h_{\overline{M}}(x_m) \geq \epsilon \frac{\overline{M} \cdot \overline{L}^n \cdot \overline{H}^{d-1}}{\deg_{\widetilde{L}}(X)} + O(\epsilon^2).$$

Then this implies

$$\lim_{m\to\infty} h_{\overline{M}}(x_m) = \frac{\overline{M} \cdot \overline{L}^n \cdot \overline{H}^{d-1}}{\deg_{\widetilde{L}}(X)}.$$

As in the proof of Theorem 5.4.3, it further implies the equidistribution theorem by taking $\overline{M}$ to be the trivial bundle on U with metrics given by model functions.

Now we prove the claim. The proof is similar to that of Theorem 5.3.2, but more delicate due to the extra term $\epsilon\overline{M}$. As in that proof, let $\overline{N} \in \widehat{\mathrm{Pic}}(B/k)_{\mathbb{Q}}$ be an element of degree 1. Here if $k = \mathbb{Z}$, then $B = \mathrm{Spec}\, O_K$ and further assume that $\overline{N}$ comes from the pull-back of $\widehat{\mathrm{Pic}}(\mathbb{Z})_{\mathbb{Q}}$; if k is a field, then B is the unique projective and regular curve over k with function field K.

We make two convenient assumptions. First, assume that $\epsilon > 0$, which can be achieved by replacing $\overline{M}$ by $-\overline{M}$ if necessary. Second, assume that $\overline{L}^{n+1}\overline{H}^{d-1} > 0$. This can be achieved by replacing $\overline{L}$ by $\overline{L} + \overline{N}$, which does not affect the inequality we want to prove.

Denote $\overline{L}' = \overline{L} - c\overline{N}$ with $c \in \mathbb{Q}$. We still have

$$\lambda_1^{\overline{H}}(X, \overline{L} + \epsilon\overline{M}) - \lambda_1^{\overline{H}}(X, \overline{L}' + \epsilon\overline{M}) = h^{\overline{H}}_{\overline{L}+\epsilon\overline{M}}(X) - h^{\overline{H}}_{\overline{L}'+\epsilon\overline{M}}(X) = c\, \widetilde{H}^{d-1}.$$

It suffices to prove that

$$\lambda_1^{\overline{H}}(X, \overline{L}' + \epsilon\overline{M}) - h^{\overline{H}}_{\overline{L}'+\epsilon\overline{M}}(X) \geq O(\epsilon^2)$$

for some rational number c.

Write $\overline{M} = \overline{A} - \overline{B}$ for nef adelic line bundles $\overline{A}, \overline{B}$ on X. Denote by $\overline{L}_K, \overline{M}_K, \overline{A}_K, \overline{B}_K$ the images of $\overline{L}, \overline{M}, \overline{A}, \overline{B}$ in $\widehat{\mathrm{Pic}}(X/K)$. Note that $\overline{M}_K = 0$ by assumption. In the following, take

$$c = c(\epsilon) = \frac{(\overline{L} + \epsilon\overline{A})^{n+1} \cdot \overline{H}^{d-1} - (n+1)(\overline{L} + \epsilon\overline{A})^n \cdot \epsilon\overline{B} \cdot \overline{H}^{d-1}}{(n+1)(\widetilde{L}^n)(\widetilde{H}^{d-1})} - \delta(\epsilon),$$

where $\delta : \mathbb{Q}_{>0} \to \mathbb{R}$ is a fixed function such that $0 < \delta(\epsilon) < \epsilon^2$ and such that $c(\epsilon)$ is always a rational number. For this choice of c, we will check that

$$h^{\overline{H}}_{\overline{L}' + \epsilon\overline{M}}(X) = O(\epsilon^2)$$

and

$$\lambda_1^{\overline{H}}(X, \overline{L}' + \epsilon\overline{M}) > 0.$$

This clearly implies the claim.

By definition, it is easy to have

$$c(\epsilon) = \frac{(\overline{L} + \epsilon\overline{M})^{n+1} \cdot \overline{H}^{d-1}}{(n+1)(\widetilde{L}^n)(\widetilde{H}^{d-1})} + O(\epsilon^2).$$

This implies

$$h^{\overline{H}}_{\overline{L}' + \epsilon\overline{M}}(X) = h^{\overline{H}}_{\overline{L} + \epsilon\overline{M}}(X) - c\,\widetilde{H}^{d-1} = O(\epsilon^2).$$

It remains to prove

$$\lambda_1^{\overline{H}}(X, \overline{L}' + \epsilon\overline{M}) > 0.$$

The assumption $\overline{H}^d = 0$ still implies that

$$\lambda_1^{\overline{H}}(X, \overline{L}' + \epsilon\overline{M}) = \lambda_1^{\overline{H}}(X, \overline{L}' + \epsilon\overline{M} + m\overline{H})$$

for all rational numbers m. Then it suffices to prove

$$\lambda_1^{\overline{H}}(X, \overline{L}' + \epsilon\overline{M} + m\overline{H}) \geq 0$$

for some m. As in the proof of Theorem 5.3.2, it suffices to prove

$$\widehat{\mathrm{vol}}(\overline{L}' + \epsilon\overline{M} + m\overline{H}) > 0$$

for sufficiently large m.

Now we estimate $\widehat{\mathrm{vol}}(\overline{L}' + \epsilon\overline{M} + m\overline{H})$ for $m > 0$. Note that $c(\epsilon) \geq 0$ when ϵ is sufficiently small, due to the assumption $\overline{L}^{n+1}\overline{H}^{d-1} > 0$. By Proposition 5.2.3,

$$\widehat{\mathrm{vol}}(\overline{L}' + \epsilon\overline{M} + m\overline{H}) \geq \widehat{\mathrm{vol}}(\overline{L} + \epsilon\overline{M} + m\overline{H}) - (d+n)\, c(\epsilon)\, \widehat{\mathrm{vol}}(\overline{L}_K + m\widetilde{H}).$$

Write

$$\overline{L} + \epsilon\overline{M} + m\overline{H} = (\overline{L} + \epsilon\overline{A} + m\overline{H}) - (\epsilon\overline{B}),$$

and apply Theorem 5.2.2 to the above terms. We have

$$\begin{aligned} & \widehat{\mathrm{vol}}(\overline{L}' + \epsilon\overline{M} + m\overline{H}) \\ \geq\ & (\overline{L} + \epsilon\overline{A} + m\overline{H})^{d+n} - (d+n)(\overline{L} + \epsilon\overline{A} + m\overline{H})^{d+n-1} \cdot \epsilon\overline{B} \\ & -(d+n)\, c(\epsilon)(\overline{L}_K + m\widetilde{H})^{d+n-1}. \end{aligned}$$

By the assumption $\overline{H}^d = 0$, the right-hand side is a polynomial in m of degree at most $d-1$, and the coefficient of m^{d-1} is equal to

$$\begin{aligned} & \binom{d+n}{d-1}(\overline{L} + \epsilon\overline{A})^{n+1}\overline{H}^{d-1} - (d+n)\binom{d+n-1}{d-1}(\overline{L} + \epsilon\overline{A})^{n} \cdot \overline{H}^{d-1} \cdot \epsilon\overline{B} \\ - \quad & (d+n)\, c(\epsilon)\binom{d+n-1}{d-1}(\widetilde{L}^n)(\widetilde{H}^{d-1}). \end{aligned}$$

This is exactly the product of $\binom{d+n}{d-1}$ with

$$(\overline{L} + \epsilon\overline{A})^{n+1}\overline{H}^{d-1} - (n+1)(\overline{L} + \epsilon\overline{A})^{n} \cdot \overline{H}^{d-1} \cdot \epsilon\overline{B} - (n+1)\, c(\epsilon)(\widetilde{L}^n)(\widetilde{H}^{d-1}).$$

It is strictly positive by the definition of $c(\epsilon)$. This finishes the proof. $\square$

Remark 5.4.7. Note that Conjecture 5.4.1 can be viewed as a fiberwise version of Theorem 5.4.6. We expect that Theorem 5.4.6 implies Conjecture 5.4.1, while the obstruction is some complicated regularization processes.

5.5 The Hodge bundle

In §2.6, we have introduced the example of Hodge bundles and mentioned that it naturally defines an adelic line bundle. The goal of this section is to state the result precisely and give a proof.

5.5.1 Hodge bundle for a general family

Recall from §2.6 that S is an integral projective scheme over $\mathbb{Z}$ or $\mathbb{Q}$, and $\pi : X \to S$ is a principally polarized abelian scheme of relative dimension g. Recall that $\omega(S) = e^*\Omega^g_{X/S}$ is the Hodge bundle on S, and the Faltings metric $\|\cdot\|_{\mathrm{Fal}}$ of $\omega(S)$ on $S(\mathbb{C})$ is defined by integration. Our precise theorem is as follows.

Theorem 5.5.1. *There is a canonically defined adelic line bundle $\overline{\omega(S)}$ on $S/\mathbb{Z}$ which extends the pair $(\omega(S), \|\cdot\|_{\mathrm{Fal}})$. Moreover,*

$$h_{\overline{\omega(S)}}(s) = h_{\mathrm{Fal}}(X_s), \quad \forall\, s \in S(\overline{\mathbb{Q}}).$$

Here we explain some terms of the theorem. First, that $\overline{\omega(S)}$ extends $(\omega(S), \|\cdot\|_{\mathrm{Fal}})$ means that the underlying line bundle of $\overline{\omega(S)}$ is $\omega(S)$, and that the metric of $\omega(S)$ on $S(\mathbb{C})$ induced by $\overline{\omega(S)}$ (via Proposition 3.5.1) is equal to $\|\cdot\|_{\mathrm{Fal}}$.

Second, by restriction, $\overline{\omega(S)}$ induces an adelic line bundle on $S_{\mathbb{Q}}$, and thus defines a height function $h_{\overline{\omega(S)}} : S_{\mathbb{Q}}(\overline{\mathbb{Q}}) \to \mathbb{R}$.

Third, the stable Faltings height $h_{\mathrm{Fal}}(X_s)$ of the abelian variety X_s over $\overline{\mathbb{Q}}$ associated to y is defined as follows. Note that X_s descends to an abelian variety G over a number field K with semi-abelian reduction. Then we define the stable Faltings height by

$$h_{\mathrm{Fal}}(X_s) = \frac{1}{[K:\mathbb{Q}]}\widehat{\deg}(\omega_{\mathcal{G}}, \|\cdot\|_{\mathrm{Fal}}).$$

Here $\omega_{\mathcal{G}} = e^*_{\mathcal{G}}\Omega^g_{\mathcal{G}/O_K}$ is the Hodge bundle of the Néron model $\mathcal{G}$ of G over O_K, where $e_{\mathcal{G}} : \mathrm{Spec}\, O_K \to \mathcal{G}$ is the identity section, and $\|\cdot\|_{\mathrm{Fal}}$ is the Faltings metric of $\omega_{\mathcal{G}}$ defined by

$$\|\alpha\|^2_{\mathrm{Fal}} = \frac{i^{g^2}}{2^g}\int_{G_\sigma(\mathbb{C})} \alpha \wedge \bar{\alpha}$$

for any embedding $\sigma : K \to \mathbb{C}$ and any element α of

$$\omega_{\mathcal{G}} \otimes_\sigma \mathbb{C} \simeq \Gamma(G_\sigma(\mathbb{C}), \Omega^g_{G_\sigma(\mathbb{C})/\mathbb{C}}).$$

The definition is independent of the choice of (G, K).

5.5.2 Hodge bundles for moduli spaces

Theorem 5.5.1 is implied by a similar result for the minimal compactification of the coarse moduli scheme of abelian varieties. To introduce it, we will start with many constructions by Faltings–Chai [FC]. We will eventually only work on schemes, but the construction is easier to describe in terms of stacks.

Denote by $\mathcal{A}_g$ the moduli stack of principally polarized abelian varieties over $\mathbb{Z}$. It is a smooth Deligne–Mumford stack over $\mathbb{Z}$, endowed with a universal abelian scheme $\mathcal{X}_g \to \mathcal{A}_g$. Denote by $\mathcal{A}'_g$ the coarse moduli scheme of $\mathcal{A}_g$, which is a flat and quasi-projective integral scheme over $\mathbb{Z}$.

By [FC, IV, Thm. 5.7], there is a toroidal compactification $\mathcal{A}_g^{\mathrm{tor}}$ of $\mathcal{A}_g$ (by choosing a suitable combinatorial datum), which is a proper Deligne–Mumford stack over $\mathbb{Z}$ containing $\mathcal{A}_g$ as an open and dense substack. Moreover, the universal abelian scheme $\mathcal{X}_g \to \mathcal{A}_g$ extends to a semi-abelian scheme $\mathcal{X}_g^{\mathrm{tor}} \to \mathcal{A}_g^{\mathrm{tor}}$.

In terms of the universal abelian scheme (resp. semi-abelian scheme), we have a Hodge bundle $\omega(\mathcal{A}_g)$ on $\mathcal{A}_g$ (resp. $\omega(\mathcal{A}_g^{\mathrm{tor}})$ on $\mathcal{A}_g^{\mathrm{tor}}$) defined similar to the Hodge bundle $\omega(S)$ on S.

By [FC, V, Thm. 2.3], there is a minimal compactification $\mathcal{A}_g^*$ of the coarse moduli scheme $\mathcal{A}'_g$. It is a normal projective scheme over $\mathbb{Z}$ defined by contracting $\mathcal{A}_g^{\mathrm{tor}}$ via linear systems associated to $\omega(\mathcal{A}_g^{\mathrm{tor}})$. As a consequence, the Hodge bundle $\omega(\mathcal{A}_g^{\mathrm{tor}})$ descends to a $\mathbb{Q}$-line bundle $\omega(\mathcal{A}_g^*)$ on $\mathcal{A}_g^*$. In fact, $\omega(\mathcal{A}_g^*)$ is just $m^{-1}\mathcal{L}$ in the notation of [FC, V, Thm. 2.3], so it is indeed a $\mathbb{Q}$-line bundle. Denote by $\omega(\mathcal{A}'_g)$ the restriction of $\omega(\mathcal{A}_g^*)$ to $\mathcal{A}'_g$.

Note that $(\mathcal{A}_g^*, \omega(\mathcal{A}_g^*))$ is constructed by choosing a toroidal compactification, but the final result does not depend on the choices.

Since $\mathcal{A}'_g$ is the coarse moduli scheme, any point $y \in \mathcal{A}'_g(\mathbb{C})$ corresponds to a complex abelian variety G. Then the fiber $\omega(\mathcal{A}_g^*)(y)^{\otimes m}$ is canonically isomorphic to the m-th tensor power of the Hodge bundle of $G/\mathbb{C}$. Then the integration on $G(\mathbb{C})$ as before induces a Faltings metric $\|\cdot\|_{\mathrm{Fal}}$ of $\omega(\mathcal{A}_g^*)(y)$. Varying y, we obtain a Faltings metric $\|\cdot\|_{\mathrm{Fal}}$ of $\omega(\mathcal{A}_g^*)$ on $\mathcal{A}'_g(\mathbb{C})$.

Consider pair $(\omega(\mathcal{A}_g^*), \|\cdot\|_{\mathrm{Fal}})$. It is similar to the original pair $(\omega(S), \|\cdot\|_{\mathrm{Fal}})$, but it has the huge advantage that $\mathcal{A}_g^*$ is projective over $\mathbb{Z}$. In particular, $(\omega(\mathcal{A}_g^*), \|\cdot\|_{\mathrm{Fal}})$ induces a metrized line bundle $\overline{\omega(\mathcal{A}'_g)}^{\text{r-an}}$ in $\mathcal{P}\mathrm{ic}(\mathcal{A}_g'^{\text{r-an}})_{\mathbb{Q}}$ with underlying $\mathbb{Q}$-line bundle $\omega(\mathcal{A}'_g)$.

The following is an analogue of Theorem 5.5.1, which is still based on the analytification functor in Proposition 3.5.1.

Theorem 5.5.2. *The metrized $\mathbb{Q}$-line bundle $\overline{\omega(\mathcal{A}'_g)}^{r\text{-}an}$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{A}'^{r\text{-}an}_g)_{\mathbb{Q}}$ is the image of a unique adelic $\mathbb{Q}$-line bundle $\overline{\omega(\mathcal{A}'_g)}$ in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{A}'_g/\mathbb{Z})_{\mathbb{Q}}$ via the analytification functor. Moreover, for any $y \in \mathcal{A}'_g(\overline{\mathbb{Q}})$ corresponding to an abelian variety G over $\overline{\mathbb{Q}}$, we have $h_{\overline{\omega(\mathcal{A}'_g)}}(y) = h_{\mathrm{Fal}}(G)$.*

Proof. Let $\mathcal{A}^{**}_g \to \mathcal{A}^*_g$ be the blowing-up of $\mathcal{A}^*_g$ along the boundary $\mathcal{A}^*_g \setminus \mathcal{A}'_g$. The exceptional divisor $\mathcal{E}$ can be extended to a boundary divisor $(\mathcal{E}, g_{\mathcal{E}})$ on $\mathcal{A}^{**}_g$. Let $\omega(\mathcal{A}^{**}_g)$ be the pull-back of $\omega(\mathcal{A}^*_g)$ to $\mathcal{A}^{**}_g$. It suffices to consider the pair $(\omega(\mathcal{A}^{**}_g), \|\cdot\|_{\mathrm{Fal}})$.

By [FC, V, Def. 4.2, Rem. 4.3, Prop. 4.5], the metric $\|\cdot\|_{\mathrm{Fal}}$ has logarithmic singularities along the boundary $\mathcal{E}(\mathbb{C})$. Namely, take any hermitian metric $\|\cdot\|'$ of $\omega(\mathcal{A}^{**}_g)$ on $\mathcal{A}^{**}_g(\mathbb{C})$. Denote

$$f = \log(\|\cdot\|_{\mathrm{Fal}}/\|\cdot\|'),$$

which is a continuous function on $\mathcal{A}'_g(\mathbb{C})$. Then the logarithmic singularity means that

$$|f| < c \log g_{\mathcal{E}}$$

over $\mathcal{A}^{**}_g(\mathbb{C})$ for some constant $c > 0$.

The "arithmetic divisor" $(0, f)$ is a natural element of $\widehat{\mathrm{Div}}(\mathcal{A}'_g)$. This is an example of the local theory in §3.6. In fact, by Theorem 3.6.4, for any real-valued continuous function f on $\mathcal{A}'_g(\mathbb{C})$ which grows as $o(g_{\mathcal{E}})$ along $\mathcal{E}(\mathbb{C})$, the pair $(0, f)$ lies in $\widehat{\mathrm{Div}}(\mathcal{A}'_g)$. The theorem is stated in the local setting (over $\mathbb{C}$), but its proof holds for the global $(0, f)$ (over $\mathbb{Z}$) since the only problem appears at the archimedean places.

Therefore, we have proved that the metrized $\mathbb{Q}$-line bundle $\overline{\omega(\mathcal{A}'_g)}^{\text{r-an}}$ comes from an adelic $\mathbb{Q}$-line bundle $\overline{\omega(\mathcal{A}'_g)}$ on $\mathcal{A}'_g$.

It remains to prove the identity $h_{\overline{\omega(\mathcal{A}'_g)}}(y) = h_{\mathrm{Fal}}(G)$ for $y \in \mathcal{A}'_g(\overline{\mathbb{Q}})$. In fact, we can assume that y corresponds to a point $y : \mathrm{Spec}\, K \to \mathcal{A}_g$ for a number field K. This induces a point $y : \mathrm{Spec}\, K \to \mathcal{A}^{\mathrm{tor}}_g$ on the proper stack $\mathcal{A}^{\mathrm{tor}}_g$ over $\mathbb{Z}$. By the properness and the valuative criterion, by enlarging K if necessary, we can assume that $y : \mathrm{Spec}\, K \to \mathcal{A}^{\mathrm{tor}}_g$ extends to a morphism $\tilde{y} : \mathrm{Spec}\, O_K \to \mathcal{A}^{\mathrm{tor}}_g$. Via the universal semi-abelian scheme $\mathcal{X}^{\mathrm{tor}}_g \to \mathcal{A}^{\mathrm{tor}}_g$, we obtain a semi-abelian scheme $\mathcal{G} = \tilde{y}^*\mathcal{X}^{\mathrm{tor}}_g$ over O_K. The generic fiber $\mathcal{G}_K$ is a descent of the abelian variety G to K. By this, we see that $h_{\mathrm{Fal}}(G)$ is equal to $\widehat{\deg}(\tilde{y}^*(\omega(\mathcal{A}^{\mathrm{tor}}_g), \|\cdot\|_{\mathrm{Fal}}))/[K : \mathbb{Q}]$. Here the Faltings metric $\|\cdot\|_{\mathrm{Fal}}$ of $\omega(\mathcal{A}^{\mathrm{tor}}_g)$ on $\mathcal{A}_g(\mathbb{C})$ is defined by integration as before. By the compatibility of the Hodge bundles, $(\omega(\mathcal{A}_g^{\mathrm{tor}}), \|\cdot\|_{\mathrm{Fal}})$ can be changed to $(\omega(\mathcal{A}_g^*), \|\cdot\|_{\mathrm{Fal}})$. This finishes the proof. $\square$

Once we have Theorem 5.5.2, the proof of Theorem 5.5.1 is immediate. In fact, the family $X \to S$ induces a moduli morphism $S \to \mathcal{A}_g$. Composing with the canonical morphism $\mathcal{A}_g \to \mathcal{A}'_g$, we obtain a morphism $S \to \mathcal{A}'_g$. Then $\overline{\omega(S)}$ is just the pull-back of $\overline{\omega(\mathcal{A}'_g)}$ to S. This pull-back is a priori only an adelic $\mathbb{Q}$-line bundle, but it is uniquely realized as an adelic line bundle since the underlying line bundle $\omega(S)$ is an integral line bundle on S.

6 Algebraic dynamics

In this section, we first develop a theory of admissible adelic line bundles for polarized algebraic dynamical systems over finitely generated fields, following the idea of [Zha2, YZ1]. Then we generalize the arithmetic Hodge index theorem of Faltings [Fal1] and Hriljac [Hri] to projective curves over finitely generated fields.

To work with adelic $\mathbb{Q}$-line bundles on flat and essentially quasi-projective integral schemes X over k, we recall the definitions of $\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q}}$, $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}}$ and $\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q},\mathrm{nef}}$ in §2.5.6. Recall the categories $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathbb{Q}}$, $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathrm{int},\mathbb{Q}}$ and $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathbb{Q},\mathrm{nef}}$ defined similarly.

6.1 Invariant adelic line bundles

Let (X, f, L) be a *polarized dynamical system* over an integral scheme S, i.e.,

(1) X is an integral scheme projective and flat over S;

(2) $f : X \to X$ is a morphism over S;

(3) $L \in \mathcal{P}\mathrm{ic}(X)_{\mathbb{Q}}$ is a $\mathbb{Q}$-line bundle on X, relatively ample over S, such that $f^*L \simeq qL$ for some rational number $q > 1$.

We refer to [Laz1, §1.7] for relative ampleness. In particular, [Laz1, Thm. 1.7.8] asserts that a line bundle on X is relatively ample over S if and only if it is ample on every fiber of X over S.

If S is the spectrum of a number field, Zhang [Zha2] applied Tate's limiting argument to construct a nef adelic $\mathbb{Q}$-line bundle $\overline{L}_f$ extending L and with $f^*\overline{L}_f \simeq q\overline{L}_f$. The goal here is to generalize the result to finitely generated fields or even essentially quasi-projective schemes S.

6.1.1 Invariant adelic line bundle

Let k be either $\mathbb{Z}$ or a field. Take the uniform terminology in §1.6. Let S be a flat and essentially quasi-projective integral scheme over k. Let (X, f, L) be a polarized dynamical system over S. Fix an isomorphism $f^*L \to qL$ with $q > 1$ by assumption.

Choose a projective model $\pi : \mathcal{X} \to \mathcal{S}$ of $X \to S$, i.e., a projective model $\mathcal{S}$ of S over k and a flat morphism $\pi : \mathcal{X} \to \mathcal{S}$ of projective varieties over k whose base change by $S \to \mathcal{S}$ is isomorphic to $X \to S$. Choose a hermitian $\mathbb{Q}$-line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$ such that $(\mathcal{X}_S, \mathcal{L}_S) \simeq (X, L)$.

For each positive integer i, consider the composition $X \xrightarrow{f^i} X \to \mathcal{X}$. Denote the normalization of the composition by $f_i : \mathcal{X}_i \to \mathcal{X}$, and denote the induced map to $\mathcal{S}$ by $\pi_i : \mathcal{X}_i \to \mathcal{S}$. Denote $\overline{\mathcal{L}}_i = q^{-i} f_i^* \overline{\mathcal{L}}$, which lies in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}_i)_{\mathbb{Q}}$.

The sequence $\{(\mathcal{X}_i, \overline{\mathcal{L}}_i)\}_{i\geq 1}$ is an adelic sequence in the sense of Moriwaki [Mor4, §3.1]. In our setting, we will complete the datum to an adelic line bundle $\overline{L}_f = (\mathcal{L}_{\mathcal{V}}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ for a quasi-projective model $\mathcal{U}$ of X over k.

In fact, there is an open subscheme $\mathcal{V}$ of $\mathcal{S}$ containing S, such that $\mathcal{U} = \mathcal{X}_{\mathcal{V}}$ is projective and flat over $\mathcal{V}$, and that $f : X \to X$ extends to a morphism $f_{\mathcal{V}} : \mathcal{U} \to \mathcal{U}$ and such that the isomorphism $f^*L \to qL$ extends to an isomorphism $f_{\mathcal{V}}^*\mathcal{L}_{\mathcal{V}} \to q\mathcal{L}_{\mathcal{V}}$ in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$. By the construction, we make identifications $\mathcal{X}_{i,\mathcal{V}} = \mathcal{X}_{\mathcal{V}} = \mathcal{U}$ and $\mathcal{L}_i|_{\mathcal{U}} = \mathcal{L}_{i,\mathcal{V}}$.

Start with the isomorphism

$$\ell : \mathcal{L}_{\mathcal{V}} \longrightarrow q^{-1} f_{\mathcal{V}}^* \mathcal{L}_{\mathcal{V}}$$

in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$. By applying $q^{-1} f_{\mathcal{V}}^*$ to ℓ successively, we obtain canonical isomorphisms

$$\mathcal{L}_{\mathcal{V}} \longrightarrow q^{-1} f_{\mathcal{V}}^* \mathcal{L}_{\mathcal{V}} \longrightarrow q^{-2} (f_{\mathcal{V}}^*)^2 \mathcal{L}_{\mathcal{V}} \longrightarrow \cdots \longrightarrow q^{-i} (f_{\mathcal{V}}^*)^i \mathcal{L}_{\mathcal{V}}$$

in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$. This induces an isomorphism

$$\ell_i : \mathcal{L}_{\mathcal{V}} \longrightarrow \mathcal{L}_{i,\mathcal{V}}$$

in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$ by the identification $\mathcal{L}_{i,\mathcal{V}} = q^{-i}(f_{\mathcal{V}}^*)^i \mathcal{L}_{\mathcal{V}}$. Then we have introduced every term in $(\mathcal{L}_{\mathcal{V}}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$.

Note that if S is already a quasi-projective variety over k, then we can simply take $(\mathcal{U}, \mathcal{V}) = (X, S)$. This is the essential case of the result.

Theorem 6.1.1. *Let k be either $\mathbb{Z}$ or a field. Let S be a flat and essentially quasi-projective integral scheme over k. Let (X, f, L) be a polarized dynamical system over S. Fix an isomorphism $f^*L \to qL$ in $\mathcal{P}\mathrm{ic}(X)_{\mathbb{Q}}$ with $q > 1$.*

The above sequence $(\mathcal{L}_{\mathcal{V}}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ converges in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathbb{Q}}$, and thus defines an object $\overline{L}_f$ of $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathbb{Q}}$. The adelic line bundle $\overline{L}_f$ is uniquely determined by $(S, X, f, L)/k$ and $f^*L \to qL$ up to isomorphism, and satisfies the following properties.

(1) $\overline{L}_f$ is f-invariant in the sense that $f^*\overline{L}_f \simeq q\overline{L}_f$ in $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathbb{Q}}$.

(2) $\overline{L}_f$ is nef in $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathbb{Q}}$. If S has an affine quasi-projective model over k, then $\overline{L}_f$ is strongly nef in $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathbb{Q}}$.

(3) If furthermore $L \in \mathcal{P}\mathrm{ic}(X)$ (instead of $\mathcal{P}\mathrm{ic}(X)_{\mathbb{Q}}$) and $q \in \mathbb{Z}_{>1}$ with $f^*L \simeq qL$ in $\mathcal{P}\mathrm{ic}(X)$, then all the results hold in $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ (instead of $\widehat{\mathcal{P}\mathrm{ic}}(X/k)_{\mathbb{Q}}$).

Proof. We first prove the existence of the limit. By blowing-up $\mathcal{S}$ along $\mathcal{S} \setminus \mathcal{V}$ if necessary, we can assume that there is a boundary divisor $(\mathcal{S}, \overline{\mathcal{E}}_0)$ of $\mathcal{V}$. Then we get a boundary divisor $(\mathcal{X}, \pi^*\overline{\mathcal{E}}_0)$ of $\mathcal{U}$.

View the isomorphism $\ell : \mathcal{L}_{\mathcal{V}} \to q^{-1}f_{\mathcal{V}}^*\mathcal{L}_{\mathcal{V}}$ as a rational map $\overline{\mathcal{L}} \dashrightarrow \overline{\mathcal{L}}_1$. This defines a model adelic divisor $\widehat{\mathrm{div}}(\ell)$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod},\mathbb{Q}}$ whose image in $\mathrm{Div}(\mathcal{U})$ is 0. Then there exists $r > 0$ such that

$$-r\pi^*\overline{\mathcal{E}}_0 \leq \widehat{\mathrm{div}}(\ell) \leq r\pi^*\overline{\mathcal{E}}_0$$

holds in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod},\mathbb{Q}}$. The existence of r can be seen in the comparison of the boundary norms in the proof of Lemma 2.4.1.

By construction, the isomorphism $\ell_{i+1}\ell_i^{-1} : \mathcal{L}_{i,\mathcal{V}} \to \mathcal{L}_{i+1,\mathcal{V}}$ is obtained from $\ell : \mathcal{L}_{\mathcal{V}} \to q^{-1}f_{\mathcal{V}}^*\mathcal{L}_{\mathcal{V}}$ by applying $(q^{-1}f_{\mathcal{V}}^*)^i$. Accordingly, the rational map $\ell_{i+1}\ell_i^{-1} : \overline{\mathcal{L}}_i \dashrightarrow \overline{\mathcal{L}}_{i+1}$ is obtained from the rational map $\ell : \overline{\mathcal{L}} \dashrightarrow \overline{\mathcal{L}}_1$ by "applying" $(q^{-1}f^*)^i$. The situation can be conveniently described by the analytification functor in Proposition 3.4.1 or the Zariski–Riemann space in §2.6.4, but we give a precise description in terms of projective models of $\mathcal{U}$ as follows.

Write $\mathcal{X}_0 = \mathcal{X}$ and $\overline{\mathcal{L}}_0 = \overline{\mathcal{L}}$ for convenience. There are projective models $\mathcal{Y}_1$ and $\mathcal{Y}_{i+1}$ of $\mathcal{U}$ over k, together with morphisms

$$\tau_1 : \mathcal{Y}_1 \to \mathcal{X}_1, \quad \tau_1' : \mathcal{Y}_1 \to \mathcal{X}_0, \quad \tau_{i+1} : \mathcal{Y}_{i+1} \to \mathcal{X}_i, \quad \tau_{i+1}' : \mathcal{Y}_{i+1} \to \mathcal{X}_{i+1}$$

extending the identity morphism $\mathcal{U} \to \mathcal{U}$, and a morphism

$$g_i : \mathcal{Y}_{i+1} \to \mathcal{Y}_1$$

extending the morphism $f_{\mathcal{V}}^i : \mathcal{U} \to \mathcal{U}$. Then the rational map $\ell : \overline{\mathcal{L}}_0 \dashrightarrow \overline{\mathcal{L}}_1$ is realized as a rational map $\ell' : \tau_1'^* \overline{\mathcal{L}}_0 \to \tau_1^* \overline{\mathcal{L}}_1$ over $\mathcal{Y}_1$; the rational map $\ell_{i+1}\ell_i^{-1} : \overline{\mathcal{L}}_i \dashrightarrow \overline{\mathcal{L}}_{i+1}$ is realized as a rational map $(\ell_{i+1}\ell_i^{-1})' : \tau_{i+1}'^* \overline{\mathcal{L}}_i \to \tau_{i+1}^* \overline{\mathcal{L}}_{i+1}$ over $\mathcal{Y}_{i+1}$. The second rational map, including its source and its target, is obtained by applying $q^{-i} g_i^*$ to the first rational map via $g_i : \mathcal{Y}_{i+1} \to \mathcal{Y}_1$. As a consequence, we have

$$\widehat{\mathrm{div}}((\ell_{i+1}\ell_i^{-1})') = q^{-i} g_i^* \widehat{\mathrm{div}}(\ell')$$

in $\widehat{\mathrm{Div}}(\mathcal{Y}_{i+1})_{\mathbb{Q}}$.

Denote by $\pi_1' : \mathcal{Y}_1 \to \mathcal{S}$ and $\pi_{i+1}' : \mathcal{Y}_{i+1} \to \mathcal{S}$ the structure morphisms. Note that $g_i^* \pi_1'^* \overline{\mathcal{E}}_0 = \pi_{i+1}'^* \overline{\mathcal{E}}_0$ is equal to $\pi^* \overline{\mathcal{E}}_0$ in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod},\mathbb{Q}}$. We obtain

$$-\frac{r}{q^i} \pi^* \overline{\mathcal{E}}_0 \leq \widehat{\mathrm{div}}(\ell_{i+1}\ell_i^{-1}) \leq \frac{r}{q^i} \pi^* \overline{\mathcal{E}}_0$$

holds in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod},\mathbb{Q}}$. As a consequence, $\{\widehat{\mathrm{div}}(\ell_i \ell_1^{-1})\}_{i \geq 1}$ is a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod},\mathbb{Q}}$.

This finishes the existence of the limit. The independence of the limit on the auxiliary data can be proved similarly, so we omit it. It remains to treat the nefness of $\overline{L}_f$ on X.

At the beginning of the construction, if we are able to choose $(\mathcal{X}, \overline{\mathcal{L}})$ such that $\overline{\mathcal{L}}$ is nef on $\mathcal{X}$, then $\overline{L}_f$ is strongly nef by definition. This happens if S has an affine quasi-projective model $\mathcal{V}$ over k. In fact, in this case, we can assume that S is an open subscheme of $\mathcal{V}$, and then relative ampleness of L on S implies the ampleness of L on X, so we can choose $(\mathcal{X}, \overline{\mathcal{L}})$ such that $\overline{\mathcal{L}}$ is nef.

However, such $(\mathcal{X}, \overline{\mathcal{L}})$ might not exist in general, and we will have to make a slightly weaker choice. Namely, we claim that there is a projective model $\pi : \mathcal{X} \to \mathcal{S}$ of $X \to S$ over k, together with a hermitian $\mathbb{Q}$-line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ extending L and a nef hermitian $\mathbb{Q}$-line bundle $\overline{\mathcal{M}}$ over $\mathcal{S}$, such that $\overline{\mathcal{L}}' = \overline{\mathcal{L}} + \pi^* \overline{\mathcal{M}}$ is nef on $\mathcal{X}$.

To prove the claim, by taking a sufficiently small quasi-projective model of $X \to S$ over k, we can assume that S is quasi-projective over k. Since L is relatively ample, there is an ample line bundle M on S such that $L + \pi^* M$

is ample on X. Take a tensor power of $L+\pi^*M$, use it embed X into $\mathbb{P}^N_k$, and take the Zariski closure of X. Then $L+\pi^*M$ extends to an ample $\mathbb{Q}$-line bundle $\mathcal{L}'$ on a projective model $\mathcal{X}$ of X over k. Extend $\mathcal{L}$ to a nef hermitian line bundle $\overline{\mathcal{L}}'$ on $\mathcal{X}$. Similarly, using a tensor power of M to embed S into $\mathbb{P}^{N'}_k$ and taking the Zariski closure, we have a projective model $\mathcal{S}$ of S such that M extends to a nef hermitian line bundle $\overline{\mathcal{M}}$ on $\mathcal{S}$. The rational map $\mathcal{X} \dashrightarrow \mathcal{S}$ extends to a morphism $\pi : \mathcal{X} \to \mathcal{S}$ by blowing-up $\mathcal{X}$, and we can further assume that $\mathcal{X} \to \mathcal{S}$ is flat by the Raynaud–Gruson flattening theorem in [RG, Thm. 5.2.2]. Finally, we set $\overline{\mathcal{L}} = \overline{\mathcal{L}}' - \pi^*\overline{\mathcal{M}}$. This proves the claim.

Now we prove that $\overline{L}_f$ is nef. Let $\overline{L}_f = (\mathcal{L}_{\mathcal{V}}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ be constructed using the new pair $(\mathcal{X}, \overline{\mathcal{L}})$ as in the claim. Note that $\overline{\mathcal{L}}' = \overline{\mathcal{L}} + \pi^*\overline{\mathcal{M}}$ is nef on $\mathcal{X}$. Then $\overline{\mathcal{L}}_i + q^{-i}\pi_i^*\overline{\mathcal{M}} = q^{-i}f_i^*\overline{\mathcal{L}}'$ is nef on $\mathcal{X}_i$ for any $i \geq 1$. It follows that for any positive integer a, the line bundle $\overline{\mathcal{L}}_{i+a} + q^{-a}\pi^*\overline{\mathcal{M}}$ is nef for any $i \geq 1$. View $\overline{L}_f$ as the limit of $(\mathcal{L}_{\mathcal{V}}, (\mathcal{X}_{i+a}, \overline{\mathcal{L}}_{i+a}, \ell_{i+a})_{i\geq 1})$. We see that $\overline{L}_f + q^{-a}\pi^*\overline{\mathcal{M}}$ is strongly nef. This proves that $\overline{L}_f$ is nef. □

For the uniqueness of $\overline{L}_f$, we actually have the following result. For convenience of applications, we do not require L to be ample.

Theorem 6.1.2. *Let k be either $\mathbb{Z}$ or a field. Let X and S be flat and essentially quasi-projective integral schemes over k. Let $\pi : X \to S$ be a projective and flat morphism with geometrically connected fibers. Let $f : X \to X$ be a morphism over S. Let $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ be an element such that $f^*L = qL$ in $\mathrm{Pic}(X)_{\mathbb{Q}}$ for some rational number $q > 1$. The following are true:*

(1) There exists a unique preimage $\overline{L}$ of L under the map $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$ such that $f^*\overline{L} = q\overline{L}$ in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$.

(2) If $f' : X \to X$ is a morphism over k such that $f'f = ff'$ and that $f'^*L = q'L$ in $\mathrm{Pic}(X)_{\mathbb{Q}}$ for some rational number $q' \neq 0$, then the adelic line bundle $\overline{L}$ defined in (1) satisfies $f'^*\overline{L} = q'\overline{L}$ in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$.

Proof. Note that (1) implies (2). In fact, $f'f = ff'$ implies $f^*(f'^*\overline{L}) = qf'^*\overline{L}$. Then $\overline{L}' = q'^{-1}f'^*\overline{L}$ is an extension of L with $f^*\overline{L}' = q\overline{L}'$. By the uniqueness in (1), we have $\overline{L}' = \overline{L}$. This proves (2).

For (1), the existence of $\overline{L}$ is the similar to Theorem 6.1.1. For the uniqueness, we can assume that $L = \mathcal{O}_X$ is the trivial line bundle.

By Proposition 3.4.1, there is a canonical injection

$$\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathbb{Q}}.$$

As $L = \mathcal{O}_X$, the image of $\overline{L}$ in $\widehat{\mathrm{Pic}}(X^{\mathrm{an}})_{\mathbb{Q}}$ is represented by an element $(0, g)$ of $\widehat{\mathrm{Div}}(X^{\mathrm{an}})_{\mathbb{Q}}$, where the underlying divisor is 0 on X, and the Green's function g is actually a continuous function on X^{an}. The condition $f^*\overline{L} = q\overline{L}$ implies in $\widehat{\mathrm{Div}}(X^{\mathrm{an}})$

$$m(0, f^*g - qg) = (\mathrm{div}(\alpha), -\log|\alpha|), \quad \alpha \in k(X)^{\times},\ m \in \mathbb{Z},\ m \neq 0.$$

This implies $\mathrm{div}(\alpha) = 0$ on X, and thus α lies in $\Gamma(X, \mathcal{O}_X^{\times}) = \Gamma(S, \mathcal{O}_S^{\times})$. As a result, the difference

$$f^*g - qg = -\frac{1}{m}\log|\alpha|$$

is constant on every fiber of $X^{\mathrm{an}} \to S^{\mathrm{an}}$.

Let $v \in S^{\mathrm{an}}$ be a point with residue field H_v. The fiber X_v^{an} of X^{an} above v is exactly the Berkovich space associated to X_{H_v} over H_v. We have that $f^*g - qg = c_v$ is constant on X_v^{an}. Denote by $g_{\max}$ and $g_{\min}$ the global maximal value and the global minimum value of the continuous function g on the compact space X_v^{an}. Note that $f : X_v^{\mathrm{an}} \to X_v^{\mathrm{an}}$ is surjective. The relation $f^*g = qg + c_v$ gives $g_{\max} = qg_{\max} + c_v$ and thus $g_{\max} = -c_v$. Similarly, $g_{\min} = -c_v$. This forces $g_{\max} = g_{\min}$ and thus g is constant on X_v^{an}.

As a consequence, $f^*g = g$ on X^{an}. The original equation gives

$$m(1-q)(0, g) = (\mathrm{div}(\alpha), -\log|\alpha|).$$

Then $\overline{L}$ is 0 in $\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q}}$. This finishes the proof. $\square$

6.1.2 Abelian schemes

The most important example of the above construction is for abelian schemes. In this case, we can prove that the adelic line bundles $\overline{L}_f$ in Theorem 6.1.2 is actually integrable (without assuming that L is relatively ample.)

Theorem 6.1.3. *Let k be either $\mathbb{Z}$ or a field. Let S be a flat and essentially quasi-projective integral scheme over k. Let $\pi : X \to S$ be an abelian scheme with the identity section $e : S \to X$. Let L be a line bundle on X with a* *rigidification, i.e., an isomorphism $e^*L \to \mathcal{O}_S$. Assume that $[-1]^*L \simeq \epsilon L$ for some $\epsilon \in \{\pm 1\}$.*

Then there is an adelic line bundle $\overline{L}$ on X extending L satisfying $[2]^*\overline{L} \simeq 4\overline{L}$ for $\epsilon = 1$ and $[2]^*L \simeq 2L$ for $\epsilon = -1$. The adelic line bundle $\overline{L}$ is uniquely determined by the rigidification.

Moreover, $\overline{L}$ is always integrable. For any integer m, $[m]^*L \simeq m^2 L$ if $\epsilon = 1$; and $[m]^*L \simeq mL$ if $\epsilon = -1$.

Proof. Set $i = 2$ for the symmetric case $\epsilon = 1$, and $i = 1$ for the anti-symmetric case $\epsilon = -1$. Note that $[-1]^*L \simeq \epsilon L$ implies that $[m]^*\overline{L} \simeq m^i\overline{L}$. In fact, we first see that $[m]^*\overline{L} - m^i\overline{L}$ is trivial on fibers of $\pi : X \to S$, and thus is isomorphic to π^*M for some $M \in \mathrm{Pic}(S)$. But M is trivial by the rigidification.

The rigidification determines a unique choice of an isomorphism $[2]^*\overline{L} \to 2^i\overline{L}$. Apply Theorem 6.1.2(1) to the dynamical system $(X, [2], L)$ over S. We obtain a unique adelic line bundle $\overline{L}$ on X extending L such that $[2]^*\overline{L} \simeq 2^i\overline{L}$. Moreover, Theorem 6.1.2(2) implies $[m]^*\overline{L} \simeq m^i\overline{L}$.

It remains to prove that $\overline{L}$ is integrable. In the case $\epsilon = 1$, if L is relatively ample, then $\overline{L}$ is nef by Theorem 6.1.1. In the case $\epsilon = 1$ for general L, we can write it as the difference of two relatively ample line bundles with rigidification, and then the integrability still follows.

Assume $\epsilon = -1$ in the following. Let $X^\vee \to S$ be the dual abelian scheme of $X \to S$. Let P be the Poincare line bundle on $X \times_S X^\vee$, with a rigidification along the identity section of $X \times_S X^\vee \to S$. Then L corresponds to a section $\sigma : S \to X^\vee$ in the sense that $L \simeq (\mathrm{id}, \sigma \circ \pi)^*P$. Here $(\mathrm{id}, \sigma \circ \pi)$ is the composition $X \to X \times_S S \stackrel{(\mathrm{id},\sigma)}{\to} X \times_S X^\vee$.

For any $m \in \mathbb{Z}$, denote by

$$[m] : X \times_S X^\vee \longrightarrow X \times_S X^\vee$$

the (total) multiplication of the abelian scheme $X \times_S X^\vee$ over S, and denote by

$$[m]' : X \times_S X^\vee \longrightarrow X \times_S X^\vee$$

the (partial) multiplication of the abelian scheme $X \times_S X^\vee$ on $X^\vee$. By the universal property, $[-1]^*P \simeq P$ and $[-1]'^*P \simeq -P$. Then there is a unique adelic line bundle $\overline{P}$ in $\widehat{\mathcal{P}\mathrm{ic}}(X \times_S X^\vee/k)$ extending P with $[2]^*\overline{P} \simeq 4\overline{P}$. It further gives $[2]'^*\overline{P} \simeq 2\overline{P}$ by Theorem 6.1.2(2). Moreover, $\overline{P}$ is integrable by the case $\epsilon = 1$.

Finally, under $L \simeq (\mathrm{id}, \sigma \circ \pi)^* P$, we have $\overline{L}' := (\mathrm{id}, \sigma \circ \pi)^* \overline{P}$ extends L and satisfies $[2]^* \overline{L}' \simeq 2\overline{L}'$. It follows that $\overline{L} \simeq \overline{L}'$ by the uniqueness. Finally, $\overline{L}$ is integrable since so is $\overline{P}$. This finishes the proof. $\square$

6.1.3 Canonical height

Let k be either $\mathbb{Z}$ or a field. Let F be a finitely generated field over k. Let (X, f, L) be a polarized dynamical system over F.

By Theorem 6.1.1, there is an f-invariant line bundle $\overline{L}_f$ in $\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q},\mathrm{nef}}$. For any closed $\overline{F}$-subvariety Z of X, define *the vector-valued canonical height of Z* as

$$\mathfrak{h}_f(Z) = \mathfrak{h}_{L,f}(Z) := \mathfrak{h}_{\overline{L}_f}(Z) \in \widehat{\mathrm{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}.$$

It gives a map $\mathfrak{h}_f : |X_{\overline{F}}| \to \widehat{\mathrm{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}$.

We can also define the canonical height by Tate's limiting argument:

$$\mathfrak{h}_f(Z) = \lim_{m\to\infty} \frac{1}{q^m} \mathfrak{h}_{(\mathcal{X},\overline{\mathcal{L}})}(f^m(Z)).$$

Here $(\mathcal{X}, \overline{\mathcal{L}})$ is any initial model of (X, L) as in the construction of $\overline{L}_f$ above. Then one can check that it is convergent in $\widehat{\mathrm{Pic}}(F/k)$ and compatible with the previous definition.

Proposition 6.1.4. *Let Z be a closed subvariety of X. Then the following are true:*

(1) The height $\mathfrak{h}_f(Z)$ lies in $\widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q},\mathrm{nef}}$.

(2) The height is f-invariant in the sense that $\mathfrak{h}_f(f(Z)) = q\ \mathfrak{h}_f(Z)$.

(3) The height $\mathfrak{h}_f(Z) = 0$ in $\widehat{\mathrm{Pic}}(F/k)_{\mathrm{int}}$ if Z is preperiodic under f. Conversely, if $\mathfrak{h}_f(Z) \equiv 0$ (numerically trivial) and Z is a point, then Z is preperiodic under f.

Proof. Since $\overline{L}_f$ is nef, the height $\mathfrak{h}_f(Z)$ is nef. The formula $\mathfrak{h}_f(f(Z)) = q\mathfrak{h}_f(Z)$ follows from the projection formula in Lemma 4.6.1(4) and the invariance of $\overline{L}_f$. Thus $\mathfrak{h}_f(Z) = 0$ if Z is preperiodic under f. The second statement of (3) follows from the Northcott property. $\square$

By choosing adelic line bundles $\overline{H}_1, \cdots, \overline{H}_{d-1} \in \widehat{\mathcal{P}\mathrm{ic}}(F/k)_{\mathbb{Q},\mathrm{nef}}$, we can form *the canonical Moriwaki height*

$$h_f^{\overline{H}_1,\cdots,\overline{H}_{d-1}}(Z) := \mathfrak{h}_f(Z) \cdot \overline{H}_1 \cdots \overline{H}_{d-1}.$$

It is a non-negative real number.

6.1.4 Néron–Tate height

Let k be either $\mathbb{Z}$ or a field. Let F be a finitely generated field over k. Let X be an abelian variety over F, $f = [2]$ be the multiplication by 2, and L be any symmetric and ample line bundle on X. Then the canonical height

$$\widehat{\mathfrak{h}}_L = \mathfrak{h}_{L,[2]} : X(\overline{F}) \longrightarrow \widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q},\mathrm{nef}},$$

as a generalization of the Néron–Tate height, is quadratic in the sense that

$$\langle x, y \rangle_L := \widehat{\mathfrak{h}}_L(x+y) - \widehat{\mathfrak{h}}_L(x) - \widehat{\mathfrak{h}}_L(y)$$

gives a bilinear map

$$X(\overline{F}) \times X(\overline{F}) \longrightarrow \widehat{\mathrm{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}.$$

It can be proved by the theorem of the cube as in the classical case over number fields. We refer to [Ser, §3.3] for the classical case, and omit the proof in the current case.

6.1.5 Equidistribution conjecture of preperiodic points

Since all preperiodic points of a polarized dynamical system have height 0, Conjecture 5.4.1 implies the following conjecture.

Conjecture 6.1.5 (equidistribution of preperiodic points)**.** *Let k be either $\mathbb{Z}$ or a field. Let F be a finitely generated field over k. Let v be a non-trivial valuation of F. Assume that the restriction of v to k is trivial if k is a field. Let (X, f, L) be a polarized dynamical system over F. Let $\{x_m\}_m$ be a generic sequence of preperiodic points in $X(\overline{F})$. Then the Galois orbit of $\{x_m\}_m$ is equidistributed in X_v^{an} with respect to the measure $d\mu_{L,f,v}$.*

Here X_v^{an} is the Berkovich space associated to X_{F_v}, where F_v is the completion of F with respect to v. The equilibrium measure is the Chambert-Loir measure

$$d\mu_{L,f,v} = \frac{1}{\deg_L(X)} c_1(L, \|\cdot\|_{f,v})^{\dim X}$$

over the analytic space X_v^{an}, where $\|\cdot\|_{f,v}$ is an f-invariant metric of L on X_v^{an} obtained by Tate's limiting argument.

One can also formulate the consequence of Theorem 5.4.6 for preperiodic points. We omit it here.

6.2 Heights of points on a subvariety

Let S be a quasi-projective variety over a number field K. Let (X, f, L) be a polarized dynamical system over S. Let $\overline{L}_f \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q},\mathrm{nef}}$ be the f-invariant extension of L. Let Y be a closed subvariety of X. The goal of this subsection is to explore properties of the height function

$$h_{\overline{L}_f} : Y(\overline{K}) \longrightarrow \mathbb{R}.$$

We consider two special cases. If Y is a section, then we have a specialization theorem. If Y is non-degenerate, then we have an equidistribution theorem.

The following exposition also works over function fields of one variables, but we restrict to number fields for simplicity.

6.2.1 Height of specialization

Now we consider the variation of the height of a section specializing in an algebraic family of algebraic dynamical systems.

Let S be a quasi-projective variety over a number field K. Let (X, f, L) be a polarized dynamical system over S. Let $\overline{L}_f \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q},\mathrm{nef}}$ be the f-invariant extension of L. Let $i : S \to X$ be a section of $\pi : X \to S$. Denote the vector-valued height

$$\overline{M} := \mathfrak{h}_{\overline{L}_f}(i) = i^*\overline{L}_f \in \widehat{\mathrm{Pic}}(S)_{\mathbb{Q},\mathrm{nef}}.$$

This gives a height function

$$h_{\overline{M}} : S(\overline{K}) \longrightarrow \mathbb{R}.$$

For any point $s \in S(\overline{K})$, denote by s' the closed point of S corresponding to s. Then $i(s) \in X(\overline{K})$ is actually a point on the polarized dynamical system $(X_{s'}, f_{s'}, L_{s'})$ over s'. Denote by $h_{\overline{L}_{s'}, f_{s'}}(i(s))$ the canonical height of $i(s)$ with respect to the polarized dynamical system $(X_{s'}, f_{s'}, L_{s'})$ over s'. Now we have the following identity.

Lemma 6.2.1 (specialization)**.** *For any point $s \in S(\overline{K})$,*

$$h_{\overline{L}_{s'}, f_{s'}}(i(s)) = h_{\overline{M}}(s).$$

Therefore, $i(s)$ is preperiodic under f if and only $h_{\overline{M}}(s) = 0$.

Proof. By definition, the $f_{s'}$-invariant extension of $L_{s'}$ on $X_{s'}$ is exactly $\overline{L}_f|_{X_{s'}}$. Then $h_{\overline{M}}(s)$ is the normalized degree of the pull-back of $\overline{L}_f$ via the composition $s' \to S \to X$, and $h_{\overline{L}_{s'}, f_{s'}}(i(s))$ is the normalized degree of the pull-back of $\overline{L}_f$ via the composition $s' \to X_{s'} \to X$. Then both terms are equal to $h_{\overline{L}_f}(i(s))$. $\square$

If X is a family of elliptic curves over a smooth curve S over K, a similar height identity was obtained by DeMarco–Mavrak [DM, Thm. 1.1]. Their approach was very different and their result is stronger in this case. In fact, they proved that there is an adelic line bundle $\overline{M}'$ on the unique smooth projective model S' of S over K such that $h_{\overline{M}'}(s) = h_{\overline{L}_f}(i(s))$ for any $s \in S(\overline{K})$. In other words, their result implies that our $\overline{M}$ lies in the image of $\widehat{\mathrm{Pic}}(S')_{\mathbb{Q},\mathrm{nef}} \to \widehat{\mathrm{Pic}}(S)_{\mathbb{Q},\mathrm{nef}}$.

As the work of [DM] is a refinement of the specialization theorem of Tate [Tat] and Silverman [Sil2, Sil3, Sil4] for elliptic surfaces, our height identity can be viewed as a generalization and new interpretation of the specialization theorem for families of algebraic dynamic systems.

6.2.2 Non-degenerate subvarieties

Let S be a quasi-projective variety over a number field K. Let (X, f, L) be a polarized dynamical system over S. Let Y be a closed subvariety of X.

Let $\overline{L}_f \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q},\mathrm{nef}}$ be the f-invariant extension of L. Denote by

$$\overline{M} := \overline{L}_f|_Y$$

the image of $\overline{L}_f$ under the pull-back map

$$\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathbb{Q},\mathrm{nef}} \longrightarrow \widehat{\mathrm{Pic}}(Y/\mathbb{Z})_{\mathbb{Q},\mathrm{nef}},$$

and denote by $\widetilde{M}$ the image of $\overline{M}$ under the canonical composition

$$\widehat{\mathrm{Pic}}(Y/\mathbb{Z})_{\mathbb{Q},\mathrm{nef}} \longrightarrow \widehat{\mathrm{Pic}}(Y/\mathbb{Q})_{\mathbb{Q},\mathrm{nef}} \longrightarrow \widehat{\mathrm{Pic}}(Y/K)_{\mathbb{Q},\mathrm{nef}}.$$

Note that the last arrow is an isomorphism. We refer to §2.5.5 for the definitions of these maps. By nefness, both self-intersection numbers

$$\widehat{\deg}_{\overline{M}}(Y) = \overline{M}^{\dim Y+1}, \qquad \deg_{\widetilde{M}}(Y) = \widetilde{M}^{\dim Y}$$

are non-negative.

We say that Y is *non-degenerate* in X if $\deg_{\widetilde{M}}(Y) > 0$. As $\widetilde{M}$ is nef on Y, the condition is equivalent to that $\widetilde{M}$ is big on Y. Another related result is Lemma 5.4.4, which asserts that, for any embedding $\sigma : K \to \mathbb{C}$,

$$\deg_{\widetilde{M}}(Y) = \int_{Y_\sigma(\mathbb{C})} c_1(\overline{L}_f)_\sigma^{\dim Y}.$$

So Y is non-degenerate if and only if the measure $c_1(\overline{L}_f)_\sigma^{\dim Y}|_{Y_\sigma(\mathbb{C})}$ is nonzero on $Y_\sigma(\mathbb{C})$. The same result holds over non-archimedean places.

If $X \to S$ is an abelian scheme over a smooth variety S over K, in terms of Tate's limiting argument, $c_1(\overline{L}_f)_\sigma$ defines a semipositive smooth $(1,1)$-form on $X_\sigma(\mathbb{C})$. In particular, it is the Betti form as defined in [CGHX, §2]. By [DGH, Prop. 2.2], $c_1(\overline{L}_f)_\sigma^{\dim Y}$ is non-zero on $Y_\sigma(\mathbb{C})$ if and only if the Betti map $Y_\sigma(\mathbb{C})_V \to (\mathbb{R}/\mathbb{Z})^{2g}$ has a full rank at some point of $Y_\sigma(\mathbb{C})_V$ for some simply connected open subset of $S_\sigma(\mathbb{C})$. Strictly speaking, the Betti form in [DGH, Prop. 2.2] is the one comes from a principal polarization (instead of a general L), but Betti forms of any two relatively ample line bundles can bound each other by positive constant multiples. Therefore, our definition of "non-degenerate" agrees with that of the loc. cit., and generalizes to families of algebraic dynamical systems.

Now we have the following theorem, which generalizes [GH, Thm. 1.4] and [DGH, Thm. 1.6] from abelian schemes to dynamical systems. Our proof follows the idea of [DGH], but is simplified significantly by our new notion of adelic line bundles.

Theorem 6.2.2 (height inequality)**.** *Let S be a quasi-projective variety over a number field K. Let (X, f, L) be a polarized dynamical system over S. Let Y be a non-degenerate closed subvariety of X over K. Let $\overline{B} \in \widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$ be an adelic $\mathbb{Q}$-line bundle on S. Then for any $c > 0$, there exist $\epsilon > 0$ and a Zariski open and dense subvariety U of Y such that*

$$h_{\overline{L}_f}(y) \geq \epsilon\, h_{\overline{B}}(\pi(y)) - c, \qquad \forall\, y \in U(\overline{K}).$$

Here $\pi : X \to S$ denotes the structure morphism.

Proof. Apply Theorem 5.3.5(2) to the morphism $Y \to S$ and the adelic line bundles $\overline{L}_f|_Y$ and $\overline{M}$. □

6.2.3 Equidistribution theorem over non-degenerate subvarieties

Restricted to the setting of non-degenerate subvarieties, we get a special example of Theorem 5.4.3.

Theorem 6.2.3 (equidistribution over non-degenerate subvarieties)**.** *Let S be a quasi-projective variety over a number field K. Let (X, f, L) be a polarized dynamical system over S. Let Y be a non-degenerate closed subvariety of X over K. Let $\{y_m\}_{m\geq 1}$ be a generic sequence of $Y(\overline{K})$ such that $h_{\overline{L}_f}(y_m) \to 0$. Then for any place v of K, the Galois orbit of $\{y_m\}_{m\geq 1}$ is equidistributed over the analytic space Y_v^{an} with respect to the canonical measure $d\mu_{\overline{L}_f|_Y, v}$.*

The theorem generalizes [DM, Cor. 1.2], which treats the family of elliptic curves described above. If $X \to S$ is an abelian scheme, the theorem confirms the conjecture (REC) of Kühne [Kuh], and our proof is independent of the slightly weaker version in [Kuh, Thm. 1]. The proof of [Kuh] is a limit version of the original proof in [SUZ] and uses a result of Dimitrov–Gao–Habegger [DGH] for uniformity in the limit process.

Note that the existence of the sequence $\{y_m\}_{m\geq 1}$ implies $h_{\overline{M}}(Y) = 0$ and thus $\widehat{\deg}_{\overline{M}}(Y) = 0$, as a consequence of Theorem 5.3.3. In the following, we make some remarks on the existence of Y satisfying the condition of the theorem.

First, the non-degeneracy of Y is easy to check if $\dim Y = \dim S = 1$. In fact, in this case, it becomes $\deg(\widetilde{M}) > 0$, and $\deg(\widetilde{M})$ is exactly the canonical height $\widehat{h}(Y_\eta)$ of the closed point Y_η with respect to the polarized dynamical system (X_η, f_η, L_η) over the generic point $\eta = \operatorname{Spec} K(S)$ of S. For example, if X is a family of abelian varieties over S with trivial $K(S)/K$-trace, then $\widehat{h}(Y_\eta) = 0$ if and only if Y_η is torsion in $X_\eta(\eta)$. See [Con, Thm. 9.15] for example.

For an abelian scheme $X \to S$ of relative dimension g with a high-dimensional base S, there are natural generalizations of the above situation by André–Corvaja–Zannier [ACZ] and Gao [Gao1]. Namely, by [ACZ, Thm. 2.3.1, Prop. 2.1.1] and [Gao1, Thm. 9.1], a closed subvariety Y of X is non-degenerate and contains a Zariski dense set of torsion points if the following conditions hold:

(1) $\dim S = g$;

(2) the morphism from S to the moduli space of abelian varieties of dimension g (with a polarization of degree equal to $\deg(L_\eta)/g!$) is generically finite;

(3) X is simple over the algebraic closure of the function field of S;

(4) Y is a non-torsion section of $X \to S$.

Contrary to the case $g = 1$, the result does not hold if we change condition (3) to the statement that X has a trivial $K(S)/K$-trace by Gao [Gao2, Thm. 1.4(ii)].

6.3 Equidistribution of PCF maps

In this subsection, we consider equidistribution of post-critically finite endomorphisms on $\mathbb{P}^n$, as another application of the equidistribution theorem (Theorem 5.4.3). The equidistribution fits perfectly to the setting of the dynamical Andre–Oort conjecture of Baker–DeMarco [BD, Conj. 1.10]. Our treatment plays a crucial role in the recent solution of the dynamical Andre–Oort conjecture for 1-dimensional families by Ji–Xie [JX].

We will only write the case of number fields, though some of the results also hold over function fields of one variables.

6.3.1 Post-critically finite maps

Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a finite separable morphism over a field. Assume that its algebraic degree d (defined by $f^*\mathcal{O}(1) \simeq \mathcal{O}(d)$) is strictly larger than 1.

Denote by $R(f)$ the ramification divisor (or critical locus) of f in $\mathbb{P}^n_k$, whose definition will be recalled below in the family version. The morphism f is said to be *post-critically finite* (PCF) if every irreducible component of $R(f)$ (with reduced structure) under f is preperiodic.

Let S be a smooth and quasi-projective variety over a number field K. Let $X = \mathbb{P}^n_S$ be the projective space over S, and let $f : X \to X$ be a finite morphism over S of algebraic degree $d > 1$ (over the fibers above S). A point $y \in S(\overline{K})$ is called *post-critically finite* (PCF) if the morphism $f_y : X_y \to X_y$ is post-critically finite.

The main result here is the construction of a natural adelic line bundle $\overline{M}$ over S, and equidistribution theorems of Galois orbits of PCF points.

6.3.2 The adelic line bundle $\overline{M}$

Let S and $f : X \to X$ be as above. Namely, S is a smooth and quasi-projective variety over a number field K, $X = \mathbb{P}^n_S$, and $f : X \to X$ is a finite morphism over S of algebraic degree $d > 1$.

Denote by $\pi : X \to S$ the structure morphism. The canonical morphism $f^*\omega_{X/S} \to \omega_{X/S}$ induces a global section δf of $\omega_f = \omega_{X/S} \otimes f^*\omega_{X/S}^\vee$ on X. The *ramification divisor* $R = R(f)$ of the finite morphism $f : X \to X$ is defined to be the divisor of the section δf. It is also viewed as a (possibly non-reduced) closed subscheme in X. By definition, we have a canonical isomorphism $\omega_f \simeq \mathcal{O}(R)$. We have the following basic result.

Lemma 6.3.1. *The scheme $R(f)$ and every irreducible component of it (with the reduced structure) are projective and flat of relative dimension $n-1$ over S. The fiber $R(f)_y$ of $R(f)$ above any point $y \in S$ is equal to the ramification divisor $R(f_y)$ of $f_y : X_y \to X_y$.*

Proof. Since the canonical map $f_y^*\omega_{X_y/y} \to \omega_{X_y/y}$ is the base change of $f^*\omega_{X/S} \to \omega_{X/S}$ via $y \to S$, we see that $\delta(f_y)$ is the base change of δf, and $R(f_y)$ is the base change of $R(f)$ via $y \to S$. Then $R(f)$ is of pure relative dimension $n-1$ over S. Since $R(f)$ is a Cartier divisor on X, it is actually Cohen–Macaulay over S. By the miracle flatness (cf. [Mat, Thm. 23.1]), the morphism $R(f) \to S$ is flat. Similarly, any irreducible component of R is flat over S. □

Let L be a $\mathbb{Q}$-line bundle on X, isomorphic to $\mathcal{O}(1)$ on fibers of S, such that $f^*L \simeq dL$. There is a unique class in $\mathrm{Pic}(X)_\mathbb{Q}$ satisfying these requirements. In fact, we can set $L = \mathcal{O}_{\mathbb{P}^1_S}(1) \otimes \pi^*N$ for a suitable $\mathbb{Q}$-line bundle N

on S. Then $f^*L \simeq dL$ becomes $f^*\mathcal{O}_{\mathbb{P}^1_S}(1) - \mathcal{O}_{\mathbb{P}^1_S}(d) = (d-1)\pi^*N$. Note that $f^*\mathcal{O}_{\mathbb{P}^1_S}(1) - \mathcal{O}_{\mathbb{P}^1_S}(d)$ is trivial on fibers of $X \to S$, and thus lies in $\pi^*\mathrm{Pic}(S)$. The equality determines the class $N \in \mathrm{Pic}(S)_{\mathbb{Q}}$ uniquely.

Denote by $\overline{L} = \overline{L}_f$ the nef f-invariant extension of L in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ such that $f^*\overline{L} \simeq d\overline{L}$, as constructed in Theorem 6.1.1. Recall that the ramification divisor R is projective and flat of pure relative dimension $n-1$ over S. Define

$$\overline{M} := \langle \overline{L}|_R \rangle^n_{R/S} = \langle \overline{L}|_R, \cdots, \overline{L}|_R \rangle_{R/S} \in \widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}.$$

Here the Deligne pairing is as in Theorem 4.1.3. Since the theorem requires R to be integral, we need to extend the definition if R is not integral. In fact, write $R = \sum_{i=1}^r m_i R_i$ in terms of distinct prime divisors $R_1, \cdots, R_r$ of X, and interpret the definition by

$$\overline{M} = \sum_{i=1}^r m_i \langle \overline{L}|_{R_i} \rangle^n_{R_i/S} \in \widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}.$$

In all cases, $\overline{M}$ is a nef adelic $\mathbb{Q}$-line bundle on S.

If $n = 1$, then R is finite and flat over S, so

$$\overline{M} = N_{R/S}(\overline{L}|_R) \in \widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$$

is actually given by the norm map.

As before, denote by

$$\overline{L} \longmapsto \widetilde{L} \longmapsto L, \quad \overline{M} \longmapsto \widetilde{M} \longmapsto M$$

the images of $\overline{L}$ and $\overline{M}$ under the maps

$$\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(X/K)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}, \quad \widehat{\mathrm{Pic}}(S)_{\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(S/K)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(S)_{\mathbb{Q}}.$$

6.3.3 The height function

Consider the height function

$$h_{\overline{M}} : S(\overline{K}) \longrightarrow \mathbb{R}.$$

It detects PCF points by the following result.

Lemma 6.3.2. *Let $y \in S(\overline{K})$ be a point. The following are true:*

(1) $h_{\overline{M}}(y) \geq 0$.

(2) If y *is PCF in* S*, then* $h_{\overline{M}}(y) = 0$.

(3) If $n = 1$*, then* y *is PCF in* S *if and only if* $h_{\overline{M}}(y) = 0$.

Proof. Part (1) holds since $\overline{M}$ is nef. For (2) and (3), for convenience, assume that y is a closed point of S instead of an algebraic point. By Theorem 4.1.3, the Deligne pairing is compatible with base change $y \to S$. It follows that

$$\overline{M}|_y = \langle \overline{L}|_R \rangle^n |_y = \langle \overline{L}|_{R_y} \rangle^n = \sum_i m_{R_{y,i}} \langle \overline{L}|_{R_{y,i}} \rangle^n.$$

Here $R_y = \sum_i m_{R_{y,i}} R_{y,i}$ is the decomposition into prime divisors in X_y. Then we have

$$\widehat{\deg}(\overline{M}|_y) = \sum_i m_{R_{y,i}} \overline{L}|^n_{R_{y,i}}.$$

In terms of heights, we have

$$h_{\overline{M}}(y) = \sum_i m'_{y,i} h_{\overline{L}}(R_{y,i}),$$

Here $m'_{y,i} = m_{R_{y,i}} n \deg_{L_y}(R_{y,i}) / \deg(y)$ is strictly positive.

Then $h_{\overline{M}}(y) = 0$ if and only if $h_{\overline{L}}(R_{y,i}) = 0$ for every irreducible component $R_{y,i}$ of R_y. This gives (2) immediately. For (3), $R_{y,i}$ is a closed point, and thus $h_{\overline{L}}(R_{y,i}) = 0$ further implies $R_{y,i}$ is preperiodic. □

Remark 6.3.3. In the case $n = 1$, the height function $h_{\overline{M}} : S(\overline{K}) \to \mathbb{R}$ is equal to the critical height as considered in Ingram [Ing] and Gauthier–Okuyama–Vigny [GOV]. This leads to a new proof of [Ing, Thm. 1] by the method of Theorem 6.2.2.

Problem 6.3.4. *We raise the question of whether Lemma 6.3.2(3) holds for* $n \geq 2$*. This amounts to ask: for a finite morphism* $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ *of algebraic degree* $d > 1$ *over a number field* K*, if every irreducible component of the ramification divisor* $R(f)$ *has canonical height 0, does it follow that every irreducible component of* $R(f)$ *is preperiodic? This is actually the dynamical Manin–Mumford conjecture for* $R(f)$ *under the dynamical system* $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$*. We refer to Ghioca–Tucker–Zhang [GTZ] for various versions and examples about the dynamical Manin–Mumford conjecture.*

6.3.4 The equidistribution theorem

With the nef adelic line bundle $\overline{M}$ over S, we have the following equidistribution theorem, which is a direct consequence of Theorem 5.4.3.

Theorem 6.3.5 (equidistribution: PCF maps on projective space). *Let S be a smooth and quasi-projective variety over a number field K. Let $X = \mathbb{P}^n_S$ be the projective space over S, and let $f : X \to X$ be a finite morphism over S of algebraic degree $d > 1$. Assume that $\deg_{\widetilde{M}}(S) > 0$. Let $\{y_m\}_m$ be a generic sequence of PCF points of $S(\overline{K})$. Then the Galois orbit of $\{y_m\}_m$ is equidistributed in S_v^{an} with respect to $d\mu_{\overline{M},v}$ for any place v of K.*

Note that the existence of a generic sequence of PCF points implies $h_{\overline{M}}(S) = 0$. This follows from the fundamental inequality

$$\liminf_{y \in S(\overline{K})} h_{\overline{M}}(y) \geq h_{\overline{M}}(S)$$

proved in Theorem 5.3.3.

The condition $\deg_{\widetilde{M}}(S) > 0$ seems very hard to check in general. However, in the case $n = 1$, it is equivalent to a very easy condition in terms of the moduli space of endomorphisms.

To describe the condition, denote by $\mathcal{M}_d^n$ the moduli space over K of endomorphisms $\mathbb{P}^n$ of algebraic degree d. The moduli space was constructed using Mumford's geometric invariant theory, by the works of Silverman [Sil5], Levy [Lev] and Petsche–Szpiro–Tepper [PST].

If $n = 1$, there is a special type of PCF morphisms $\mathbb{P}^1 \to \mathbb{P}^1$, called the flexible Lattès maps, which are descended from multiplication morphisms of elliptic curves. We refer to Silverman [Sil6, §6.5] for the basics of the flexible Lattès maps. In $\mathcal{M}_d^1$, there is a distinguished closed subvariety, called the flexible Lattès locus, parametrizing the flexible Lattès maps on $\mathbb{P}^1$. The flexible Lattès locus is empty if d is not a perfect square, and has dimension 1 if d is a perfect square.

Return to the dynamical system $f : X \to X$ for $X = \mathbb{P}^1_S$. Recall

$$\overline{M} = N_{R/S}(\overline{L}_f|_R) \in \widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}.$$

By the moduli property, there is a morphism $S \to \mathcal{M}_d^1$. Finally, the main result here is the following variant of Theorem 6.3.5.

Theorem 6.3.6 (equidistribution: PCF maps on projective line)**.** *Let S be a smooth and quasi-projective variety over a number field K. Let $X = \mathbb{P}^1_S$ be the projective line over S, and let $f : X \to X$ be a finite morphism over S of algebraic degree $d > 1$. Assume that the morphism $S \to \mathcal{M}^1_d$ is generically finite and its image is not contained in the flexible Lattès locus. Let $\{y_m\}_m$ be a generic sequence of PCF points of $S(\overline{K})$. Then the Galois orbit of $\{y_m\}_m$ is equidistributed in S_v^{an} with respect to $d\mu_{\overline{M},v}$ for any place v of K.*

If S is a family of polynomial maps on $\mathbb{P}^1$, the theorem was previously proved by Favre–Gauthier [FG]. Their strategy is to reduce the problem to the equidistribution of Yuan [Yua1], which works for the polynomial maps, but not for rational maps.

As a dilation, we remark that the nef adelic line bundle $\overline{L}_f$ for general n is actually strongly nef. For this, it suffices to treat the case that S is the moduli space $\mathcal{M}^n_d$. Note that the corresponding moduli space over $\mathbb{Z}$ is affine (cf. [Lev, Thm. 1.1]). Then $\overline{L}_f$ is strongly nef by Theorem 6.1.1.

6.3.5 The Lyapunov exponent

Let us recall the classical Lyapunov exponent in the current setting. For completeness, we will include both the archimedean case and the non-archimedean case.

Let K be a complete field with a non-trivial absolute value $|\cdot|$. Let $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ be a finite and separable morphism of algebraic degree $d > 1$. Recall that the ramification divisor $R(f) = \mathrm{div}(\delta f)$, where δf is the section of $\omega_f = \omega_{\mathbb{P}^n_K} \otimes f^*\omega^\vee_{\mathbb{P}^n_K}$ induced by the canonical morphism $f^*\omega_{\mathbb{P}^n_K} \to \omega_{\mathbb{P}^n_K}$. This definition gives a canonical isomorphism $\omega_f \simeq \mathcal{O}(R(f))$.

Fix a continuous metric $\|\cdot\|_0$ of $\omega_{\mathbb{P}^n_K}$ on the analytic space $\mathbb{P}^{n,\mathrm{an}}_K$ over the valued field $(K, |\cdot|)$, and take the metric $f^*\|\cdot\|_0$ of $f^*\omega_{\mathbb{P}^n_K}$ on $\mathbb{P}^{n,\mathrm{an}}_K$. Then we have the quotient metric $\|\cdot\|_1$ of $\omega_f = \omega_{\mathbb{P}^n_K} \otimes f^*\omega^\vee_{\mathbb{P}^n_K}$ on $\mathbb{P}^{n,\mathrm{an}}_K$. The function $-\log\|\delta f\|_1$ is a Green's function of $R(f)$ on $\mathbb{P}^{n,\mathrm{an}}_K$. The *Lyapunov exponent* of f is defined by

$$\mathrm{Ly}(f) = \int_{\mathbb{P}^{n,\mathrm{an}}_K} \log\|\delta f\|_1 \, d\mu_f.$$

Here $d\mu_f = c_1(\overline{\mathcal{O}(1)}_f)^n$ is the f-invariant probability measure on $\mathbb{P}^{n,\mathrm{an}}_K$. The definition is independent of choice of the metric $\|\cdot\|_0$ of $\omega_{\mathbb{P}^n_K}$. In fact, if $\|\cdot\|_0' = \|\cdot\|_0 e^h$ is a different choice for a continuous function h on $\mathbb{P}^{n,\mathrm{an}}_K$, then the integral of

$$\log \|\delta f\|_1' - \log \|\delta f\|_1 = h - f^* h$$

with respect to μ_f is 0 by $f_* d\mu_f = d\mu_f$.

It is convenient to choose $\|\cdot\|_0$ to be an f-invariant metric of $\omega_{\mathbb{P}^n_K} \simeq \mathcal{O}(-n-1)$ on $\mathbb{P}^{n,\mathrm{an}}_K$. This metric is unique up to constant multiples, but then the induced metric on ω_f does not depend on the constant multiple, and we will denote this metric by $\|\cdot\|_f$. Then the Lyapunov exponent is just

$$\mathrm{Ly}(f) = \int_{\mathbb{P}^{n,\mathrm{an}}_K} \log \|\delta f\|_f \, d\mu_f.$$

Now we assume that K is a number field, and that $f : \mathbb{P}^n_K \to \mathbb{P}^n_K$ is still a finite morphism of algebraic degree $d > 1$. Choose $\|\cdot\|_0 = \{\|\cdot\|_{0,v}\}_v$ to be an f-invariant adelic metric of $\omega_{\mathbb{P}^n_K} \simeq \mathcal{O}(-n-1)$ on $\mathbb{P}^{n,\mathrm{an}}_K$ in the classical sense of [Zha2]. This induces an f-invariant adelic metric $\|\cdot\|_f = \{\|\cdot\|_{f,v}\}_v$ of $\omega_f = \omega_{\mathbb{P}^n_K} \otimes f^* \omega_{\mathbb{P}^n_K}^\vee \simeq \mathcal{O}((n+1)(d-1))$ on $\mathbb{P}^{n,\mathrm{an}}_K$. Write $\overline{\omega}_f = (\omega_f, \|\cdot\|_f)$ for the adelic line bundle. Via these metrics, the arithmetic intersection number

$$\overline{\omega}_f \cdot \overline{\mathcal{O}(1)}_f^n = (n+1)(d-1)\overline{\mathcal{O}(1)}_f^{n+1} = 0.$$

On the other hand, we can apply [CT, Thm. 1.4] to the section δf of ω_f to compute the arithmetic intersection number. It gives

$$\overline{\omega}_f \cdot \overline{\mathcal{O}(1)}_f^n = \left(\overline{\mathcal{O}(1)}_f|_{R(f)}\right)^n - \sum_v \int_{\mathbb{P}^{n,\mathrm{an}}_{K_v}} \log \|\delta f\|_f \, c_1(\overline{\mathcal{O}(1)}_f)_v^n.$$

This gives the height formula

$$\left(\overline{\mathcal{O}(1)}_f|_{R(f)}\right)^n = \sum_v \mathrm{Ly}(f_{K_v}).$$

6.3.6 The bifurcation measure

Return to the situation of Theorem 6.3.5, where $f : X \to X$ and $X = \mathbb{P}^n_S$ are over a number field K.

It turns out that the equilibrium measure at a complex place v in the theorem is exactly the probability measure associated to the bifurcation measure introduced by DeMarco [DeM1, DeM2] for $n = 1$ and further studied by Bassanelli–Berteloot [BB, §5] for general n. The goal here is to explore this relation in our adelic setting, which implies the identity of the measures in both the archimedean setting and the non-archimedean setting. The exposition here is a family version of the above height formula in terms of the Lyapunov exponents.

Let v be a place of K. The Lyapunov exponent defines a function

$$\mathrm{Ly}_v : S_v^{\mathrm{an}} \longrightarrow \mathbb{R}, \quad y \longmapsto \mathrm{Ly}(f_y).$$

If v is archimedean, the pull-back of Ly_v to $S_v(\mathbb{C})$ is continuous and psh; if v is non-archimedean, then Ly_v is locally psh-approachable on S_v^{an} in the sense of [CD, 6.3.1, Def. 5.6.3, Def. 5.5.1]. The archimedean case of this statement can be derived from [DeM1, DeM2, BB], and we will present an approach including both cases. The *bifurcation measure* of (X, f) over S_v^{an} is defined to be the Monge–Ampère measure

$$d\mu_{\mathrm{bif},v} = (dd^c \mathrm{Ly}_v)^{\dim S}.$$

Now we have the following description of the equilibrium measure in the setting of Theorem 6.3.5.

Theorem 6.3.7 (bifurcation measure)**.** *Let v be a place of K. As $(1,1)$-currents on S_v^{an},*

$$c_1(\overline{M})_v = dd^c \mathrm{Ly}_v.$$

As measures on S_v^{an},

$$c_1(\overline{M})_v^{\dim S} = (dd^c \mathrm{Ly}_v)^{\dim S},$$

and

$$d\mu_{\overline{M},v} = \frac{1}{\deg_{\widetilde{M}}(S)} (dd^c \mathrm{Ly}_v)^{\dim S}.$$

Note that the third equality follows from the second one. In fact, by Lemma 5.4.4, we have

$$\deg_{\widetilde{M}}(S) = \int_{S_v^{\mathrm{an}}} c_1(\overline{M})_\sigma^{\dim S} = \int_{S_v^{\mathrm{an}}} (dd^c \mathrm{Ly}_v)^{\dim S}.$$

This also implies that the integral on the right-hand side is independent of v.

The complex version of Theorem 6.3.7 is essentially Bassanelli–Berteloot [BB, Cor. 4.6]. The following theorem is an adelic treatment of the situation, which actually asserts that the Lyapunov exponents for all places v can be glued together to form an adelic divisor.

Theorem 6.3.8. *(1) There is a unique adelic divisor $\overline{D}_{\mathrm{bif}}$ on S with underlying divisor 0 whose total Green's function $\widetilde{g}_{\overline{D}_{\mathrm{bif}}}: S^{\mathrm{an}} \to \mathbb{R}$ satisfies $\widetilde{g}_{\overline{D}_{\mathrm{bif}}}|_{S_v^{\mathrm{an}}} = \mathrm{Ly}_v$ on S_v^{an} for every place v of K.*

(2) For the above adelic divisor $\overline{D}_{\mathrm{bif}}$, we have $\overline{M} = \mathcal{O}(\overline{D}_{\mathrm{bif}})$ in $\widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$.

It is easy to see that Theorem 6.3.8 implies Theorem 6.3.7. It also implies the continuity and reasonable psh properties of $\mathrm{Ly}_v : S_v^{\mathrm{an}} \to \mathbb{R}$, since $\overline{M}$ is nef by construction.

Proof of Theorem 6.3.8. The uniqueness in (1) follows from Proposition 3.5.1. The major part of the proof follows from an adelic version of the above construction to derive the height formula in terms of $\sum_v \mathrm{Ly}(f_{K_v})$ on single dynamical systems.

Recall that $\overline{L} = \overline{L}_f \in \widehat{\mathcal{P}\mathrm{ic}}(X)_{\mathbb{Q}}$ is a nef f-invariant extension of L with $f^*\overline{L} \simeq d\overline{L}$, and $\overline{M} \in \widehat{\mathcal{P}\mathrm{ic}}(S)_{\mathbb{Q}}$ is the Deligne pairing $\langle \overline{L}|_R \rangle^n$. Here we assume that $\overline{L}$ and $\overline{M}$ are adelic $\mathbb{Q}$-line bundles instead of just isomorphism classes.

Recall that $R = \mathrm{div}(\delta f)$ is the divisor of the canonical global section δf of $\omega_f = \omega_{X/S} - f^*\omega_{X/S}$ on X. Here we write the operation of line bundles additively again. By comparing the fibers above S, there is an isomorphism $\tau_0 : \omega_{X/S} \to -(n+1)L + \pi^* N$ for some $\mathbb{Q}$-line bundle $N \in \mathcal{P}\mathrm{ic}(S)_{\mathbb{Q}}$. This induces an isomorphism $\tau_1 : \omega_f \to (n+1)(d-1)L$, which does not depend on the choice of τ_0. In fact, different choices of τ_0 (for fixed L, N) are up to multiples by elements of $\Gamma(S, \mathcal{O}_S^{\times})$, and these elements are killed in the definition of τ_1.

Denote by $\overline{\omega}_f$ the adelic line bundle on X with underlying line bundle ω_f, such that $\tau_1 : \omega_f \to (n+1)(d-1)L$ induces an isomorphism $\overline{\omega}_f \to (n+1)(d-1)\overline{L}$. Note that the extension $\overline{\omega}_f$ of ω_f is unique up to unique isomorphism.

With these extensions, the Deligne pairing

$$\langle \overline{\omega}_f, \overline{L}^n \rangle_{X/S} = (n+1)(d-1)\langle \overline{L} \rangle_{X/S}^{n+1} = 0$$

in $\mathrm{Pic}(S)_{\mathbb{Q}}$. Here the last equality is similar to Proposition 6.1.4(3), as a consequence of the projection formula in Lemma 4.6.1(4) and the invariant property $f^*\overline{L} \simeq d\overline{L}$.

On the other hand, using the section δf of ω_f to compute the Deligne pairing, we have a canonical isomorphism

$$\langle \omega_f, L^n \rangle_{X/S} \longrightarrow \langle L|_R \rangle_{R/S}^n = M.$$

This gives a canonical section $t_{\rm bif}$ of $M - \langle\omega_f, L^n\rangle_{X/S}$, and thus an adelic $\mathbb{Q}$-divisor $\overline{D}_{\rm bif} = \widehat{\rm div}(t_{\rm bif})$ on S with respect to $\overline{M} - \langle\overline{\omega}_f, \overline{L}^n\rangle_{X/S}$. The underlying divisor $D_{\rm bif} = 0$ on S by definition. By construction, we have

$$\overline{M} \simeq \langle\overline{\omega}_f, \overline{L}^n\rangle_{X/S} + \mathcal{O}(\overline{D}_{\rm bif}) \simeq \mathcal{O}(\overline{D}_{\rm bif})$$

in $\mathcal{P}{\rm ic}(S)_{\mathbb{Q}}$.

Now we can compute the total Green's function $\widetilde{g}_{\overline{D}_{\rm bif}}$ on $S_v^{\rm an}$. If v is archimedean, for any $y \in S_v(\mathbb{C})$, the fiberwise formula in §4.2.2 gives

$$\widetilde{g}_{\overline{D}_{\rm bif}}(y) = -\log\|t_{\rm bif}\|(y) = \int_{X_y^{\rm an}} \log\|\delta f\|_y\, c_1(\overline{L})_y^n = {\rm Ly}_v(y).$$

The formula also holds for non-archimedean v by Theorem 4.6.2. This proves Theorem 6.3.8. □

6.3.7 Bigness problem

In the case $n = 1$, to deduce Theorem 6.3.6 from Theorem 6.3.5, it suffices to check that the total volume of $\mu_{{\rm bif},\sigma}$ is strictly positive if and only if the morphism $S \to \mathcal{M}_d^1$ is generically finite and its image is not contained in the flexible Lattès locus. The essential part of the statement is the "if" part when $S \to \mathcal{M}_d$ is a closed immersion, which follows from [BB, Prop. 6.3] and [GOV, Lem. 6.8]. This proves Theorem 6.3.6.

In the case $n = 1$, it is well-known that the set of PCF points are dense in $\mathcal{M}_d^1$. See [DeM3, Thm. A] for example.

In the case $n > 1$, the situation is very different. In fact, by the work of Ingram–Ramadas–Silverman [IRS], PCF points in $\mathcal{M}_d^n$ are expected to be very sparse in some sense. As in [IRS, Question 5], we do not know if the set of PCF points in $\mathcal{M}_d^n$ is Zariski dense. Then we raise the following question.

Problem 6.3.9. *Assume $n \geq 2$, $S = \mathcal{M}_d^n$ and $f : \mathbb{P}_S^n \to \mathbb{P}_S^n$ is the universal family. Is $\widetilde{M}$ big on S? Is $\overline{M}$ big on S?*

The bigness of $\widetilde{M}$ is equivalent to $\deg_{\widetilde{M}}(S) > 0$, which is a condition of the equidistribution theorem. The bigness of $\overline{M}$ is equivalent to $\widehat{\deg}_{\overline{M}}(S) > 0$, which becomes $h_{\overline{M}}(S) > 0$ assuming $\deg_{\widetilde{M}}(S) > 0$. It is further related to the existence of a generic and small sequence for $h_{\overline{M}}$ considering Theorem 5.3.3. In particular, if $\overline{M}$ is big, then the set of PCF points in $\mathcal{M}_d^n$ is not Zariski dense.

6.4 Admissible extensions of line bundles

Let (X, f, L) be a polarized dynamical system over a finitely generated field F over $\mathbb{Q}$. Assume that X is normal. We have already constructed an adelic line bundle $\overline{L}_f \in \widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q},\mathrm{nef}}$ extending L and with $f^*\overline{L}_f = q\overline{L}_f$. Following the idea of [YZ1], we can construct an admissible extension in $\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q},\mathrm{int}}$ for any line bundle $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$.

Our exposition is sketchy, and we refer to [YZ1, §4.3] for the common arguments, but we will explain the difference of the current case. Moreover, we will only restrict to the arithmetic case ($k = \mathbb{Z}$), and refer to [Car1, Car2] for the counterparts in the geometric case, where the extra argument is to treat contribution of the F/k-image of $\underline{\mathrm{Pic}}^0_{X/F}$.

6.4.1 Semisimplicity

The pull-back map f^* preserves the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

We refer to [YZ1, Appendix 1] for a list of properties of this sequence. In particular, $\mathrm{NS}(X)$ is a finitely generated $\mathbb{Z}$-module. By the Lang–Néron theorem (cf. [Con, Thm. 2.1]), $\mathrm{Pic}^0(X)$ is also a finitely generated $\mathbb{Z}$-module, since it is the Mordell–Weil group of the Picard variety representing the functor $\underline{\mathrm{Pic}}^0_{X/F}$ over the finitely generated field F. The counterpart of [YZ1, Theorem 4.7] is as follows.

Theorem 6.4.1. *Let (X, f, L) be a polarized dynamical system over a finitely generated field F over $\mathbb{Q}$. Assume that X is normal.*

(1) The operator f^* is semisimple on $\mathrm{Pic}^0(X)_{\mathbb{C}}$ (resp. $\mathrm{NS}(X)_{\mathbb{C}}$) with eigenvalues of absolute values $q^{1/2}$ (resp. q).

(2) The operator f^* is semisimple on $\mathrm{Pic}(X)_{\mathbb{C}}$ with eigenvalues of absolute values $q^{1/2}$ or q.

Proof. The proof is similar to its counterpart. The only difference is some extra work to prove that f^* is semisimple on $\mathrm{Pic}^0(X)_{\mathbb{C}}$ with eigenvalues of absolute values $q^{1/2}$. We describe it briefly here.

As before, (X, f, L) extends to a dynamical system (U, f, L_V) over a smooth quasi-projective variety V over $\mathbb{Q}$ with function field F. Here $U \to V$

is a projective and flat morphism with generic fiber $X \to \operatorname{Spec} F$, $f : U \to U$ is a V-morphism extending $f : X \to X$, and L_V is a $\mathbb{Q}$-line bundle on U, relatively ample over V, and with $f^*L_V = qL_V$. We can further assume that all the fibers of $U \to V$ are normal. We claim that there is a closed point $v \in V$ such that the reduction map $\mathrm{Pic}^0(X)_{\mathbb{C}} \to \mathrm{Pic}^0(U_v)_{\mathbb{C}}$ is injective. If this holds, then the result follows from its counterpart over number fields.

Note that the Picard functor $\underline{\mathrm{Pic}}_{U/V}$ is representable by a group scheme by [BLR, §8.2, Thm. 1]. Its relative identity component $\underline{\mathrm{Pic}}^0_{U/V}$ is an abelian scheme over V by [Kle, Thm. 9.5.4]. Then the injectivity is a consequence of the specialization theorem of Wazir [Waz], which is a generalization of the specialization theorem of Silverman [Sil1] using the Moriwaki height. $\square$

By the theorem above, the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}} \longrightarrow \mathrm{NS}(X)_{\mathbb{Q}} \longrightarrow 0.$$

has an f^*-equivariant splitting

$$\ell_f : \mathrm{NS}(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}.$$

Denote by $\mathrm{Pic}_f(X)_{\mathbb{Q}}$ the image of ℓ_f.

We say an element of $\mathrm{Pic}(X)_{\mathbb{Q}}$ is *f-pure of weight* 1 *(resp. f-pure of weight* 2*)* if it lies in $\mathrm{Pic}^0(X)_{\mathbb{Q}}$ (resp. $\mathrm{Pic}_f(X)_{\mathbb{Q}}$).

6.4.2 Admissible extensions

The action $f^* : \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ is compatible with the action $f^* : \mathrm{Pic}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$. The goal is to study the spectral theory of this action. The following result is the generalization of [YZ1, Thm. 4.9].

Theorem 6.4.2. *Let (X, f, L) be a polarized dynamical system over a finitely generated field F over $\mathbb{Q}$. Assume that X is normal. The projection*

$$\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}$$

has a unique section

$$M \longmapsto \overline{M}_f$$

as f^*-modules. The image $\overline{M}_f$ is always integrable. If $M \in \mathrm{Pic}_f(X)_{\mathbb{Q}}$ is ample, then $\overline{M}_f$ is nef.

We call $\overline{M}_f$ *the f-admissible extension of M* in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$. An adelic line bundle in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ which is isomorphic to some $\overline{M}_f$ is called *f-admissible*.

Note that the theorem for abelian schemes is actually Theorem 6.1.3. In fact, any $\mathbb{Q}$-line bundle L on an abelian scheme X can be written as the sum of the symmetric $\mathbb{Q}$-line bundle $(L+[-1]^*L)/2$ with the anti-symmetric $\mathbb{Q}$-line bundle $(L-[-1]^*L)/2$.

As in the case of number fields, we also have the following result, as the counterpart of [YZ1, Cor. 4.11].

Corollary 6.4.3. *For $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$, the following are true:*

(1) If $f^*M = \lambda M$ for some $\lambda \in \mathbb{Q}$, then $f^*\overline{M}_f = \lambda \overline{M}_f$ in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$.

(2) For any $x \in \mathrm{Prep}(f)$, one has $\overline{M}_f|_{x'} = 0$ in $\widehat{\mathrm{Pic}}(x')_{\mathbb{Q}}$. Here x' is the closed point of X corresponding to x. Hence, the height function $\mathfrak{h}_{\overline{M}_f}$ is zero on $\mathrm{Prep}(f)$.

Now we sketch a proof of Theorem 6.4.2, following the line of that of [YZ1, Thm. 4.9].

Proof of Theorem 6.4.2. Assume that X is geometrically connected over F, which can be achieved by replace F by its algebraic closure in $F(X)$. Let $\mathcal{V}$ be a quasi-projective model of $\mathrm{Spec}\, F$ over $\mathbb{Z}$, and $(\mathcal{U}, f, \mathcal{L})$ be a polarized dynamical system over $\mathcal{V}$ whose generic fiber is the polarized dynamical system (X, f, L) over $\mathrm{Spec}\, F$.

Step 1. We claim that there is an affine open subscheme $\mathcal{V}'$ of $\mathcal{V}$ such that the canonical map $\mathrm{Pic}(\mathcal{U}_{\mathcal{V}'}) \to \mathrm{Pic}(X)$ is an isomorphism. This is a well-known fact, but we provide a proof due to lack of precise reference.

(1) There is an open subscheme $\mathcal{V}'$ of $\mathcal{V}$ such that $\mathcal{V}'$ is regular and $\mathcal{U}_{\mathcal{V}'} \to \mathcal{V}'$ has geometrically connected fibers.

(2) We can assume that $\mathrm{Pic}(\mathcal{V}')$ is trivial by [Lan, Chap. 2, Cor. 7.7]. Then $\mathrm{Pic}(\mathcal{V}'')$ is trivial for any open subscheme $\mathcal{V}''$ of $\mathcal{V}'$ since $\mathrm{Pic}(\mathcal{V}') \to \mathrm{Pic}(\mathcal{V}'')$ is surjective by passing to Weil divisors, where the key is that $\mathcal{V}'$ is regular.

(3) The canonical map $\mathrm{Pic}(\mathcal{U}_{\mathcal{V}''}) \to \mathrm{Pic}(X)$ is injective for any open subscheme $\mathcal{V}''$ of $\mathcal{V}'$. It suffices to prove that $\mathrm{CaCl}(\mathcal{U}_{\mathcal{V}''}) \to \mathrm{CaCl}(X)$ is injective for the class groups of Cartier divisors. Then it suffices to prove

that $\mathrm{Cl}(\mathcal{U}_{\mathcal{V}''}) \to \mathrm{Cl}(X)$ is injective for the class groups of Weil divisors. If a Weil divisor of $\mathcal{U}_{\mathcal{V}''}$ is trivial on X, then it is vertical in the sense that it is the pull-back of a Weil divisor from $\mathcal{V}''$, which is linearly equivalent to 0 by $\mathrm{Pic}(\mathcal{V}'') = 0$.

(4) The canonical map $\varinjlim_{\mathcal{V}''} \mathrm{Pic}(\mathcal{U}_{\mathcal{V}''}) \to \mathrm{Pic}(X)$ is an isomorphism by [EGA, IV-3, Thm. 8.5.2].

(5) By (3) and (4), $\mathrm{Pic}(\mathcal{U}_{\mathcal{V}''}) \to \mathrm{Pic}(X)$ is an isomorphism for sufficiently small open subscheme $\mathcal{V}''$ of $\mathcal{V}'$, since $\mathrm{Pic}(X)$ is finitely generated.

Therefore, we can assume that the canonical map $\mathrm{Pic}(\mathcal{U}) \to \mathrm{Pic}(X)$ is an isomorphism, by replacing $\mathcal{V}$ by a sufficiently small affine open subscheme.

Let $\pi : \mathcal{X} \to \mathcal{S}$ be a projective model of $\mathcal{U} \to \mathcal{V}$; i.e., $\mathcal{X}$ and $\mathcal{S}$ are projective models of $\mathcal{U}$ and $\mathcal{V}$ respectively, and $\mathcal{X} \to \mathcal{S}$ is a morphism extending $\mathcal{U} \to \mathcal{V}$. We can further assume that there is a strictly effective arithmetic divisor $\overline{\mathcal{E}}_0$ on $\mathcal{S}$, whose finite part has support equal to $\mathcal{S} \setminus \mathcal{V}$. Use the boundary divisor $(\mathcal{X}, \pi^*\overline{\mathcal{E}}_0)$ to define the boundary topology of $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{Q}}$.

Step 2. Consider the exact sequence

$$0 \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(\mathcal{U})_{\mathbb{Q}} \longrightarrow 0.$$

Here $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}}$ is defined by the left exactness. For the right exactness, it suffices to prove that any effective Cartier divisor $\mathcal{D}$ on $\mathcal{U}$ can be extended to a projective model $\mathcal{X}'$ of $\mathcal{U}$. This is easy, by setting $\mathcal{X}'$ to be the blowing-up of $\mathcal{X}$ along the Zariski closure of $\mathcal{D}$ in $\mathcal{X}$.

Denote by $R(t)$ the characteristic polynomial of f^* on the finite-dimensional vector space $\mathrm{Pic}(\mathcal{U})_{\mathbb{Q}} = \mathrm{Pic}(X)_{\mathbb{Q}}$. We claim that

$$R(f^*) : \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}}$$

is surjective.

Contrary to the proof of [YZ1, Thm. 4.9], we do not use the interpretation of the metrics on Berkovich analytic spaces as in Proposition 3.5.1, since it would be hard to control the convergence in terms of the boundary topology.

Define $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}}$ by the left exactness of

$$0 \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}} \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{Q}} \longrightarrow \mathrm{Div}(\mathcal{U})_{\mathbb{Q}} \longrightarrow 0.$$

In terms of Proposition 2.5.1, there is a canonical surjection

$$\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}}.$$

It suffices to prove that

$$R(f^*) : \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}} \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}}$$

is surjective.

Take the Taylor expansion at $t = 0$ by

$$\frac{1}{R(t)} = \sum_{m=0}^{\infty} a_m t^m, \quad a_m \in \mathbb{Q}.$$

By Theorem 6.4.1, the roots of the polynomial $R(t)$ have absolute values equal to q or $q^{1/2}$. Using partial fractions to expand $1/R(t)$, there is a polynomial $Q(t)$ of rational coefficients such that

$$|a_m| \leq Q(m) q^{-m/2}, \ \forall m.$$

Denote

$$S_i(t) = \sum_{m=0}^{i} a_m t^m, \quad i \geq 1.$$

To prove the surjectivity, take any $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}}$. We claim that the sequence $\{S_i(f^*)\overline{\mathcal{D}}\}_i$ converges in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}}$. If so, then the limit gives an inverse image of $\overline{\mathcal{D}}$ under $R(f^*)$.

For the convergence, note that there is positive rational constant c such that

$$-c\,\pi^*\overline{\mathcal{E}}_0 \leq \overline{\mathcal{D}} \leq c\,\pi^*\overline{\mathcal{E}}_0.$$

This holds automatically if $\overline{\mathcal{D}}$ lies in the kernel of $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod},\mathbb{Q}} \to \mathrm{Div}(\mathcal{U})_{\mathbb{Q}}$. In general, $\overline{\mathcal{D}}$ is a limit of such elements, but then the Cauchy condition of $\overline{\mathcal{D}}$ gives the constant c.

For any $i > j \geq 1$, we have

$$S_i(f^*)\overline{\mathcal{D}} - S_j(f^*)\overline{\mathcal{D}} = \sum_{m=j+1}^{i} a_m \,(f^*)^m \overline{\mathcal{D}} \leq c \sum_{m=j+1}^{i} |a_m| \,\pi^*\overline{\mathcal{E}}_0.$$

We similarly have

$$S_i(f^*)\overline{\mathcal{D}} - S_j(f^*)\overline{\mathcal{D}} \geq -c \sum_{m=j+1}^{i} |a_m|\, \pi^*\overline{\mathcal{E}}_0.$$

By the bound of a_m, we see that $\{S_i(f^*)\overline{\mathcal{D}}\}_i$ converges in $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}}$.

Step 3. The remaining part of the proof is almost identical to that of [YZ1, Thm. 4.9]. In fact, for any $M \in \mathrm{Pic}(\mathcal{U})_{\mathbb{Q}}$, take any extension $\overline{M}^0$ of M in $\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{int},\mathbb{Q}}$, and set

$$\overline{M}_f = \overline{M}^0 - R(f^*)|^{-1}_{\widehat{\mathrm{Pic}}(\mathcal{U})_{\mathrm{vert},\mathbb{Q}}}(R(f^*)\overline{M}^0).$$

The proof of the nefness of $\overline{M}_f$ under the ampleness of M on $\mathcal{U}$, though lengthy, is similar to that in [YZ1, Thm. 4.9], so we omit it. $\square$

6.5 Néron-Tate height on a curve

When X is a projective curve over a finitely generated field, we present a theorem (Theorem 6.5.1) which interprets the intersection numbers in terms of the Néron–Tate height. It generalizes the result of Faltings [Fal1] and Hriljac [Hri] to finitely generated fields.

6.5.1 The arithmetic Hodge index theorem

Let k be either $\mathbb{Z}$ or a field. Take the uniform terminology in §1.6. Let F be a finitely generated field over k, and let $\pi : X \to \mathrm{Spec}\, F$ be a smooth, projective, and geometrically connected curve of genus $g > 0$. We first introduce the canonical height function

$$\hat{\mathfrak{h}} : \mathrm{Pic}^0(X_{\overline{F}}) \longrightarrow \widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q},\mathrm{nef}}.$$

Denote by $J = \underline{\mathrm{Pic}}^0_{X/F}$ the Jacobian variety of X. Denote by Θ the symmetric line bundle on J associated to the theta divisor. Namely, choose a point $x_0 \in X(\overline{F})$ and denote by $j : X_{\overline{F}} \hookrightarrow J_{\overline{F}}$ the embedding $x \mapsto [x - x_0]$. Denote by θ the image of the composition $X_{\overline{F}}^{g-1} \hookrightarrow J_{\overline{F}}^{g-1} \to J_{\overline{F}}$. The second map is the sum under the group law. Then θ is a divisor of $J_{\overline{F}}$. Denote by Θ the line bundle on $J_{\overline{F}}$ associated to $\theta + [-1]^*\theta$. The isomorphism class of Θ

does not depend on the choice of x_0, so it is Galois invariant and descends to a line bundle on J. See [Ser, §5.6] for more details about the construction.

By the symmetric and ample line bundle Θ on J, we have the canonical height

$$\hat{\mathfrak{h}}_\Theta : J(\overline{F}) \longrightarrow \widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q},\mathrm{nef}}.$$

By convention, we set

$$\hat{\mathfrak{h}} = \frac{1}{2}\hat{\mathfrak{h}}_\Theta.$$

The goal of this section is to prove the following extension of the arithmetic Hodge index theorem of Faltings [Fal1] and Hriljac [Hri] to finitely generated fields.

Theorem 6.5.1 (arithmetic Hodge index theorem)**.** *Let k be either $\mathbb{Z}$ or a field. Let F be a finitely generated field over k, and let $\pi : X \to \operatorname{Spec} F$ be a smooth, projective, and geometrically connected curve. Let M be a line bundle on X with $\deg M = 0$. Then there is an adelic line bundle $\overline{M}_0 \in \widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}}$ with underlying line bundle M such that*

$$\pi_*\langle \overline{M}_0, \overline{V}\rangle = 0, \quad \forall\, \overline{V} \in \widehat{\mathrm{Pic}}(X/k)_{\mathrm{vert},\mathbb{Q}}.$$

Moreover, for such an adelic line bundle,

$$\pi_*\langle \overline{M}_0, \overline{M}_0\rangle = -2\,\widehat{\mathfrak{h}}(M)$$

in $\widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q}}$.

In the theorem, $\pi_*\langle\cdot,\cdot\rangle$ denotes the Deligne pairing

$$\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}} \times \widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}} \longrightarrow \widehat{\mathrm{Pic}}(F/k)_{\mathrm{int},\mathbb{Q}}$$

introduced in Theorem 4.1.3. And $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{vert},\mathbb{Q}}$ denotes the kernel of the forgetful map $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$.

If we fix a polarization of F/k, and intersect both sides of the equality with the polarization, then we obtain an equality about the Moriwaki heights. This was proved by Moriwaki [Mor2, Thm. B].

Remark 6.5.2. We will see in [YZ2] that the extension $\overline{M}_0$ is unique up to translation by $\pi^*\widehat{\mathrm{Pic}}(F/k)_{\mathrm{int}}$.

6.5.2 The universal adelic line bundle

Now we construct the extension $\overline{M}_0$ in Theorem 6.5.1. It is written almost the same as the number field case. We include it here briefly. For basic geometric results on abelian varieties and Jacobian varieties, we refer to Mumford [Mum1] and Serre [Ser].

Denote by $p_1 : X \times J \to X$ and $p_2 : X \times J \to J$ the projections. Via p_1, we view $X \times J$ as an abelian scheme on X. Denote by $[m]_X : X \times J \to X \times J$ the multiplication by an integer m as abelian schemes on X, i.e., the map sending (x, y) to (x, my).

We claim that there is a universal line bundle $Q \in \mathrm{Pic}(X \times J)_{\mathbb{Q}}$ satisfying the following properties:

(1) For any $\alpha \in J(\overline{F})$, the $\mathbb{Q}$-line bundle $Q|_{X\times\alpha}$ on $X \times \alpha = X_{\overline{F}}$ is equal to α in $\mathrm{Pic}^0(X_{\overline{F}})_{\mathbb{Q}}$.

(2) For any integer m, $[m]_X^* Q = mQ$ in $\mathrm{Pic}(X \times J)_{\mathbb{Q}}$.

The line bundle Q is unique up to translation by $p_2^*\mathrm{Pic}^0(J)_{\mathbb{Q}}$.

In fact, let α_0 be a line bundle on X of degree $d > 0$. Denote the canonical morphism

$$i_0 : X \longrightarrow J, \quad x \longmapsto dx - \alpha_0.$$

Denote by

$$(i_0, \mathrm{id}) : X \times J \longrightarrow J \times J$$

the natural morphism. Set

$$Q = \frac{1}{d}(i_0, \mathrm{id})^* P,$$

where P is the Poincaré line bundle on $J \times J$.

If there is a line bundle on X of degree 1, we can choose Q to be an integral line bundle on $X \times J$. If $X(F)$ is non-empty, take $x_0 \in X(F)$ and use it to define $i_0 : X \to J$. Then Q is an integral line bundle on $X \times J$ such that $Q_{x_0\times J} = 0$ and that for any $\alpha \in J(\overline{F})$, the line bundle $Q|_{X\times\alpha}$ on $X \times \alpha = X_{\overline{F}}$ is equal to α in $\mathrm{Pic}^0(X_{\overline{F}})$. These properties determine Q uniquely.

With the universal line bundle $Q \in \mathrm{Pic}(X \times J)_{\mathbb{Q}}$, by Theorem 6.1.3, there is a unique extension $\overline{Q} \in \widehat{\mathrm{Pic}}(X \times_F J/k)_{\mathrm{int},\mathbb{Q}}$ of Q such that $[2]_X^*\overline{Q} = 2\overline{Q}$.

Let α be the point of $J(F)$ represented by the line bundle $M \in \mathrm{Pic}^0(X)$. Set

$$\overline{M}_0 := \overline{Q}|_{X\times\alpha} \in \widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}}.$$

We need to prove that $\overline{M}_0$ satisfies the requirement of Theorem 6.5.1; i.e.,

$$\pi_*\langle \overline{M}_0, \overline{V}\rangle = 0, \quad \forall \overline{V} \in \widehat{\mathrm{Pic}}(X/k)_{\mathrm{vert},\mathbb{Q}}.$$

Consider the adelic line bundle

$$\overline{R} := p_{2,*}\langle \overline{Q}, p_1^*\overline{V}\rangle$$

in $\widehat{\mathrm{Pic}}(J/k)_{\mathrm{int},\mathbb{Q}}$. Note that $\overline{R}$ is universal in the sense that the pull-back of $\overline{R}$ via $\alpha : \mathrm{Spec}(F) \to J$ is exactly $\pi_*\langle \overline{M}_0, \overline{V}\rangle$. Thus it suffices to prove that the adelic line bundle $\overline{R} = 0$ in $\widehat{\mathrm{Pic}}(J/k)_{\mathrm{int},\mathbb{Q}}$.

This is a consequence of Theorem 6.1.2 by noting the following two properties:

(1) the underlying line bundle $R = 0$ in $\mathrm{Pic}(X \times J)_{\mathbb{Q}}$, as a consequence of the underlying line bundle $V = 0$;

(2) $[2]_X^*\overline{R} = 2\overline{R}$ in $\widehat{\mathrm{Pic}}(J/k)_{\mathbb{Q}}$ by the dynamical property of $\overline{Q}$;

6.5.3 The height equality

It remains to prove

$$\pi_*\langle \overline{M}_0, \overline{M}_0\rangle = -2\,\widehat{\mathfrak{h}}(M).$$

Replacing the field F by a finite extension if necessary, we can assume that $X(F)$ is non-empty. We first express the left-hand side as a height function.

Take $x_0 \in X(F)$. Use x_0 to define the canonical embedding $i_0 : X \to J$, and identity X as a subvariety of J. As before, let Q be the restriction of the Poincaré line bundle P from $J \times J$ to $X \times J$.

Note that P is symmetric on $J \times J$. Thus $[2]^*P = 4P$ and we can extend it to $[2]^*\overline{P} = 4\overline{P}$ for some $\overline{P} \in \widehat{\mathrm{Pic}}(J \times_F J/k)_{\mathrm{int}}$ by Theorem 6.1.3. We claim that $\overline{Q} = \overline{P}|_{X\times J}$ in $\widehat{\mathrm{Pic}}(X \times_F J/k)_{\mathbb{Q}}$.

In fact, note that $[2] : J \times J \to J \times J$ is multiplication by 2 on both components, while $[2]_X : X \times J \to X \times J$ is only the multiplication by 2 on the second component. Denote by $[2]_2 : J \times J \to J \times J$ the multiplication by 2 on the second component. By Theorem 6.1.2(2), $[2]^*\overline{P} = 4\overline{P}$ implies

$[2]_2^*\overline{P} = 2\overline{P}$. This argument was used in the proof of Theorem 6.1.3. This implies $\overline{Q} = \overline{P}|_{X\times J}$ by the uniqueness of $\overline{Q}$ in Theorem 6.1.2(1). All these equalities are viewed as isomorphism classes of adelic $\mathbb{Q}$-line bundles.

Lemma 6.5.3. *For any $\alpha, \beta \in J(F)$, we have*

$$\pi_*\langle \overline{P}_\alpha, \overline{P}_\beta\rangle = \mathfrak{h}_{\overline{P}}(\alpha, \beta).$$

Here $\overline{P}_\alpha = \overline{P}|_{X\times\alpha}$ and $\overline{P}_\beta = \overline{P}|_{X\times\beta}$ are viewed as adelic line bundles on X.

Proof. Note both sides are bilinear in (α, β). We can assume that α represents the divisor $x - x_0$ on X. Then $\alpha = j(x)$. Here we assume $x \in X(F)$ by replacing F by a finite extension if necessary. Then we have

$$\pi_*\langle \overline{P}_\alpha, \overline{P}_\beta\rangle = \pi_*\langle \hat{x} - \hat{x}_0, \overline{P}_\beta\rangle = \pi_*\langle \hat{x}, \overline{P}_\beta\rangle - \pi_*\langle \hat{x}_0, \overline{P}_\beta\rangle.$$

Here $\hat{x}$ and $\hat{x}_0$ are any extensions of x and x_0 in $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}}$. Note that $\overline{P}_\beta$ has zero intersection with any vertical classes. The above becomes

$$\pi_*(\overline{P}|_{x\times\beta}) - \pi_*(\overline{P}|_{x_0\times\beta}) = \pi_*(\overline{P}|_{x\times\beta}) = \mathfrak{h}_{\overline{P}}(\alpha, \beta).$$

□

Now we are ready to prove

$$\pi_*\langle \overline{M}_0, \overline{M}_0\rangle = -2\,\widehat{\mathfrak{h}}(M).$$

By the lemma, it suffices to prove

$$\mathfrak{h}_{\overline{P}}(\alpha, \alpha) = -\mathfrak{h}_{\overline{\Theta}}(\alpha), \quad \forall\ \alpha \in J(F).$$

It is well known that the Poincaré bundle on $J \times J$ has the expression

$$P = p_1^*\theta + p_2^*\theta - m^*\theta.$$

Here $m, p_1, p_2 : J \times J \to J$ denotes the addition law and the projections. It induces

$$2P = p_1^*\Theta + p_2^*\Theta - m^*\Theta.$$

We use Θ because it is also symmetric. It follows that

$$2\overline{P} = p_1^*\overline{\Theta} + p_2^*\overline{\Theta} - m^*\overline{\Theta}.$$

Computing heights using the identity, we have

$$2\mathfrak{h}_{\overline{P}}(\alpha, \alpha) = \mathfrak{h}_{\overline{\Theta}}(\alpha) + \mathfrak{h}_{\overline{\Theta}}(\alpha) - \mathfrak{h}_{\overline{\Theta}}(2\alpha) = -2\mathfrak{h}_{\overline{\Theta}}(\alpha).$$

This finishes the proof of Theorem 6.5.1.

6.5.4 High-dimensional bases

The above setting treats $X \to \operatorname{Spec} F$ for a finitely generated field F over k. We can replace $\operatorname{Spec} F$ by an essentially quasi-projective scheme S over k, but due to flatness problem, we have to restrict to vector-valued height of sections of the relative Jacobian scheme.

Let k be either $\mathbb{Z}$ or a field. Let S be a normal integral scheme, flat and essentially quasi-projective over k. Let $\pi : X \to S$ be a projective and smooth morphism, whose fibers are smooth and geometrically connected curves. Denote by $\operatorname{Pic}^0(X/S)$ the group of line bundles on X with degree 0 on the fibers of $X \to S$. We first introduce a canonical height function

$$\hat{\mathfrak{h}} : \operatorname{Pic}^0(X/S) \longrightarrow \widehat{\operatorname{Pic}}(S/k)_{\mathbb{Q},\mathrm{nef}}.$$

This is obtained as a slight generalization of the case $S = \operatorname{Spec} F$, and is thus compatible with the latter.

Denote by $J = \operatorname{Pic}^0_{X/S}$ the Jacobian scheme of X over S. For basics of Jacobian schemes, we refer to [MFK, Chap. 6]. By [MFK, §6.1, Prop. 6.9], there is a canonical principal polarization $\lambda_1 : J \to J^\vee$ over S. By the construction of [MFK, §6.2, Prop. 6.10], there is a symmetric line bundle Θ on J such that the polarization $\lambda_\Theta : J \to J^\vee$ corresponding to Θ is exactly twice of $\lambda_1 : J \to J^\vee$. To relate it to our previous case of fields, Θ recovers that on each fiber of $J \to S$. We can uniquely determine Θ by the rigidification $e^*\Theta \simeq \mathcal{O}_S$ for the identity section $e : S \to J$.

Finally, by the symmetric and relatively ample line bundle Θ on J, we have a unique extension $\overline{\Theta}$ of Θ in $\widehat{\operatorname{Pic}}(J/k)_{\mathbb{Q},\mathrm{nef}}$ such that $[2]^*\overline{\Theta} = 4\overline{\Theta}$. Then we have the vector-valued height function

$$\hat{\mathfrak{h}}_{\overline{\Theta}} : J(S) \longrightarrow \widehat{\operatorname{Pic}}(S/k)_{\mathbb{Q},\mathrm{nef}}.$$

By convention, we set

$$\hat{\mathfrak{h}} = \frac{1}{2}\hat{\mathfrak{h}}_{\overline{\Theta}}.$$

By the canonical map $\operatorname{Pic}^0(X/S) \to J(S)$, we obtain

$$\hat{\mathfrak{h}} : \operatorname{Pic}^0(X/S) \longrightarrow \widehat{\operatorname{Pic}}(S/k)_{\mathbb{Q},\mathrm{nef}}.$$

As in the classical case, this height function is also quadratic.

As before, we have a universal $\mathbb{Q}$-line bundle $Q \in \operatorname{Pic}(X \times_S J)_{\mathbb{Q}}$ satisfying the following properties:

(1) For any base change $X' \to S'$ of $X \to S$, and for any $\alpha \in \mathrm{Pic}(X')$ of degree 0 on fibers of $X' \to S'$, the pull-back of the $\mathbb{Q}$-line bundle Q via $(\mathrm{id}, \alpha) : X \times_S S' \to X \times_S J$ is equal to α in $\mathrm{Pic}(X')_{\mathbb{Q}}$.

(2) For any integer m, $[m]_X^* Q = mQ$ in $\mathrm{Pic}(X \times_S J)_{\mathbb{Q}}$.

The line bundle Q is unique up to translation by $p_2^*\mathrm{Pic}(J)_{\mathbb{Q}}$. There is a unique extension $\overline{Q} \in \widehat{\mathrm{Pic}}(X \times_S J/k)_{\mathrm{int},\mathbb{Q}}$ of Q such that $[2]_X^* \overline{Q} = 2\overline{Q}$.

The following is a variant of Theorem 6.5.1 over high-dimensional bases.

Theorem 6.5.4. *Let k be either $\mathbb{Z}$ or a field. Let S be a normal integral scheme, flat and essentially quasi-projective over k. Let $\pi : X \to S$ be a projective and smooth morphism of relative dimension 1 with geometrically connected fibers. Let M be a line bundle on X with degree 0 on fibers of $X \to S$. Then there is an adelic line bundle $\overline{M}_0 \in \widehat{\mathrm{Pic}}(X/k)_{\mathrm{int},\mathbb{Q}}$ with underlying line bundle M such that*

$$\pi_*\langle \overline{M}_0, \overline{V} \rangle = 0, \quad \forall \overline{V} \in \widehat{\mathrm{Pic}}(X/k)_{\mathrm{vert},\mathbb{Q}}.$$

Moreover, for such an adelic line bundle,

$$\pi_*\langle \overline{M}_0, \overline{M}_0 \rangle = -2\,\widehat{\mathfrak{h}}(M)$$

in $\widehat{\mathrm{Pic}}(S/k)_{\mathbb{Q}}$.

Proof. The existence of $\overline{M}_0$ can be obtained by generalizing the construction by the universal line bundle $\overline{Q}$ in Theorem 6.5.1 to the general base S. Namely, let $i : S \to J$ be the morphism corresponding to $[M] \in J(S)$, i.e., by the morphism $(\mathrm{id}, i \circ \pi) : X \to X \times_S J$, the pull-back $(\mathrm{id}, i \circ \pi)^* Q = M + \pi^* N$ for some $N \in \mathrm{Pic}(S)_{\mathbb{Q}}$. Then we set

$$\overline{M}_0 = (\mathrm{id}, i \circ \pi)^* \overline{Q} - \pi^* \overline{N}$$

for any $\overline{N} \in \mathrm{Pic}(S/k)_{\mathrm{int},\mathbb{Q}}$ extending N. For the height identities are consequences of Theorem 6.5.1, as $\widehat{\mathrm{Pic}}(S/k)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(F/k)_{\mathbb{Q}}$ is injective by Corollary 3.4.2. Here $F = k(S)$ is the function field. □

The following universal Hodge index theorem is essentially equivalent to Theorem 6.5.4.

Corollary 6.5.5. *By the second projection $p_2 : X \times_S J \to J$,*

$$p_{2*}\langle \overline{Q}, \overline{Q} \rangle = -\overline{\Theta}$$

in $\widehat{\mathrm{Pic}}(J/k)_{\mathbb{Q}}$.

Proof. Let S' be a normal integral scheme, flat and essentially quasi-projective over k, endowed with a k-morphism $S' \to S$. Denote by $\pi' : X' \to S'$ the base change of $\pi : X \to S$ by $S' \to S$. Let $i : S' \to J$ be a morphism over S. Consider the morphism $(\mathrm{id}, i) : X \times_S S' \to X \times_S J$. Apply Theorem 6.5.4 to $\pi' : X' \to S'$ and the line bundle $M = (\mathrm{id}, i)^* Q$ in $\mathrm{Pic}(X')_{\mathbb{Q}}$. We obtain

$$\pi'_* \langle (\mathrm{id}, i)^* \overline{Q}, (\mathrm{id}, i)^* \overline{Q} \rangle = -i^* \overline{\Theta}.$$

Set $S' = J$ and set $i : S' \to J$ to be the identity morphism. □

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