Arithmetic bigness and a uniform Bogomolov-type result

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Contents

1 Introduction

In this paper, we study some positivity properties in Arakelov geometry, and apply them to obtain a uniform Diophantine property on curves. More precisely, the paper achieves the following three goals.

(1) Extend Zhang's theory of admissible metrics from projective curves to families of projective curves (cf. §2).

(2) Prove the bigness of the admissible canonical bundle of the universal family over the moduli space of curves introduced in (1) (cf. §3). This is the main theorem of this paper.

(3) Prove a uniform Bogomolov-type theorem for curves over global fields as a consequence of the bigness result in (2) (cf. §4).

Our whole treatment is based on the recent theory of adelic line bundles of Yuan–Zhang [YZ2], a limit version of the intersection theory in algebraic geometry and the arithmetic intersection theory of Arakelov [Ara] and Gillet–Soulé [GS]. Our proof of the bigness in (2) relies on the results of Zhang [Zha1, Zha3], Cinkir [Cin1] and de Jong [dJo2] on lower bounds of the self-intersection numbers of the admissible canonical bundles of curves over global fields.

Our uniform Bogomolov-type theorem generalizes and strengthens the uniform Bogomolov-type theorem of Dimitrov–Gao–Habegger [DGH1] and Kühne [Kuh]. This gives an alternative proof of their result on the uniform Mordell–Lang problem proposed by Mazur [Maz]. Our approach does not use the o-minimality theory, and works over function fields of arbitrary characteristics, but it relies on more algebraic geometry and Arakelov geometry.

1.1 Uniform Bogomolov-type result

Let C be a smooth projective curve of genus $g > 1$ over $\overline{\mathbb{Q}}$. In our convention, all curves are assumed to be geometrically connected over the base field. The Bogomolov conjecture proved by Ullmo [Ull] asserts that for any divisor α on C of degree 1, there is a constant $c > 0$ such that

$$\#\{x \in C(\overline{\mathbb{Q}}) : \widehat{h}(x - \alpha) \le c\} < \infty.$$

Here $\widehat{h} : J(\overline{\mathbb{Q}}) \to \mathbb{R}$ denotes the Néron–Tate height function on the Jacobian variety J of C. The proof of the loc. cit. is based on the celebrated equidistribution theorem of Szpiro–Ullmo–Zhang [SUZ].

Recently, a uniform version of this theorem was proved by Dimitrov–Gao–Habegger [DGH1] and Kühne [Kuh]. In fact, the new gap principle in [Gao2, Thm. 4.1], as a combination of [DGH1, Prop. 7.1] and [Kuh, Thm. 3], asserts that there are constants $c_1, c_2 > 0$ depending only on $g > 1$ such that for any smooth projective curve C over $\overline{\mathbb{Q}}$ of genus g, and for any $y \in C(\overline{\mathbb{Q}})$,

$$\#\{x \in C(\overline{\mathbb{Q}}) : \widehat{h}(x - y) \leq c_1 \max\{h_{\mathrm{Fal}}(C), 1\}\} \leq c_2.$$

Here $h_{\mathrm{Fal}}(C) = h_{\mathrm{Fal}}(J)$ denotes the stable Faltings height of the Jacobian variety J.

The new gap principle has a significant consequence to the uniform Mordell–Lang problem proposed by Mazur [Maz, pp.234]. Recall that the Mordell conjecture was proved by Faltings [Fal1], and a different proof was given by Vojta [Voj]. Vojta's proof was simplified and extended by Faltings [Fal2] to prove the Mordell–Lang conjecture for subvarieties of abelian varieties, and was further simplified by Bombieri [Bom] in the original case of curves. The proofs of [Voj, Fal2, Bom] actually gave an upper bound of the number of points of large heights, which was further refined by de Diego [dDi] and Rémond [Rem]. Combining the upper bound with the new gap principle, we obtain the uniform bound on the number of rational points in [Kuh, Thm. 4], which asserts that there is a constant $c > 0$ depending only on $g > 1$ such that for any smooth projective curve C of genus g over an algebraically closed field F of characteristic 0, for any $y \in C(F)$, and for any subgroup $\Gamma \subset J(F)$ of finite rank,

$$\#\{x \in C(F) : x - y \in \Gamma\} \leq c^{1 + \mathrm{rank}\,\Gamma}.$$

Our first result is the following uniform version of the Bogomolov conjecture, which strengthens and generalizes the new gap principle of [DGH1, Kuh].

Theorem 1.1 (Theorem 4.7). *Let $g > 1$ be an integer. Then there are constants $c_1, c_2 > 0$ depending only on g satisfying the following properties. Let K be either a number field or a function field of one variable over a field k. Then for any geometrically integral, smooth and projective curve C of genus g over K, and for any line bundle $\alpha \in \mathrm{Pic}(C_{\overline{K}})$ of degree 1, with the*

extra assumption that $(C_{\overline{K}}, \alpha)$ is non-isotrivial over $\overline{k}$ in the case that K is a function field of one variable over a field k, one has

$$\# \left\{ x \in C(\overline{K}) : \widehat{h}(x - \alpha) \le c_1 \big(\max\{h_{\mathrm{Fal}}(C), 1\} + \widehat{h}((2g-2)\alpha - \omega_{C/K}) \big) \right\} \le c_2.$$

Here a *number field* means a finite extension of $\mathbb{Q}$, and a *function field of one variable over k* means a finitely generated field extension of k of transcendence degree 1. In the function field case, we say that $(C_{\overline{K}}, \alpha)$ is *non-isotrivial* over $\overline{k}$ if it is not isomorphic to the base change from $\overline{k}$ to $\overline{K}$ of any pair (C_0, α_0) consisting of a smooth projective curve C_0 over $\overline{k}$ and a line bundle $\alpha_0 \in \mathrm{Pic}(C_0)$ of degree 1. We refer to §4.3.1 for the normalization of the heights involved in the theorem.

Our theorem is stronger and more general than the new gap principle of [DGH1, Kuh] by the following aspects:

(1) it has an extra non-negative term $\widehat{h}((2g-2)\alpha - \omega_{C/K})$ in the formula;

(2) it allows α to be in $\mathrm{Pic}^1(C_{\overline{K}})$ instead of just in $C(\overline{K})$;

(3) it is valid for global fields of all characteristics instead of just number fields, and the constants c_1 and c_2 are uniform for all these fields.

Our proof is very different from that of [DGH1, Kuh], and the key ingredient is a bigness result of adelic line bundles on the universal curve. We will come back to that in the next subsection.

In the function field case, Looper–Silverman–Wilms [LSW, Thm. 1.2] proves a similar bound independently. By Theorem 1.4 below, their bound can be converted to our bound without the extra term $\widehat{h}((2g-2)\alpha - \omega_{C/K})$. We refer to the explanation after Theorem 1.4 for more details on their result.

It is also worth noting that an explicit uniform bound on the Mordell–Lang problem over function fields of positive characteristic $p > 0$ was previously obtained by Buium–Voloch [BV] using jet schemes.

1.2 Potential bigness

Our exposition is based on the theory of adelic line bundles on quasi-projective varieties of Yuan–Zhang [YZ2], which generalizes the projective case of Zhang [Zha2]. In [YZ2], the base ring k is set to be either $\mathbb{Z}$ or an arbitrary field, and the main definitions give a notion of adelic line bundles on X/k for any quasi-projective integral scheme X over k. Roughly speaking, an adelic line

bundle on X is a reasonable limit of a sequence of hermitian line bundles (or usual line bundles if k is a field) on projective models of X over k. As a convention, we only require hermitian line bundles to have continuous metrics (instead smooth metrics). The adelic line bundles have nice functorial properties, analytification properties, intersection theory, positivity properties, and volume theory.

Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$, i.e., a smooth projective morphism whose fibers are curves of genus g. Denote by $J \to S$ the relative Jacobian scheme over S.

Our Néron–Tate height is based on a canonically defined line bundle Θ on J satisfying the following properties:

(1) Θ is symmetric and rigidified along the identity section of $J \to S$;

(2) the restriction of Θ to geometric fibers of $J \to S$ are algebraically equivalent to twice of theta divisors.

As a consequence, Θ is relatively ample. The construction of Θ is known to experts and reviewed in Definition 2.4.

By [YZ2, Thm. 6.1.1], there is a nef adelic line bundle $\overline{\Theta}$ on J/k extending Θ such that $[2]^*\overline{\Theta} = 4\overline{\Theta}$ in $\widehat{\mathrm{Pic}}(J/k)$. Here $\widehat{\mathrm{Pic}}(J/k)$ is the group of adelic line bundles on J/k in the sense of [YZ2], and an adelic line bundle is *nef* if it is the limit of nef hermitian (or usual) line bundles on projective models of J over k in a suitable sense. This generalizes the construction of Zhang [Zha2] from the case $S = \mathrm{Spec}\,\mathbb{Z}$ to general base S.

Let α be a line bundle on X of degree 1 on fibers of $X \to S$. Then we have an immersion
$$i_\alpha : X \longrightarrow J, \quad x \longmapsto x - \alpha.$$
We obtain an adelic line bundle $i_\alpha^*\overline{\Theta}$ on X by pull-back.

On the other hand, consider the morphism
$$\tau : J \times_S X \longrightarrow J \times_S J, \quad (y, x) \longmapsto (y, y + (2g-2)x - \omega_{X/S}).$$

This morphism agrees with the J-morphism
$$i_{\omega-Q} : X_J \longrightarrow J_J, \quad x \longmapsto (2g-2)x - (\omega_{X_J/J} - Q).$$

Here we denote $X_J = J \times_S X$ and $J_J = J \times_S J$, viewed as J-schemes via the first projections $q_1 : X_J \to J$ and $p_1 : J_J \to J$. Here J_J is canonically

isomorphic to the Jacobian scheme of X_J over J, and Q is a universal line bundle on $J \times_S X$ in a suitable sense. Denote by $\overline{\Theta}_J = p_2^* \overline{\Theta}$ the adelic line bundle on J_J by the pull-back via the second projection $p_2 : J \times_S J \to J$. Then we obtain an adelic line bundle $\tau^*(\overline{\Theta}_J)$ on X_J by pull-back.

Finally, we have the following bigness result.

Theorem 1.2 (Theorem 4.5)**.** *Let k be either $\mathbb{Z}$ or a field. Assume that the family $\pi : X \to S$ has maximal variation. Then the following hold:*

(1) The adelic line bundle $\pi_* \langle i_\alpha^* \overline{\Theta}, i_\alpha^* \overline{\Theta} \rangle$ is nef and big on S.

(2) The adelic line bundle $q_{1*} \langle \tau^*(\overline{\Theta}_J), \tau^*(\overline{\Theta}_J) \rangle$ is nef and big on J.

Here we say that the family $\pi : X \to S$ has *maximal variation* if the moduli morphism $S \to M_{g,k}$ is generically finite, where $M_{g,k}$ denotes the coarse moduli scheme of smooth curves of genus g over k.

Recall that a nef adelic line bundle is *big* if its top self-intersection number (as defined in [YZ2, Prop. 4.1.1]) is strictly positive. The notation

$$\pi_* \langle \cdot, \cdot \rangle : \widehat{\mathrm{Pic}}(X) \times \widehat{\mathrm{Pic}}(X) \longrightarrow \widehat{\mathrm{Pic}}(S)$$

denotes the Deligne pairing introduced in [YZ2, Thm. 4.1.3].

Let us explain how the bigness result implies our uniform Bogomolov-type theorem in Theorem 1.1. Consider the line bundle $\overline{L} = \tau^*(\overline{\Theta}_J)$ on X_J. Over the m-fold fiber product

$$(X_J)_{/J}^m = (X_J) \times_J \cdots \times_J (X_J),$$

we have a nef adelic line bundle

$$\overline{L}_m = \overline{L}^{\boxtimes m} = p_1^* \overline{L} + \cdots + p_m^* \overline{L}.$$

Denote $d = \dim J$. If $m \geq d$, expand the top self-intersection number

$$\overline{L}_m^{d+m} = (p_1^* \overline{L} + p_2^* \overline{L} + \cdots + p_m^* \overline{L})^{d+m}.$$

The expansion includes the term

$$(p_1^* \overline{L})^2 \cdots (p_d^* \overline{L})^2 \cdot (p_{d+1}^* \overline{L}) \cdots (p_m^* \overline{L}) = a^{m-d}(q_{1*} \langle \tau^*(\overline{\Theta}_J), \tau^*(\overline{\Theta}_J) \rangle)^d > 0.$$

Here a denotes the degree of $\overline{L}$ on the generic fiber of $q_1 : X_J \to J$, and the top self-intersection number $(q_{1*} \langle \tau^*(\overline{\Theta}_J), \tau^*(\overline{\Theta}_J) \rangle)^d$ on J is strictly positive by Theorem 1.2(2).

Therefore, $\overline{L}_m$ is big on $(X_J)^m_{/J}$ for all integers $m \geq d$. We say that $\overline{L}$ is *potentially big* on $X_J/J/k$ in this situation. We remark that our notion of *potential bigness* has some similarity with the notion of *correlation* of Caporaso–Harris–Mazur [CHM].

With the bigness result, our next key step to prove Theorem 1.1 is to apply the height inequality (and some variants) in [YZ2, Thm. 5.3.5]. To illustrate the idea, we only consider the case that K is a number field (and thus $k = \mathbb{Z}$). Take S to be a fine moduli space of smooth curves of genus g over $\mathbb{Z}[1/N]$ with a suitable full level-N structure, and take $X \to S$ to be the universal curve. Fix an integer $m \geq d$. As a consequence of the height inequality, for any adelic line bundle $\overline{M}$ on $J/\mathbb{Z}$, there is a non-empty Zariski open subscheme U of $(X_J)^m_{/J}$ such that

$$h_{\overline{L}_m}(x) \geq \epsilon\, h_{\overline{M}}(\pi_m(x)), \quad \forall x \in U(\overline{\mathbb{Q}}).$$

Here $\pi_m : (X_J)^m_{/J} \to J$ denotes the structure morphism.

For convenience, we sketch an idea to prove the height inequality taking advantage of the condition that $\overline{L}_m$ is nef and big. We can first reduce to the case that $\overline{M}$ is nef by [YZ2, Lem. 5.1.6(1)]. Then the key is [YZ2, Thm. 5.2.2(2)], the adelic version of the bigness theorems of Siu [Siu] and Yuan [Yua]. It implies that for any nef adelic line bundle $\overline{M}$ on J, viewed as an adelic line bundle on $(X_J)^m_{/J}$ via pull-back, we have

$$\widehat{\mathrm{vol}}(\overline{L}_m - \epsilon\overline{M}) \geq \overline{L}^{d+m}_m - (d+m)\overline{L}^{d+m-1}_m \cdot \epsilon\overline{M} > 0$$

for some rational number $\epsilon > 0$. Then some multiple of $\overline{L}_m - \epsilon\overline{M}$ has a nonzero effective section. Away from the zero locus Z of this effective section, we have

$$h_{\overline{L}_m - \epsilon\overline{M}}(x) \geq 0, \quad \forall x \in ((X_J)^m_{/J} \setminus Z)(\overline{\mathbb{Q}}).$$

Once we have the height inequality, choose the adelic line bundle $\overline{M}$ on J by

$$\overline{M} = \overline{\Theta} + \overline{\lambda}_S + \mathcal{O}(c).$$

Here $\overline{\Theta}$ is as above, $\overline{\lambda}_S$ is the (hermitian) Hodge (line) bundle on S associated to the abelian scheme $J \to S$, and $\mathcal{O}(c)$ with $c > 0$ is a hermitian line bundle on $\mathrm{Spec}\,\mathbb{Z}$ of arithmetic degree c. Here $\overline{\lambda}_S$ and $\mathcal{O}(c)$ are viewed as adelic line bundles on $J/\mathbb{Z}$ by pull-back. For any point $y \in J(\overline{K})$ with image $s \in S(\overline{K})$, we have

$$h_{\overline{M}}(y) = 2\,\widehat{h}(y) + h_{\mathrm{Fal}}(X_s) + c.$$

This essentially dominates the term

$$\max\{h_{\mathrm{Fal}}(C), 1\} + \widehat{h}((2g - 2)\alpha - \omega_{C/K})$$

in Theorem 1.1.

Finally, with an induction and some extra arguments, this eventually implies Theorem 1.1. This gives the idea of the proof of the theorem in the number field case.

The function field case is proved similarly by applying the above argument to moduli spaces over k (instead of over $\mathbb{Z}$) and paying special attention to definitions of height functions and non-isotriviality. The uniformity of (c_1, c_2) on K is obtained by applying the above arguments for all K to a single moduli space over $\mathbb{Z}$.

There are two other consequences of the bigness of $\overline{L}_m$ for $m \geq \dim J$. The first consequence is that in the arithmetic case $k = \mathbb{Z}$, the morphism $(X_J)^m_{/J} \times_{\mathbb{Z}} \mathbb{Q} \to (J_J)^m_{/J} \times_{\mathbb{Z}} \mathbb{Q}$ for all integers $m \geq \dim J$ satisfies the relative Bogomolov conjecture proposed by [DGH2, Conj.1.2]. This can be easily seen from the above height inequality. The second consequence of the bigness is that if k is a field of any characteristic, the morphism $(X_J)^m_{/J} \to (J_J)^m_{/J}$ is non-degenerate for all integers $m \geq \dim J$, in terms of the definition of non-degeneracy in [YZ2, §6.2.2].

The morphism $X^m_{/S} \to J^m_{/S}$ also satisfies similar properties for $m \geq \dim S$. Moreover, by a similar method, we deduce that if k is a field, and $\pi : X \to S$ has maximal variation, then the Faltings–Zhang morphism

$$i_{\mathrm{FZ},m} : X^{m+1}_{/S} \longrightarrow J^m_{/S}, \quad (x_0, \cdots, x_m) \longmapsto (x_1 - x_0, \cdots, x_m - x_0)$$

is non-degenerate for all $m \geq \dim S + 1$. These non-degeneracy results generalize [Gao1, Thm. 1.2(i), Thm 1.2'] to base fields k of all characteristics. We refer to §4.6 for more details.

The proof of the non-degeneracy in [Gao1] is based on the o-minimality theory and works only in characteristic 0, while our result is valid for families of curves in all characteristics. The proof of the new gap principle in [DGH1, Kuh] depends crucially on the non-degeneracy of the Faltings–Zhang map proved in [Gao1], while our proof of Theorem 1.1 does not need the non-degeneracy, but deduces the non-degeneracy as a consequence of the potential bigness.

Another crucial tool of the proof of [Kuh] is an equidistribution theorem, while our crucial tool is the arithmetic bigness. For a more precise comparison, recall that the original Bogomolov conjecture was proved by Ullmo [Ull]

in terms of the equidistribution theorem of [SUZ], and a second proof in terms of bounding the self-intersection number of the admissible canonical bundle was obtained along the line of Zhang [Zha1, Zha3], Cinkir [Cin1] and de Jong [dJo2]. Then the treatment of [Kuh] is a family version of that of [SUZ, Ull], and our treatment is a family version of that of [Zha1, Zha3, Cin1, dJo2].

1.3 Admissible canonical bundle

Let K be a number field or a function field of one variable. Let C be a smooth projective curve of genus $g > 1$ over K. Let $\omega_{C/K}$ be the relative dualizing sheaf, and $\Delta \subset C^2$ be the diagonal divisor. By the construction of Zhang [Zha1], there are a canonical adelic line bundle $\overline{\omega}_{C/K,a}$ on C extending $\omega_{C/K}$, and a canonical adelic line bundle $\overline{\mathcal{O}}(\Delta)_a$ on C^2 extending $\mathcal{O}(\Delta)$. The metrics of $\overline{\omega}_{C/K,a}$ and $\overline{\mathcal{O}}(\Delta)_a$ at an archimedean place are the Arakelov metrics introduced by Arakelov [Ara], and the metrics of $\overline{\omega}_{C/K,a}$ and $\overline{\mathcal{O}}(\Delta)_a$ at a non-archimedean place are the admissible metrics introduced by Zhang [Zha1].

The adelic line bundles $\overline{\omega}_{C/K,a}$ and $\overline{\mathcal{O}}(\Delta)_a$ satisfy many nice properties. For example, the canonical isomorphism $\mathcal{O}(\Delta)|_\Delta \to \omega_{C/K}^\vee$ is an isometry. The curvatures of $\overline{\mathcal{O}}(\Delta)_a$ on closed fibers of the projections $p_i : C^2 \to C$ are proportional to the curvature of $\overline{\omega}_{C/K,a}$ on C at all places.

Note that [Zha1] was written before [Zha2], but the treatment works without much difficulty in the terminology of [Zha2] as we review in §A. One goal of this paper is to introduce a family version of the admissible canonical bundle in the terminology of [YZ2].

Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective normal scheme over k. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$. Let $\omega_{X/S}$ be the relative dualizing sheaf, and $\Delta \subset X_{/S}^2$ be the diagonal divisor. In Theorem 2.3, we will introduce a canonical adelic line bundle $\overline{\omega}_{X/S,a}$ on X extending $\omega_{X/S}$, and a canonical adelic line bundle $\overline{\mathcal{O}}(\Delta)_a$ on $X_{/S}^2$ extending $\mathcal{O}(\Delta)$. The extensions are uniquely determined by the properties that at any point $s \in S$, whose residue field is a number field if $k = \mathbb{Z}$ and is a function field of one variable over k if k is a field, the pull-backs $\overline{\omega}_{X/S,a}|_{X_s}$ and $\overline{\mathcal{O}}(\Delta)_a|_{X_s^2}$ are canonically isomorphic to the admissible adelic line bundles $\overline{\omega}_{X_s/s,a}$ on X_s and $\overline{\mathcal{O}}(\Delta_s)_a$ on X_s^2 of [Zha1].

While the construction of Zhang [Zha1] uses graph theory, our construction uses the canonical metrics of line bundles on the relative Jacobian

scheme. In fact, we can define $\overline{\omega}_{X/S,a}$ by the formula

$$\overline{\omega}_{X/S,a} = \frac{1}{4g(g-1)} i_\omega^* \overline{\Theta} + \frac{1}{64g^2(g-1)^4} \pi^* \pi_* \langle i_\omega^* \overline{\Theta}, i_\omega^* \overline{\Theta} \rangle$$

in $\widehat{\mathrm{Pic}}(X/k)_{\mathbb{Q}}$. Here $\overline{\Theta}$ is the adelic line bundle on the relative Jacobian J as above, and i_ω is the canonical morphism

$$i_\omega : X \longrightarrow J, \quad x \longmapsto (2g-2)x - \omega_{X/S}.$$

By the formula, we immediately see that $\overline{\omega}_{X/S,a}$ is nef on X. We refer to Theorem 2.10 for more formulae of this type.

1.4   Bigness of admissible canonical bundle

The following is our main theorem of this paper.

Theorem 1.3 (Theorem 3.1)**.** *Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$ with maximal variation. Then the adelic line bundle $\overline{\omega}_{X/S,a}$ is nef and big on X.*

Once $\overline{\omega}_{X/S,a}$ is nef and big on X, the Deligne pairing $\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$ is nef and big on S. Then Theorem 1.2(2) is obtained by the identity

$$q_{1*} \langle \tau^*(\overline{\Theta}_J), \tau^*(\overline{\Theta}_J) \rangle = 16(g-1)^3 \overline{\Theta} + 16g(g-1)^3 \pi_J^* \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$$

in $\widehat{\mathrm{Pic}}(J/k)$. A similar formula gives Theorem 1.2(1).

Now we sketch our proof of Theorem 1.3. The process is to align the relevant results of [Zha1, Zha3, Cin1, dJo2] into a family. With some effort, we can reduce it to the statement that $\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$ is nef and big on S.

Let us first sketch the geometric case that k is a field. Replacing S by a finite extension if necessary, we can assume that $\pi : X \to S$ has a stable compactification $\overline{\pi} : \overline{X} \to \overline{S}$, i.e. a projective variety $\overline{S}$ over k with an open immersion $S \to \overline{S}$, a stable relative curve $\overline{\pi} : \overline{X} \to \overline{S}$ of genus g, and an open immersion $X \to \overline{X}$ compatible with the previous morphisms. Our proof takes the following steps.

Step 1. There is an effective adelic divisor $\overline{E}$ on S such that

$$\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle = \overline{\pi}_* \langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}} \rangle - \mathcal{O}(\overline{E})$$

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in $\widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$. Here $\overline{\pi}_* \langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}} \rangle$ is viewed as an element of $\widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$ by the natural map $\mathrm{Pic}(\overline{S})_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$. The divisor $\overline{E}$ is in fact defined by the identity. Then it has underlying divisor $0 \in \mathrm{Div}(S)$, and thus is totally determined by its Green's function $g_{\overline{E}}$ on the Berkovich analytic space S^{an}. We have an "explicit" description of $g_{\overline{E}}$ in that its value at any discrete valuation v of $k(S)/k$ is given by the ϵ-invariant of the curve X_{H_v} over the valuation field H_v of v defined by Zhang [Zha1] in terms of graph theory. This determines $g_{\overline{E}}$ by continuity, and we say that $\overline{E}$ is the globalization of the ϵ-invariant.

Step 2. Noether's formula gives

$$\overline{\pi}_* \langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}} \rangle = 12 \lambda_{\overline{S}} - \mathcal{O}(\Delta_{\overline{S}})$$

in $\mathrm{Pic}(\overline{S})_{\mathbb{Q}}$. Here $\lambda_{\overline{S}} = \det \overline{\pi}_* \omega_{\overline{X}/\overline{S}}$ is the Poincaré bundle of $\overline{X}$ over $\overline{S}$, and $\Delta_{\overline{S}}$ is the divisor of $\overline{S}$ with support equal to $\overline{S} \setminus S$ measuring the singularity of $\overline{X}$ over $\overline{S}$. Both $\lambda_{\overline{S}}$ and $\Delta_{\overline{S}}$ are the well-known tautological divisors in the theory of moduli space of curves.

Step 3. The difference $(2g-2)\Delta_{\overline{S}} - \overline{E}$ is an effective adelic divisor in $\widehat{\mathrm{Div}}(S)$. As both $\Delta_{\overline{S}}$ and $\overline{E}$ have underlying divisor $0 \in \mathrm{Div}(S)$, it suffices to check $(2g-2)g_{\Delta_{\overline{S}}} \geq g_{\overline{E}}$ on the Berkovich analytic space S^{an}. By continuity, we only need to check it at any discrete valuation v of $k(S)/k$, or equivalently compare the ϵ-invariant of the curve X_{H_v} defined by Zhang [Zha1] and the classical δ-invariant of X_{H_v} counting the number of nodes of reduction. The comparison is done by graph theory.

Combining these steps, we have

$$\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle = 12 \lambda_{\overline{S}} - (2g-1)\mathcal{O}(\Delta_{\overline{S}}) + \mathrm{eff}.$$

Here "eff" means an effective adelic divisor, and later we will write "nef" for a nef adelic line bundle.

Step 4. There is an effective adelic divisor $\overline{\Phi}$ on S such that

$$\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle = \frac{2}{3g-1} \mathcal{O}(\overline{\Phi}) + \mathrm{nef}.$$

Similarly to $\overline{E}$, the arithmetic divisor $\overline{\Phi}$ has underlying divisor $0 \in \mathrm{Div}(S)$, and its Green's function $g_{\overline{\Phi}}$ at any discrete valuation v of $k(S)/k$ is given by the φ-invariant of the curve X_{H_v} defined by Zhang [Zha3] in terms of

graph theory. Then $\overline{\Phi}$ is the globalization of Zhang's φ-invariant. The above equality is a family version of de Jong [dJo2, Thm. 8.1].

Step 5. The difference $\overline{\Phi} - \frac{1}{39}\Delta_{\overline{S}}$ is an effective adelic divisor in $\widehat{\mathrm{Div}}(S)$. Similar to Step 3, it suffices to compare the values of the Green's function at all discrete valuations v of $k(S)/k$. Then it follows from the result of Cinkir [Cin1, Thm. 2.11] in graph theory.

As a consequence, we have

$$\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle = \frac{2}{39(3g-1)}\mathcal{O}(\Delta_{\overline{S}}) + \mathrm{nef} + \mathrm{eff}.$$

Step 6. Take a linear combination of the equalities respectively at the end of Step 3 and Step 5 to cancel the term $\mathcal{O}(\Delta_{\overline{S}})$. We obtain

$$\left(1 + \frac{39(3g-1)}{2}(2g-1)\right)\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle = 12\lambda_{\overline{S}} + \mathrm{nef} + \mathrm{eff}.$$

Then the bigness of $\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle$ follows from the classical result that the Hodge (line) bundle $\lambda_{\overline{S}}$ is nef and big on $\overline{S}$.

Now we prove the bigness in the arithmetic case $k = \mathbb{Z}$. The above proof is not valid in this case as Step 2 becomes rather subtle in the arithmetic case. However, we already know the bigness of the image of $\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle$ in $\widehat{\mathrm{Pic}}(S_{\mathbb{Q}}/\mathbb{Q})$ by the above proof. Then it suffices to prove that

$$\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle = \mathcal{O}(c) + \mathrm{eff} + \mathrm{nef}$$

for some rational number $c > 0$. Here $\mathcal{O}(c)$ is the trivial line bundle on $\mathbb{Z}$ with metric given by $\|1\| = e^{-c}$, and viewed as an adelic line bundle on S by pull-back.

Step 4 still works in the arithmetic case, so it suffices to prove that there exists $c(g) > 0$ such that $\overline{\Phi} - c(g)$ is an effective adelic divisor on S. It suffices to prove that there exists $c(g) > 0$ such that $\varphi(X_s) \geq c(g)$ for any $s \in S(\mathbb{C})$. Let M_g be a fine moduli space of curves of genus g over $\mathbb{C}$ with a suitable level structure, and $\overline{M}_g$ be a suitable compactification of M_g. By Zhang [Zha3], we know that $\varphi > 0$ on M_g. To prove that $\inf \varphi > 0$ on M_g, we prove that φ tends to infinity along the boundary $\overline{M}_g \setminus M_g$. This is a consequence of Step 5 for $k = \mathbb{C}$, as the image of $\overline{\Phi}$ in $\mathrm{Div}(M_g/\mathbb{C})$ governs its growth behavior along $\overline{M}_g \setminus M_g$.

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1.5 Uniform fiberwise bigness

By Zhang [Zha1], the Bogomolov conjecture for a single smooth projective curve C over a number field K (or a function field of one variable) is equivalent to $\overline{\omega}^2_{C/K,a} > 0$. The following theorem is a uniform version of this positivity and also a fiberwise version of Theorem 1.3. We thank Ziyang Gao for raising the question of proving such a bound.

Theorem 1.4 (Theorem 4.11, Theorem 4.12). *Let $g > 1$ be an integer. Then there is a constants $c_3 > 0$ depending only on g satisfying the following properties. Let K be either a number field or a function field of one variable over a field k. Then for any geometrically connected, smooth and projective curve C over K of genus g, with the extra assumption that $C_{\overline{K}}$ is non-isotrivial over $\overline{k}$ in the function field case, one has*

$$[\overline{\omega}^2_{C/K,a}] > c_3 \cdot \max\{h_{\mathrm{Fal}}(C), 1\}.$$

The stable Faltings height $h_{\mathrm{Fal}}(C)$ is normalized as in §4.3.1. The term $\overline{\omega}_{C/K,a}$ is Zhang's admissible canonical bundle of C over K, and the normalized intersection number $[\overline{\omega}^2_{C/K,a}]$ is understood as follows. If K is a number field,

$$[\overline{\omega}^2_{C,a}] = \frac{1}{[K:\mathbb{Q}]}\overline{\omega}^2_{C/K,a},$$

where $\overline{\omega}^2_{C/K,a}$ is the arithmetic intersection number. If K is a function field of one variable over k,

$$[\overline{\omega}^2_{C,a}] = \overline{\omega}^2_{C/K,a},$$

where the intersection number $\overline{\omega}^2_{C/K,a}$ on C is normalized by multiplicity functions given by degrees over k.

This theorem is stated in a style similar to that of Theorem 1.1, and the proof is also similar. It is still a consequence of Theorem 1.3, but the function field case has an explicit bound and a more direct proof. We refer to the proofs of Theorem 4.11 and Theorem 4.12 for more details.

As mentioned above, Looper–Silverman–Wilms [LSW] proves a uniform Bogomolov-type theorem over function fields, which is closely related to Theorem 1.1. Namely, in the setting of Theorem 1.1 and Theorem 1.4, in the function field case case, [LSW, Thm. 1.2] takes the form

$$\#\left\{x \in C(\overline{K}) : \widehat{h}(x - \alpha) \le c'_1 \overline{\omega}^2_{C,a}\right\} \le c'_2.$$

The constants (c'_1, c'_2) depend only on g and are amazingly explicit. With Theorem 1.4, we can replace the term $\overline{\omega}^2_{C,a}$ by $\max\{h_{\mathrm{Fal}}(C), 1\}$ and get a bound closer to that in Theorem 1.1. The proof in the loc. cit. is also based on Zhang's admissible pairings, but its arguments are on individual curves instead of on moduli spaces of curves.

1.6 Notations and terminology

By a *variety* over a field k, we mean an integral scheme, separated of finite type over k.

By a *curve* over a field k, we mean a geometrically connected projective scheme C of dimension 1 over a field k. The curve is called *semistable* if $C_{\bar{k}}$ is reduced and all singular points of $C_{\bar{k}}$ are ordinary double points. The curve is called *stable* if it is semistable, and any rational irreducible component of $C_{\bar{k}}$ intersects other irreducible components at three or more points.

By a *relative curve over a scheme S*, we mean a projective and flat morphism $\pi : X \to S$ of purely relative dimension 1 with geometrically connected fibers. The curve is called *smooth (resp. stable, semistable)*, if every fiber of $\pi : X \to S$ is smooth (resp. stable, semistable). It is said to be *of genus g*, if every fiber of $\pi : X \to S$ has arithmetic genus g.

Let S be a quasi-projective scheme over a base ring R, and let $\pi : X \to S$ be a smooth relative curve of genus $g > 1$. We say that $\pi : X \to S$ has *maximal variation* (or *maximal variation of moduli*) if the moduli morphism $S \to M_{g,R}$ associated to π is generically finite. Here $M_{g,R}$ denotes the coarse moduli scheme of smooth curves of genus g over R.

By a *line bundle* on a scheme, we mean an invertible sheaf on the scheme. We often write or mention tensor products of line bundles additively, so $aL - bM$ means $L^{\otimes a} \otimes M^{\otimes(-b)}$ for line bundles L, M and integers a, b.

Let $X \to S$ be a morphism of schemes. For $m > 0$, we denote by

$$X^m_{/S} = X \times_S X \times_S \cdots \times_S X$$

the m-fold fiber product of X over S. We take the convention $X^0_{/S} = S$. If L is a line bundle on X, then *the box tensor*

$$m_{\boxtimes} L = p_1^* L + p_2^* L + \cdots + p_m^* L$$

is a line bundle on $X^m_{/S}$. Here $p_i : X^m_{/S} \to X$ is the projection to the i-th component.

To treat adelic line bundles of [YZ2], we follow the uniform terminology of [YZ2, §1.5]. In particular, most of the time, our base ring k is $\mathbb{Z}$ or a field.

Acknowledgements

2 Admissible adelic line bundles

The goal of this section is to generalize the theory of admissible line bundles of Zhang [Zha1] from curves to relative curves using the terminology of Yuan–Zhang [YZ2]. Note that the original theory of [Zha1] is reviewed in our appendix §A. In §2.1, we review the theory of adelic line bundles of [YZ2]. In §2.2, we introduce our theory of admissible line bundles on relative curves. In §2.3, we consider some formulae on intersections and pull-back of admissible adelic line bundles.

2.1 Review on adelic line bundles

The goal of this subsection is to sketch the notion of adelic line bundles on quasi-projective varieties of Yuan–Zhang [YZ2], which generalizes the more classical adelic line bundles of projective varieties over number fields of Zhang [Zha2]. In the end, we prove two basic results about adelic line bundles which will be used in this paper.

Adelic divisors

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{U}$ be a flat and quasi-projective integral scheme over k. Let us first recall the definition of adelic divisors on $\mathcal{U}/k$.

Let $\mathcal{X}$ be a *projective model* of $\mathcal{U}$ over k, i.e., a projective integral scheme over k with an open immersion $\mathcal{U} \to \mathcal{X}$ over k. In the spirit of [YZ2, §2.2], take the fiber product

$$\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}) = \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}} \times_{\mathrm{Div}(\mathcal{U})_{\mathbb{Q}}} \mathrm{Div}(\mathcal{U}),$$

whose elements are arithmetic divisors of mixed coefficients. In the arithmetic case (that $k = \mathbb{Z}$), $\widehat{\mathrm{Div}}(\mathcal{X})$ is the group of arithmetic divisors on $\mathcal{X}$, where the Green's functions are assumed to be continuous (away from the singularities). In the geometric case (that k is a field), $\widehat{\mathrm{Div}}(\mathcal{X})$ means the usual $\mathrm{Div}(\mathcal{X})$.

By abuse of terminology, in the following, if k is a field, then "arithmetic divisor" (resp. "hermitian line bundle") means "divisor" (resp. "line bundle").

Define the group of *model adelic divisors* by

$$\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}} = \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U}).$$

Here the limit is over the system of projective models $\mathcal{X}$ of $\mathcal{U}$ over k. An element of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ is *effective* if it is the image of an effective element of some $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$, where an element of $\widehat{\mathrm{Div}}(\mathcal{X},\mathcal{U})$ is *effective* if its images in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ and $\mathrm{Div}(\mathcal{U})$ are both effective.

Fix a *boundary divisor* $(\mathcal{X}_0, \overline{\mathcal{D}}_0)$ of $\mathcal{U}$, i.e, a projective model $\mathcal{X}_0$ of $\mathcal{U}$ and a strictly effective arithmetic divisor $\overline{\mathcal{D}}_0$ on $\mathcal{X}_0$ such that the support of the finite part $\mathcal{D}_0$ is exactly $\mathcal{X}_0 \setminus \mathcal{U}$. We have a *boundary norm*

$$\| \cdot \|_{\overline{\mathcal{D}}_0} : \widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}} \longrightarrow [0, \infty]$$

by

$$\|\overline{\mathcal{E}}\|_{\overline{\mathcal{D}}_0} := \inf\{\epsilon \in \mathbb{Q}_{>0} : -\epsilon\overline{\mathcal{D}}_0 \le \overline{\mathcal{E}} \le \epsilon\overline{\mathcal{D}}_0\}.$$

Here the inequalities are defined in terms of effectivity. It further induces a *boundary topology* on $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$, which does not depend on the choice of $(\mathcal{X}_0, \overline{\mathcal{D}}_0)$.

Let $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ be the *completion* of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ with respect to the boundary topology. An element of $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ is called an *adelic divisor* on $\mathcal{U}/k$.

By definition, an adelic divisor is represented by a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$, i.e., a sequence $\{\overline{\mathcal{E}}_i\}_{i\geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ satisfying the property that there is a sequence $\{\epsilon_i\}_{i\geq 1}$ of positive rational numbers converging to 0 such that

$$-\epsilon_i \overline{\mathcal{D}}_0 \leq \overline{\mathcal{E}}_{i'} - \overline{\mathcal{E}}_i \leq \epsilon_i \overline{\mathcal{D}}_0, \quad i' \geq i \geq 1.$$

There is a canonical map

$$\widehat{\mathrm{Div}}(\mathcal{U}/k) \longrightarrow \mathrm{Div}(\mathcal{U}), \quad \{\overline{\mathcal{E}}_i\}_{i\geq 1} \longmapsto \mathcal{E}_1|_{\mathcal{U}}.$$

We usually write $\overline{\mathcal{E}} = \{\overline{\mathcal{E}}_i\}_{i\geq 1}$ and $\mathcal{E} = \mathcal{E}_1|_{\mathcal{U}}$, and call $\mathcal{E}$ the *underlying divisor* of $\overline{\mathcal{E}}$.

Adelic line bundles

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{U}$ be a flat and quasi-projective integral scheme over k.

Let $\mathcal{X}$ be a projective model of $\mathcal{U}$ over k. In the spirit of [YZ2, §2.2], let $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ be the category of hermitian line bundles on $\mathcal{X}$, and $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathbb{Q}}$ be the category of hermitian $\mathbb{Q}$-line bundles on $\mathcal{X}$. In the arithmetic case (that $k = \mathbb{Z}$), $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ is the category of hermitian line bundles with continuous metrics on $\mathcal{X}$. In the geometric case (that k is a field), $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$ means the usual $\mathcal{P}\mathrm{ic}(\mathcal{X})$.

As a convention, categories of various line bundles are defined to be groupoids; i.e., the morphisms in them are defined to be isomorphisms (or isometries) of the line bundles. To illustrate the category of various $\mathbb{Q}$-line bundles, take $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathbb{Q}}$ for example. An object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})_{\mathbb{Q}}$ is a pair $(a, \mathcal{L})$ (or just written as $a\mathcal{L}$) with $a \in \mathbb{Q}$ and $\mathcal{L} \in \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$, and a morphism of two such objects is defined to be an element of

$$\mathrm{Hom}(a\mathcal{L}, a'\mathcal{L}') = \varinjlim_m \mathrm{Hom}(am\mathcal{L}, a'm\mathcal{L}'),$$

where m runs through positive integers such that am and $a'm$ are both integers, so that $am\mathcal{L}$ and $a'm\mathcal{L}'$ are viewed as objects of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$, and "Hom" on the right-hand side are viewed in $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X})$.

Let $(\mathcal{X}_0, \overline{\mathcal{D}}_0)$ be a boundary divisor as above. Define the *category* $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ *of adelic line bundles* on $\mathcal{U}/k$ as follows. An object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ is a pair $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i\geq 1})$ where:

(1) $\mathcal{L}$ is an object of $\mathcal{P}\mathrm{ic}(\mathcal{U})$, i.e., a line bundle on $\mathcal{U}$;

(2) $\mathcal{X}_i$ is a projective model of $\mathcal{U}$ over k;

(3) $\overline{\mathcal{L}}_i$ is an object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{X}_i)_{\mathbb{Q}}$, i.e. a hermitian $\mathbb{Q}$-line bundle on $\mathcal{X}_i$;

(4) $\ell_i : \mathcal{L} \to \mathcal{L}_i|_{\mathcal{U}}$ is an isomorphism in $\mathcal{P}\mathrm{ic}(\mathcal{U})_{\mathbb{Q}}$, where $\mathcal{L}_i$ is the underlying $\mathbb{Q}$-line bundle of $\overline{\mathcal{L}}_i$ on $\mathcal{X}_i$.

The sequence is required to satisfy the *Cauchy condition* that the sequence $\{\widehat{\mathrm{div}}(\ell_i \ell_1^{-1})\}_{i \geq 1}$ is a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ under the boundary topology.

A morphism from an object $(\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i \geq 1})$ to another $(\mathcal{L}', (\mathcal{X}'_i, \overline{\mathcal{L}}'_i, \ell'_i)_{i \geq 1})$ is an isomorphism $\iota : \mathcal{L} \to \mathcal{L}'$ of the integral line bundles on $\mathcal{U}$ such that the sequence $\{\widehat{\mathrm{div}}(\ell'_i \ell_1'^{-1}) - \widehat{\mathrm{div}}(\ell_i \ell_1^{-1}) + \widehat{\mathrm{div}}(\iota_1)\}_{i \geq 1}$ of $\widehat{\mathrm{Div}}(\mathcal{U}/k)_{\mathrm{mod}}$ converges to 0 in $\widehat{\mathrm{Div}}(\mathcal{U}/k)$ under the boundary topology.

An object of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ is called an *adelic line bundle* on $\mathcal{U}$. Define $\widehat{\mathrm{Pic}}(\mathcal{U}/k)$ to be the *group* of isomorphism classes of objects of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$.

There is a canonical forgetful functor

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k) \longrightarrow \mathcal{P}\mathrm{ic}(\mathcal{U}), \quad (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i \geq 1}) \longmapsto \mathcal{L}$$

We usually write $\overline{\mathcal{L}} = (\mathcal{L}, (\mathcal{X}_i, \overline{\mathcal{L}}_i, \ell_i)_{i \geq 1})$ and call $\mathcal{L}$ the *underlying line bundle* of $\overline{\mathcal{L}}$.

There is a canonical surjection

$$\widehat{\mathrm{Div}}(\mathcal{U}/k) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U}/k).$$

The kernel is the image of the group of principal arithmetic divisors on projective models of $\mathcal{U}$.

An adelic divisor is called *effective* if it is equal to a limit of effective arithmetic divisors (of mixed coefficients). An adelic line bundle is called *effective* if it is the image of an effective adelic divisor. An adelic line bundle (resp. adelic divisor) is called *strongly nef* if it is isomorphic (resp. equal) to a limit of nef hermitian line bundles (resp. adelic divisors) under the boundary topology. An adelic line bundle (resp. adelic divisor) $\overline{\mathcal{L}}$ on $\mathcal{U}$ is *nef* if there exists a strongly nef adelic line bundle (resp. adelic divisor) $\overline{\mathcal{M}}$ on $\mathcal{U}$ such that $a\overline{\mathcal{L}} + \overline{\mathcal{M}}$ is strongly nef for all positive integers a. An adelic line bundle (resp. adelic divisor) is *integrable* if it is isomorphic (resp. equal) to the difference of two strongly nef ones.

Denote by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{nef}}$ (resp. $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{int}}$) the subcategory of $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$ consisting of nef (resp. integrable) adelic line bundles. Denote by $\widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{nef}}$ (resp. $\widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{int}}$) the subsets of nef (resp. integrable) elements in $\widehat{\mathrm{Pic}}(\mathcal{U}/k)$.

If the base ring k is clear, we usually omit the dependence on k of the groups or categories of adelic objects. For example, $\widehat{\mathrm{Div}}(\mathcal{U}/k)$, $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)$, $\widehat{\mathrm{Pic}}(\mathcal{U}/k)$ are written as $\widehat{\mathrm{Div}}(\mathcal{U})$, $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U})$, $\widehat{\mathrm{Pic}}(\mathcal{U})$.

Analytification

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{U}$ be a flat and quasi-projective integral scheme over k. There is a natural *Berkovich analytic space* $\mathcal{U}^{\mathrm{an}} = (\mathcal{U}/k)^{\mathrm{an}}$. In fact, if $\mathcal{U}$ has an open affine cover $\{\mathrm{Spec}\, A_i\}_i$, then $\mathcal{U}^{\mathrm{an}} = \cup_i \mathcal{M}(A_i)$, where $\mathcal{M}(A_i)$ is the set of multiplicative semi-norms on A_i, assumed to be trivial over k if k is a field. The set $\mathcal{U}^{\mathrm{an}}$ is endowed with the coarsest topology such that every $\mathcal{M}(A_i)$ is open in $\mathcal{U}^{\mathrm{an}}$ and that $|f| : \mathcal{M}(A_i) \to \mathbb{R}$ is continuous for all $f \in A_i$. Then $\mathcal{U}^{\mathrm{an}}$ is Hausdorff, path-connected and locally compact.

An *arithmetic divisor on $\mathcal{U}^{\mathrm{an}}$* is a pair $\overline{\mathcal{D}} = (\mathcal{D}, g_{\mathcal{D}})$ consisting of a divisor $\mathcal{D}$ on $\mathcal{U}$ and a Green's function $g_{\mathcal{D}}$ of $\mathcal{D}$ on $\mathcal{U}^{\mathrm{an}}$, i.e., a continuous function $g_{\mathcal{D}} : \mathcal{U}^{\mathrm{an}} \setminus |\mathcal{D}|^{\mathrm{an}} \to \mathbb{R}$ with logarithmic singularity along $\mathcal{D}$ in the sense that, for any rational function f on a Zariski open subset $\mathcal{V}$ of $\mathcal{U}$ satisfying $\mathrm{div}(f) = \mathcal{D}|_{\mathcal{U}}$, the function $g_{\mathcal{D}} + \log|f|$ can be extended to a continuous function on $\mathcal{U}^{\mathrm{an}}$. Note that the definition uses the term $g_{\mathcal{D}} + \log|f|$ instead of $g_{\mathcal{D}} + 2\log|f|$, so our convention is different from that used by Gillet–Soulé.

A *metrized line bundle on $\mathcal{U}^{\mathrm{an}}$* is a pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ consisting of a line bundle $\mathcal{L}$ on $\mathcal{U}$ and a continuous metric $\|\cdot\|$ of $\mathcal{L}$ on $\mathcal{U}^{\mathrm{an}}$; i.e., an assignment of a H_v-metric $\|\cdot\|_v$ to the fiber $\mathcal{L} \otimes H_v$ over every $v \in \mathcal{U}^{\mathrm{an}}$, assumed to vary continuously as v varies. Here H_v denotes the completed residue field of v.

Denote by $\widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}})$ the group of arithmetic divisors on $\mathcal{U}^{\mathrm{an}}$. Denote by $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})$ (resp. $\widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}})$) the category (resp. group of isometry classes) of metrized line bundles on $\mathcal{U}^{\mathrm{an}}$.

By [YZ2, Prop. 3.3.1, Prop. 3.4.1], there are injective analytification maps

$$\widehat{\mathrm{Div}}(\mathcal{U}) \longrightarrow \widehat{\mathrm{Div}}(\mathcal{U}^{\mathrm{an}}),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}}),$$

$$\widehat{\mathrm{Pic}}(\mathcal{U}) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{U}^{\mathrm{an}}).$$

These maps will bring lots of convenience in our treatment later.

Intersection theory

Here we recall the intersection theory in [YZ2, §4.1].

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{U}$ be a flat and quasi-projective integral scheme over k of absolute dimension d. There is an intersection pairing

$$\widehat{\mathrm{Pic}}(\mathcal{U}/k)_{\mathrm{int}}^d \longrightarrow \mathbb{R}, \qquad (\overline{\mathcal{H}}_1, \cdots, \overline{\mathcal{H}}_d) \longmapsto \overline{\mathcal{H}}_1 \cdots \overline{\mathcal{H}}_d,$$

defined as limits of the arithmetic (or geometric) intersection pairings of the projective case.

Let k be either $\mathbb{Z}$ or a field. Let $f : \mathcal{U} \to \mathcal{V}$ be a projective flat morphism of relative dimension n of flat and quasi-projective integral schemes over k. Assume that $\mathcal{V}$ is normal. There is a relative intersection pairing

$$\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k)_{\mathrm{int}}^{n+1} \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{V}/k)_{\mathrm{int}}, \qquad (\overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1}) \longmapsto f_* \langle \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1} \rangle.$$

This is defined as the limit of the Deligne pairing. The pairing of nef adelic line bundles is still nef. For simplicity, we may abbreviate $f_* \langle \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1} \rangle$ as $\langle \overline{\mathcal{L}}_1, \cdots, \overline{\mathcal{L}}_{n+1} \rangle$ if no confusion can occur.

Volume and bigness

Here we recall the notions of volume and bigness of adelic line bundles in [YZ2, §5.1-5.2].

Let k be either $\mathbb{Z}$ or a field. Let $\mathcal{U}$ be a flat and quasi-projective integral scheme over k. Let $\overline{\mathcal{L}}$ be an adelic line bundle on $\mathcal{U}$ with underlying line bundle $\mathcal{L}$ on $\mathcal{U}$. Define

$$\widehat{H}^0(\mathcal{U}, \overline{\mathcal{L}}) := \{ s \in H^0(\mathcal{U}, \mathcal{L}) : \|s(x)\| \leq 1, \forall x \in \mathcal{U}^{\mathrm{an}} \}.$$

Here the metric $\|s(x)\|$ on the Berkovich space $\mathcal{U}^{\mathrm{an}}$ is defined via the analytification functor $\widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}) \to \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}^{\mathrm{an}})$. Elements of $\widehat{H}^0(\mathcal{U}, \overline{\mathcal{L}})$ are called *effective sections* of $\overline{\mathcal{L}}$ on $\mathcal{U}$.

If $k = \mathbb{Z}$, then $\widehat{H}^0(\mathcal{U}, \overline{\mathcal{L}})$ is a finite set, and we denote

$$\widehat{h}^0(\mathcal{U}, \overline{\mathcal{L}}) := \log \# \widehat{H}^0(\mathcal{U}, \overline{\mathcal{L}});$$

if k is a field, then $\widehat{H}^0(\mathcal{U}, \overline{\mathcal{L}})$ is a finite-dimensional vector space over k, and we denote

$$\widehat{h}^0(\mathcal{U}, \overline{\mathcal{L}}) := \dim_k \widehat{H}^0(\mathcal{U}, \overline{\mathcal{L}}).$$

Define the *volume*

$$\widehat{\mathrm{vol}}(\mathcal{U}, \overline{\mathcal{L}}) := \lim_{m \to \infty} \frac{d!}{m^d} \widehat{h}^0(\mathcal{U}, m\overline{\mathcal{L}}).$$

Here d is the absolute dimension of $\mathcal{U}$. Then the limit $\widehat{\mathrm{vol}}(\mathcal{U}, \overline{\mathcal{L}})$ always exists. The adelic line bundle $\overline{\mathcal{L}}$ is said to be *big* on $\mathcal{U}$ if $\widehat{\mathrm{vol}}(\mathcal{U}, \overline{\mathcal{L}}) > 0$.

If $\overline{\mathcal{L}}$ is *nef*, the adelic Hilbert–Samuel formula asserts that

$$\widehat{\mathrm{vol}}(\mathcal{U}, \overline{\mathcal{L}}) = \overline{\mathcal{L}}^d.$$

In this case, $\overline{\mathcal{L}}$ is big if and only if $\overline{\mathcal{L}}^d > 0$.

Varieties over global fields

By further direct limits, the above definitions and notations are extended to flat and essentially quasi-projective integral schemes over k in [YZ2]. We do not need this generality in this paper, but we only need the case of quasi-projective varieties over a number field or a function field of one variable.

Let k be either $\mathbb{Z}$ or a field. If k is a field, let K be the function field of a projective regular curve B over k. If $k = \mathbb{Z}$, let K be a number field and denote $B = \mathrm{Spec}\, O_K$.

Let X be a quasi-projective variety over K, which is viewed as a scheme over k. By a *quasi-projective model* $\mathcal{U}$ of X over k, we mean a flat and quasi-projective integral scheme $\mathcal{U}$ over B, together with an open immersion $X \to \mathcal{U}_K$ over K. Define

$$\widehat{\mathrm{Div}}(X/k) = \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}/k),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X/k) = \varinjlim_{\mathcal{U}} \widehat{\mathcal{P}\mathrm{ic}}(\mathcal{U}/k),$$

$$\widehat{\mathrm{Pic}}(X/k) = \varinjlim_{\mathcal{U}} \widehat{\mathrm{Pic}}(\mathcal{U}/k).$$

Here the limit is over all quasi-projective models $\mathcal{U}$ of X over k. Again, we often abbreviate the terms as

$$\widehat{\mathrm{Div}}(X), \quad \widehat{\mathcal{P}\mathrm{ic}}(X), \quad \widehat{\mathrm{Pic}}(X).$$

If X is projective over K, then $\widehat{\mathcal{P}\mathrm{ic}}(X)$ is essentially the category of adelic line bundles on X introduced in Zhang [Zha2].

The notions of effectivity, nefness and integrability of adelic line bundles (or adelic divisors) are also transferred to quasi-projective varieties over K by direct limits. The intersection pairings are also valid in the current situation by direct limits.

The Berkovich analytic space $X^{\mathrm{an}} = (X/k)^{\mathrm{an}}$ is defined similar to $\mathcal{U}^{\mathrm{an}} = (\mathcal{U}/k)^{\mathrm{an}}$, which does not use the property that $\mathcal{U}$ is quasi-projective over k. The notions of metrized line bundles and arithmetic divisors on X^{an} are defined similarly. Then the direct limit process induces injective analytification maps

$$\widehat{\mathrm{Div}}(X) \longrightarrow \widehat{\mathrm{Div}}(X^{\mathrm{an}}),$$

$$\widehat{\mathcal{P}\mathrm{ic}}(X) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}}),$$

$$\widehat{\mathrm{Pic}}(X) \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}}).$$

Define the set of *effective sections* of $\overline{L}$ on X by

$$\widehat{H}^0(X, \overline{L}) := \{s \in H^0(X, L) : \|s(x)\| \leq 1, \forall x \in X^{\mathrm{an}}\}.$$

If $k = \mathbb{Z}$, denote

$$\widehat{h}^0(X, \overline{L}) := \log \# \widehat{H}^0(X, \overline{L});$$

if k is a field, denote

$$\widehat{h}^0(X, \overline{L}) := \dim_k \widehat{H}^0(X, \overline{L}).$$

Then $\widehat{h}^0(X, \overline{L})$ is a well-defined (finite) number. Define the *volume*

$$\widehat{\mathrm{vol}}(X, \overline{L}) := \lim_{m \to \infty} \frac{d!}{m^d} \widehat{h}^0(X, m\overline{L}).$$

Here $d = \dim X + 1$. Then the limit $\widehat{\mathrm{vol}}(X, \overline{L})$ always exists. The adelic line bundle $\overline{L}$ is said to be *big* on X if $\widehat{\mathrm{vol}}(X, \overline{L}) > 0$.

If $\overline{L}$ is *nef*, the adelic Hilbert–Samuel formula asserts that

$$\widehat{\mathrm{vol}}(X, \overline{L}) = \overline{L}^d.$$

In this case, $\overline{L}$ is big if and only if $\overline{L}^d > 0$.

Some basic results

We will need the following basic properties of adelic line bundles, which are not in [YZ2].

Lemma 2.1. *Let k be either $\mathbb{Z}$ or a field. Let X and S be quasi-projective and flat integral schemes over k. Let $\pi : X \to S$ be a projective and flat morphism over k with geometrically connected fibers. Then the following are true:*

(1) *The canonical map $\pi^* : \widehat{\mathrm{Pic}}(S) \to \widehat{\mathrm{Pic}}(X)$ is injective.*

(2) *Assume that S is normal or $\pi : X \to S$ admits a section. Let $\overline{L}$ be an integrable adelic line bundle on X such that the underlying line bundle $L = \pi^* M$ for some line bundle M on S. Assume that for each $v \in S^{\mathrm{an}}$, under an isomorphism $L|_{X_{H_v}} \simeq \mathcal{O}_{X_{H_v}}$ induced by $L = \pi^* M$, the metric of $L|_{X_{H_v}}$ on X_v^{an} induced by $\overline{L}$ corresponds to a constant multiple of the trivial metric of $\mathcal{O}_{X_{H_v}}$ on X_v^{an}. Then there is an integrable adelic line bundle $\overline{M}$ on S with underlying line bundle M such that $\overline{L}$ is isomorphic to $\pi^* \overline{M}$.*

Proof. By [YZ2, Prop. 3.4.1], there is an injective analytification map

$$\widehat{\mathrm{Pic}}(X) \longrightarrow \widehat{\mathrm{Pic}}(X^{\mathrm{an}}).$$

By [BLR, §8.1, Prop. 4], $\pi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(X)$ is injective. Then (1) follows from the fact that the maps $C(S^{\mathrm{an}}) \to C(X^{\mathrm{an}})$ between the space of continuous functions is injective.

For (2), if $\pi : X \to S$ admits a section $x : S \to X$, then we can set $\overline{M} = x^* \overline{L}$, and check that it satisfies the requirement.

Assume that S is normal in the following. Since the integral line bundle M is given, it suffices to prove the existence of an integrable adelic $\mathbb{Q}$-line bundle $\overline{M}$ on S extending M such that $\overline{L}$ is isomorphic to $\pi^* \overline{M}$ as adelic $\mathbb{Q}$-line bundles.

Let $\overline{N}$ be an integrable adelic line bundle on X. Denote by d the degree of the underlying line bundle N of $\overline{N}$ on the generic fiber of $X \to S$, and assume that $d > 0$. Set

$$\overline{M} = d^{-1} \langle \overline{L}, \overline{N}, \cdots, \overline{N} \rangle \in \widehat{\mathcal{P}\mathrm{ic}}(S)_{\mathbb{Q}}.$$

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The motivation of this definition comes from [YZ2, Lem. 4.6.1(2)].

To check that $\pi^*\overline{M}$ is isomorphic to $\overline{L}$, we first note that their underlying line bundles are canonically isomorphic by [MG, Prop. 5.2.1.a]. By the analytification map, it suffices to check that the metrics of $L = \pi^*M$ over X^{an} induced by $\pi^*\overline{M}$ and $\overline{L}$ are equal. For any $v \in S^{\mathrm{an}}$, the metric of π^*M is equal to that of L on the fiber X_v^{an} by the integration formula in [YZ2, Thm. 4.6.2] and [YZ2, §4.2.2]. This finishes the proof. $\qquad\square$

Lemma 2.2. *Let k be either $\mathbb{Z}$ or a field. Let $f : X \to Y$ be a projective and flat morphism of relative dimension n over k. Here X, Y are quasi-projective and flat integral schemes over k. Assume that Y is normal of dimension $d \geq 1$. Let $\overline{L}_1, \cdots, \overline{L}_{n+1}$ be integrable adelic line bundles on X.*

(1) If $\overline{L}_1, \cdots, \overline{L}_{n+1}$ are nef and big, then $f_*\langle \overline{L}_1, \cdots, \overline{L}_{n+1}\rangle$ is nef and big on Y.

(2) If $\overline{L}_1$ is effective and $\overline{L}_2, \cdots, \overline{L}_{n+1}$ are nef, then $f_*\langle \overline{L}_1, \cdots, \overline{L}_{n+1}\rangle$ is pseudo-effective on Y in the sense that

$$f_*\langle \overline{L}_1, \cdots, \overline{L}_{n+1}\rangle \cdot \overline{M}_1 \cdots \overline{M}_{d-1} \geq 0$$

for any nef adelic line bundles $\overline{M}_1 \cdots, \overline{M}_{d-1}$ on Y.

Proof. Part (2) is a consequence of the identity

$$f_*\langle \overline{L}_1, \cdots, \overline{L}_{n+1}\rangle \cdot \overline{M}_1 \cdots \overline{M}_{d-1} = \overline{L}_1 \cdots \overline{L}_{n+1} \cdot f^*\overline{M}_1 \cdots f^*\overline{M}_{d-1}.$$

If $k = \mathbb{Z}$, the identity is a consequence of [YZ2, Lem. 4.6.1(1)] applying to the composition $X \to Y \to S$ with $S = \mathrm{Spec}\,\mathbb{Z}$. If k is a field, by blowing-up Y, we can obtain a fibration $Y \to S$ for $S = \mathbb{P}^1_k$. Then the identity still follows.

For part (1), note that $f_*\langle \overline{L}_1, \cdots, \overline{L}_{n+1}\rangle$ is nef by [YZ2, Thm. 4.1.3]. Let $\overline{M}$ be a nef and big adelic line bundle on Y. By [YZ2, Thm. 5.2.2(2)],

$$\widehat{\mathrm{vol}}(\overline{L}_1 - \epsilon f^*\overline{M}) \geq \overline{L}_1^{d+n} - (d+n)\overline{L}_1^{d+n-1} \cdot \epsilon f^*\overline{M} > 0$$

for some rational number $\epsilon > 0$. Then some positive multiple of $\overline{N} = \overline{L}_1 - \epsilon f^*\overline{M}$ is effective. It follows that

$$f_*\langle \overline{L}_1, \cdots, \overline{L}_{n+1}\rangle = f_*\langle \overline{N}, \overline{L}_2, \cdots, \overline{L}_{n+1}\rangle + \epsilon\, f_*\langle f^*\overline{M}, \overline{L}_2, \cdots, \overline{L}_{n+1}\rangle$$

By (2), $\overline{M}_1 = f_*\langle \overline{N}, \overline{L}_2, \cdots, \overline{L}_{n+1}\rangle$ is pseudo-effective. By [YZ2, Lem. 4.6.1(2)], $\overline{M}_2 = \epsilon f_*\langle f^*\overline{M}, \overline{L}_2, \cdots, \overline{L}_{n+1}\rangle$ is a positive multiple of $\overline{M}$, and thus it is big and nef. It follows that

$$(\overline{M}_1 + \overline{M}_2)^d \geq \overline{M}_2 \cdot (\overline{M}_1 + \overline{M}_2)^{d-1} \geq \overline{M}_2^2(\overline{M}_1 + \overline{M}_2)^{d-2} \geq \cdots \geq \overline{M}_2^d > 0.$$

This finishes the proof. □

2.2 Admissible adelic line bundles

We refer to §A for a review of admissible metrics of Arakelov [Ara] and Zhang [Zha1]. The goal of this subsection is to extend the theory to quasi-projective families of curves. Our main result here is the following family version of Theorem A.1.

Theorem 2.3. *Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve of genus $g > 0$. Denote by $\Delta : X \to X \times_S X$ the diagonal morphism. Then the following are true:*

(a) *There is an adelic line bundle $\overline{\omega}_{X/S,a}$ in $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ with underlying line bundle $\omega_{X/S}$, such that for any $v \in S^{\mathrm{an}}$, the metric of $\omega_{X_{H_v}/H_v}$ on $X_{H_v}^{\mathrm{an}}$ induced by $\overline{\omega}_{X/S,a}$ is equal to the canonical admissible metric $\|\cdot\|_a$. Moreover, $\overline{\omega}_{X/S,a}$ is nef and unique up to isomorphism.*

(b) *There is an adelic line bundle $\overline{\mathcal{O}}(\Delta)_a$ in $\widehat{\mathcal{P}\mathrm{ic}}(X \times_S X/k)$ with underlying line bundle $\mathcal{O}(\Delta)$, such that for any $v \in S^{\mathrm{an}}$, the metric of $\mathcal{O}(\Delta_{H_v})$ on $(X_{H_v}^2)^{\mathrm{an}}$ induced by $\overline{\mathcal{O}}(\Delta)_a$ is equal to the canonical admissible metric $\|\cdot\|_{\Delta,a}$. Moreover, $\overline{\mathcal{O}}(\Delta)_a$ is integrable and unique up to isomorphism.*

Moreover, the adelic line bundles satisfy the following extra properties:

(1) *The canonical isomorphism*

$$\omega_{X/S} \longrightarrow \Delta^*\mathcal{O}(-\Delta)$$

induces an isomorphism

$$\overline{\omega}_{X/S,a} \longrightarrow \Delta^*\overline{\mathcal{O}}(-\Delta)_a.$$

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(2) The canonical isomorphisms

$$p_{1*}\langle \mathcal{O}(\Delta), p_2^*\omega_{X/S}\rangle \longrightarrow \Delta^* p_2^* \omega_{X/S} \longrightarrow \omega_{X/S}$$

induce an isomorphism

$$p_{1*}\langle \overline{\mathcal{O}}(\Delta)_a, p_2^*\overline{\omega}_{X/S,a}\rangle \longrightarrow \overline{\omega}_{X/S,a}.$$

Here $p_1, p_2 : X \times_S X \to X$ denote the two projections.

A few terms in the above statement require explanation. By abuse of notations, we write the diagonal divisor $\Delta(X)$ as Δ. It is actually a Cartier divisor in $X \times_S X$ (cf. [BLR, §8.2, Lem. 6]), so $\mathcal{O}(\Delta)$ is a line bundle on $X \times_S X$.

Recall that §2.1 contains a brief review of the Berkovich space S^{an}. For any $v \in S^{\mathrm{an}}$, H_v denotes the completed residue field of v. We refer to [YZ2, §3.1] for more details.

The canonical admissible metrics are introduced for all $v \in S^{\mathrm{an}}$ in §A.6. They are the Arakelov metrics in the archimedean case, the Zhang metrics in the non-archimedean case, and the canonical metric in the trivially valued case.

Finally, the uniqueness in (a) and (b) is a consequence of [YZ2, Prop. 3.4.1], which gives a canonical fully faithful analytification functor

$$\widehat{\mathcal{P}\mathrm{ic}}(X) \longrightarrow \widehat{\mathcal{P}\mathrm{ic}}(X^{\mathrm{an}}).$$

The proof of the existence is similar to that of Theorem A.1. We first prove weak versions of (a) and (b), and then modify them to satisfy (1) and (2). These weak versions are the admissible extensions generalizing the notions in §A.3.

2.2.1 Basics on Jacobian schemes

Here we present some basic constructions of abelian schemes and Jacobian schemes. For more details, we refer to [MFK, Chap. 6] and [BLR, Chap. 9]. These constructions will be used in the proof of Theorem A.1 and elsewhere of this paper.

Let S be a noetherian scheme. Let $\pi : X \to S$ be a smooth relative curve of genus $g > 0$. Denote by $J = \mathrm{Pic}^0_{X/S}$ the Jacobian scheme over X over S.

Let α be a line bundle on X of degree d on the fibers of $X \to S$. We have a finite S-morphism

$$i_\alpha : X \longrightarrow J, \quad x \longmapsto dx - \alpha.$$

The right-hand side is understood in terms of functor points. We will also need the morphism

$$(i_\alpha, i_\alpha) : X \times_S X \longrightarrow J \times_S J.$$

Now we consider line bundles on J. Without taking a base change of S, we have to deal with the case that there is no line bundle on X of degree 1 on fibers on S. Then it is not possible to define a theta divisor as in the case of algebraically closed fields.

However, there is still a canonical principal polarization $J \to J^\vee$ over S in this generality, as proved in [MFK, §6.1, Prop. 6.9]. Then we have a *Poincaré line bundle* P on $J \times_S J$, obtained by the Poincaré line bundle of $J \times_S J^\vee$ via the polarization. As a convention, the Poincaré line bundle P is rigidified along zero sections in the sense that there are fixed isomorphisms $(e, \mathrm{id})^* P \to \mathcal{O}_J$ and $(\mathrm{id}, e)^* P \to \mathcal{O}_J$. Here $e : S \to J$ is the identity section, and (e, id) and (id, e) are the morphisms $J \to J \times_S J$ by the identity section. This determines the class of P in $\mathrm{Pic}(J \times_S J)$ uniquely.

Definition 2.4. Define a line bundle on J by

$$\Theta = \Delta_J^*(P^\vee),$$

where $\Delta_J : J \to J \times_S J$ is the diagonal morphism.

This is the construction outlined right before [MFK, §6.2, Prop. 6.10]. By construction, Θ is symmetric and rigidified along the identity section $e : S \to J$, since the similar property holds for P. By Lemma A.3(3), Θ is algebraically equivalent to twice of a theta divisor on each geometric fiber of $J \to S$. Therefore, Θ is relatively ample over S.

2.2.2 Admissible extensions

Recall that in §A.3 (together with §A.1, §A.6), we have a notion of admissible metrics of line bundles on abelian varieties, smooth curves, square of smooth curves over a complete valuation field. Now we extend this to family version as follows.

Resume the notation of Theorem 2.3. Namely, let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve of genus $g > 0$. We introduce the following definition.

Definition 2.5. Let W be X, $X \times_S X$, or an abelian scheme over S. Let $\overline{L}$ be an integrable adelic line bundle on W/k with underlying line bundle L. We say that $\overline{L}$ is *admissible* if for any $v \in S^{\mathrm{an}}$, the metric of $L|W_{H_v}$ on $W_{H_v}^{\mathrm{an}}$ induced by $\overline{L}$ is admissible. In this case, we say that $\overline{L}$ is an *admissible adelic extension* of L.

Proposition 2.6. *Let W be X, $X \times_S X$, or an abelian scheme over S. Then any line bundle L on W admits an admissible adelic extension in $\widehat{\mathrm{Pic}}(W/k)_{\mathrm{int}}$, unique up to translation by $\ker(\widehat{\mathrm{Pic}}(S/k)_{\mathrm{int}} \to \mathrm{Pic}(S))$.*

Proof. The uniqueness follows from Lemma 2.1(2). The existence is similar to the exposition in §A.3. For example, the case of abelian schemes is given by the dynamically invariant extension constructed in [YZ2, Thm. 6.1.1].

To treat the other cases, let us first introduce some notations. For any projective and flat S-schemes Y, denote by $\mathrm{Pic}^0(Y)$ the subgroup of $\mathrm{Pic}(Y)$ of line bundles on Y which are algebraically equivalent to 0 on fibers of $Y \to S$. Denote by $\mathrm{Pic}_{Y/S}(S)$ the relative Picard group, and denote by $\mathrm{Pic}^0_{Y/S}(S)$ the subgroup of line bundles which are algebraically equivalent to 0 on fibers of $Y \to S$.

For any projective and flat S-schemes Y_1, Y_2, denote by $\mathrm{Pic}^{00}(Y_1 \times_S Y_2)$ the subgroup of $\mathrm{Pic}(Y_1 \times_S Y_2)$ of line bundles which are algebraically equivalent to 0 on fibers of both projections $p_i : Y_1 \times_S Y_2 \to Y_i$.

For the case $W = X$, let α be a line bundle on X of degree $d > 0$ on the fibers of $X \to S$. It defines a finite morphism $i_\alpha : X \to J$. By pullback of an admissible adelic extension of the line bundle Θ from J to X, it suffices to prove that the map $i_\alpha^* : \mathrm{Pic}^0(J) \to \mathrm{Pic}^0(X)$ has a torsion cokernel, or equivalently that the map $i_\alpha^* : \mathrm{Pic}^0(J)/\mathrm{Pic}(S) \to \mathrm{Pic}^0(X)/\mathrm{Pic}(S)$ has a torsion cokernel. As a consequence of the Leray spectral sequence (cf. [BLR, §8.1, Prop. 4]), there is a canonical isomorphism $\mathrm{Pic}^0(J)/\mathrm{Pic}(S) \to \mathrm{Pic}^0_{J/S}(S)$ and a canonical injection $\mathrm{Pic}^0(X)/\mathrm{Pic}(S) \to \mathrm{Pic}^0_{X/S}(S)$. Then its suffices to prove that the map $i_\alpha^* : \mathrm{Pic}^0_{J/S}(S) \to \mathrm{Pic}^0_{X/S}(S)$ has a torsion cokernel.

Note that we have a canonical isomorphism $\mathrm{Pic}^0_{J/S} \to \mathrm{Pic}^0_{X/S}$, which can be established via flat descent. Under the isomorphism, $i_\alpha^* : \mathrm{Pic}^0_{J/S} \to \mathrm{Pic}^0_{X/S}$

is isomorphic to the multiplication $[d] : \mathrm{Pic}^0_{X/S} \to \mathrm{Pic}^0_{X/S}$, which can be verified fiberwise. This proves the case $W = X$.

For the case $W = X \times_S X$, let α be as above, and consider the morphism $(i_\alpha, i_\alpha) : X \times_S X \to J \times_S J$. By pull-back of admissible extensions of $p_i^* \Theta$ from $J \times_S J$ to $X \times_S X$, it suffices to prove that the canonical map

$$i_\alpha^* : \mathrm{Pic}^{00}(J \times_S J) \longrightarrow \mathrm{Pic}^{00}(X \times_S X)$$

has a torsion cokernel.

Apply the above result to the relative curve $p_1 : X \times_S X \to X$, whose relative Jacobian is just $q_1 : X \times_S J \to X$. We see that the canonical map

$$\mathrm{Pic}^{00}(X \times_S J) \longrightarrow \mathrm{Pic}^{00}(X \times_S X)$$

has a torsion cokernel. Similarly, apply the above result to the relative curve $q_2 : X \times_S J \to J$, whose relative Jacobian is just $p_2 : J \times_S J \to J$. We see that the canonical map

$$\mathrm{Pic}^{00}(J \times_S J) \longrightarrow \mathrm{Pic}^{00}(X \times_S J)$$

has a torsion cokernel. The result is proved by combining these two results. $\square$

2.2.3 Construction of canonical admissible extensions

Now we prove Theorem 2.3. Let $X/S/k$ be as in the theorem.

By Proposition 2.6, there is an admissible adelic extension $\overline{\omega}'_{X/S,a}$ of $\omega_{X/S}$ in $\widehat{\mathcal{P}\mathrm{ic}}(X/k)$ and an admissible adelic extension $\overline{\mathcal{O}}(\Delta)'_a$ of $\mathcal{O}(\Delta)$ in $\widehat{\mathcal{P}\mathrm{ic}}(X \times_S X/k)$. We will modify $(\overline{\omega}'_{X/S,a}, \overline{\mathcal{O}}(\Delta)'_a)$ to meet Theorem 2.3. Namely, we will prove that there are integrable adelic line bundles $\overline{\beta}_1$ and $\overline{\beta}_2$ in $\widehat{\mathcal{P}\mathrm{ic}}(S)$ with underlying line bundles β_1 and β_2 in $\mathcal{P}\mathrm{ic}(S)$, together with isomorphisms $\beta_1 \to \mathcal{O}_S$ and $\beta_2 \to \mathcal{O}_S$, such that the adelic line bundles

$$\overline{\omega}_{X/S,a} := \overline{\omega}'_{X/S,a} + \pi^* \overline{\beta}_1, \qquad \overline{\mathcal{O}}(\Delta)_a := \overline{\mathcal{O}}(\Delta)'_a + (\pi, \pi)^* \overline{\beta}_2$$

satisfy (1) and (2) of Theorem 2.3. Here $(\pi, \pi) : X \times_S X \to S$ is the structure morphism.

We first look at (2), which requires an isomorphism

$$p_{1*} \langle \overline{\mathcal{O}}(\Delta)_a, p_2^* \overline{\omega}_{X/S,a} \rangle \longrightarrow \overline{\omega}_{X/S,a}$$

extending the canonical isomorphism of the corresponding underlying line bundles. By [YZ2, Lem. 4.6.1(2)], there is a canonical isomorphism

$$p_{1*}\langle \overline{\mathcal{O}}(\Delta)_a, (\pi, \pi)^*\overline{\beta}_1\rangle \longrightarrow \pi^*\overline{\beta}_1.$$

Thus (2) is equivalent to an isomorphism

$$p_{1*}\langle \overline{\mathcal{O}}(\Delta)_a, p_2^*\overline{\omega}'_{X/S,a}\rangle \longrightarrow \overline{\omega}'_{X/S,a}.$$

By [YZ2, Lem. 4.6.1(2)] again, it suffices to have a canonical isomorphism

$$p_{1*}\langle \overline{\mathcal{O}}(\Delta)'_a, \ p_2^*\overline{\omega}'_{X/S,a}\rangle + (2g-2)\overline{\beta}_2 \longrightarrow \overline{\omega}'_{X/S,a}.$$

Define $\overline{\beta}_2 \in \widehat{\mathcal{P}\mathrm{ic}}(S)$ with trivial underlying line bundle satisfying the above isomorphism.

This establishes (2), which determines $\overline{\beta}_2$ and thus $\overline{\mathcal{O}}(\Delta)_a$. Similarly, we use (1) to determine $\overline{\beta}_1$ and thus $\overline{\omega}_{X/S,a}$. This proves Theorem 2.3.

2.3 Intersection and pull-back

The goal of this subsection is to consider some properties of admissible adelic line bundles. We first present the adjunction formula and the arithmetic Hodge index theorem, and then prove some explicit formulae about pull-back of admissible adelic line bundles from the Jacobian schemes.

For convenience, we introduce the notations for this subsection. Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve of genus $g > 0$. Denote by $\Delta : X \to X \times_S X$ the diagonal morphism. Let $\overline{\omega}_{X/S,a}$ and $\overline{\mathcal{O}}(\Delta)_a$ be the canonical admissible extensions in Theorem 2.3.

For any section $x : S \to X$ of $\pi : X \to S$, note that $x(S)$ is a Cartier divisor on X (cf. [BLR, §8.2, Lem. 6]). Denote by $\mathcal{O}(x)$ the line bundle on X associated to the Cartier divisor $x(S)$. Denote by $\overline{\mathcal{O}}(x)_a$ the pull-back of $\overline{\mathcal{O}}(\Delta)_a$ via the morphism $(x, \mathrm{id}) : X \to X \times_S X$, which is an admissible adelic extension of $\mathcal{O}(x)$ on X/k. We call $\overline{\mathcal{O}}(x)_a$ the *canonical admissible adelic extension* of $\mathcal{O}(x)$ on X/k.

By Definition 2.4, we have a symmetric, relatively ample and rigidified line bundle Θ on J. By [YZ2, Thm. 6.1.1(3)], there is a nef adelic line bundle $\overline{\Theta}$ on J/k extending Θ such that $[2]^*\overline{\Theta} = 4\overline{\Theta}$ in $\widehat{\mathrm{Pic}}(J/k)$.

2.3.1 Adjunction formula and Hodge index theorem

We first have the following adjunction formula, which generalizes [Zha1, Thm. 4.2].

Proposition 2.7. *Let $x : S \to X$ be a section of $\pi : X \to S$.*

(1) There is a canonical isomorphism

$$\pi_* \langle \overline{\mathcal{O}}(x)_a, \overline{L} \rangle \longrightarrow x^* \overline{L}$$

for any admissible adelic line bundle $\overline{L}$ on X/k.

(2) There is a canonical isomorphism

$$\pi_* \langle \overline{\mathcal{O}}(x)_a, \overline{\mathcal{O}}(x)_a \rangle \longrightarrow -\pi_* \langle \overline{\mathcal{O}}(x)_a, \overline{\omega}_{X/S,a} \rangle.$$

Proof. We first prove (1). There is a canonical isomorphism $\pi_* \langle \mathcal{O}(x), L \rangle \to x^* L$. By the analytification functor $\widehat{\mathcal{P}\mathrm{ic}}(S) \to \widehat{\mathcal{P}\mathrm{ic}}(S^{\mathrm{an}})$, it suffices to prove that the norm of this isomorphism is 1 at any $v \in S^{\mathrm{an}}$. By the integration formulae in [YZ2, Thm. 4.6.2] and [YZ2, §4.2.2], the logarithm of this norm is given by

$$\int_{X_v^{\mathrm{an}}} \log \|1\|_v c_1(\overline{L})_v.$$

Here 1 is the canonical section of $\mathcal{O}(x)$. This vanishes by Theorem A.1.

For (2), recall that Theorem 2.3(1) gives an isomorphism $\overline{\omega}_{X/S,a} \to \Delta^* \overline{\mathcal{O}}(-\Delta)_a$. Take the pull-back of this isomorphism via $x : S \to X$, we obtain an isomorphism $x^* \overline{\omega}_{X/S,a} \to x^* \overline{\mathcal{O}}(-x)_a$. Combining with (1), this gives (2). $\qquad \square$

Now we have the following consequence of the arithmetic Hodge index theorem of [YZ2].

Theorem 2.8. *Let M be a line bundle on X with degree 0 on fibers of $X \to S$. Let $\overline{M}$ be an admissible adelic extension of M on X. Then the following hold.*

(1) Denote by $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{vert}}$ the kernel of the forgetful map $\widehat{\mathrm{Pic}}(X/k)_{\mathrm{int}} \to \mathrm{Pic}(X)$. Then for any $\overline{V} \in \widehat{\mathrm{Pic}}(X/k)_{\mathrm{vert}}$,

$$\pi_* \langle \overline{M}, \overline{V} \rangle = 0$$

in $\widehat{\mathrm{Pic}}(S/k)$.

(2) *Denote by $\iota : S \to J$ the section of $J \to S$ corresponding to the line bundle M on X. Then*

$$\pi_* \langle \overline{M}, \overline{M} \rangle = -\iota^* \Theta$$

in $\widehat{\mathrm{Pic}}(S/k)_{\mathbb{Q}}$.

Proof. This is a variant of the arithmetic Hodge index theorem of [YZ2, Thm. 6.5.4]. It suffices to prove (1). The proof is similar to that of Proposition 2.7(1). In fact, fix an isomorphism $V \simeq \mathcal{O}_X$. The isomorphism induces a canonical isomorphism $\pi_* \langle \overline{M}, \overline{V} \rangle \to \mathcal{O}_S$. By the analytification functor $\widehat{\mathcal{P}\mathrm{ic}}(S) \to \widehat{\mathcal{P}\mathrm{ic}}(S^{\mathrm{an}})$, it suffices to prove that the norm of this isomorphism is 1 at any $v \in S^{\mathrm{an}}$. By definition, the logarithm of this norm is given by

$$\int_{X_v^{\mathrm{an}}} \log \|1\|_v c_1(\overline{M})_v.$$

Here 1 is viewed as a section of V via $V \simeq \mathcal{O}_X$. By Theorem A.1, $c_1(\overline{M})_v = 0$ since $\deg(M_v) = 0$. This finishes the proof. $\qquad\square$

2.3.2 Pull-back formula: general

Resume the above notations. Namely, let k be either $\mathbb{Z}$ or a field, let S be a quasi-projective and flat normal integral scheme over k, and let $\pi : X \to S$ be a smooth relative curve of genus $g > 0$.

We first present the following general pull-back formula, which generalizes its primitive version in Lemma A.3.

Theorem 2.9. *Let α be a line bundle on X of degree d on the fibers of $\pi : X \to S$. Let $\overline{\alpha}$ be an admissible adelic extension of α on X.*

(1) *There is an identity*

$$i_\alpha^* \overline{\Theta} = d^2 \overline{\omega}_{X/S,a} + 2d\overline{\alpha} - \pi^* \pi_* \langle \overline{\alpha}, \overline{\alpha} \rangle$$

in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$.

(2) *There are identities*

$$\pi_* \langle i_\alpha^* \overline{\Theta}, i_\alpha^* \overline{\Theta} \rangle = d^4 \pi_* \langle \overline{\omega}_{X/S,a} \, \overline{\omega}_{X/S,a} \rangle + 4d^3 \pi_* \langle \overline{\omega}_{X/S,a} \, \overline{\alpha} \rangle - (4g-4)d^2 \pi_* \langle \overline{\alpha}, \overline{\alpha} \rangle$$

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and

$$(g-1)\pi_*\langle i_\alpha^*\overline{\Theta}, i_\alpha^*\overline{\Theta}\rangle = gd^4\pi_*\langle\overline{\omega}_{X/S,a}\,\overline{\omega}_{X/S,a}\rangle + d^2\iota_\alpha^*\overline{\Theta}$$

in $\widehat{\mathrm{Pic}}(S)_\mathbb{Q}$. *Here* $\iota_\alpha : S \to J$ *is the section corresponding to the line bundle* $(2g-2)\alpha - d\omega_{X/S}$ *on* X.

(3) *Assume* $\alpha = \mathcal{O}(x)$ *for a section* $x : S \to X$ *of* $\pi : X \to S$. *Then*

$$i_\alpha^*\overline{\Theta} = \overline{\omega}_{X/S,a} + 2\overline{\mathcal{O}}(x)_a + \pi^*x^*\overline{\omega}_{X/S,a}$$

in $\widehat{\mathrm{Pic}}(X)_\mathbb{Q}$, *and*

$$\pi_*\langle i_\alpha^*\overline{\Theta}, i_\alpha^*\overline{\Theta}\rangle = \pi_*\langle\overline{\omega}_{X/S,a}\,\overline{\omega}_{X/S,a}\rangle + 4g\,x^*\overline{\omega}_{X/S,a}$$

in $\widehat{\mathrm{Pic}}(S)_\mathbb{Q}$.

Proof. It is easy to see that (3) is a consequence of (1) and (2) for $\overline{\alpha} = \overline{\mathcal{O}}(x)_a$ by the adjunction formula in Proposition 2.7.

To prove (2) by (1), by [YZ2, Lem. 4.6.1(2)],

$$\langle i_\alpha^*\overline{\Theta}, i_\alpha^*\overline{\Theta}\rangle = \langle d^2\overline{\omega}_{X/S,a} + 2d\overline{\alpha},\ d^2\overline{\omega}_{X/S,a} + 2d\overline{\alpha}\rangle - 4gd^2\langle\overline{\alpha}, \overline{\alpha}\rangle.$$

This gives

$$\langle i_\alpha^*\overline{\Theta}, i_\alpha^*\overline{\Theta}\rangle = d^4\langle\overline{\omega}_{X/S,a}\,\overline{\omega}_{X/S,a}\rangle + 4d^3\langle\overline{\omega}_{X/S,a}\,\overline{\alpha}\rangle - (4g-4)d^2\langle\overline{\alpha}, \overline{\alpha}\rangle.$$

and

$$(g-1)\langle i_\alpha^*\overline{\Theta}, i_\alpha^*\overline{\Theta}\rangle = gd^4\langle\overline{\omega}_{X/S,a}\,\overline{\omega}_{X/S,a}\rangle - d^2\langle(2g-2)\overline{\alpha} - d\overline{\omega}_{X/S,a},\ (2g-2)\overline{\alpha} - d\overline{\omega}_{X/S,a}\rangle.$$

Then it suffices to prove

$$\langle(2g-2)\overline{\alpha} - d\overline{\omega}_{X/S,a},\ (2g-2)\overline{\alpha} - d\overline{\omega}_{X/S,a}\rangle = -\iota_\alpha^*\overline{\Theta}$$

in $\widehat{\mathrm{Pic}}(S)_\mathbb{Q}$. This follows from the arithmetic Hodge index theorem in Theorem 2.8.

It remains to prove (1). Denote

$$\overline{N} = i_\alpha^*\overline{\Theta} - d^2\overline{\omega}_{X/S,a} - 2d\overline{\alpha} + \pi^*\pi_*\langle\overline{\alpha}, \overline{\alpha}\rangle.$$

We need to prove that $\overline{N} = 0$ in $\widehat{\mathrm{Pic}}(X)_\mathbb{Q}$.

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First, we prove that for any section $x : S \to X$ of $\pi : X \to S$, the pull-back $x^*\overline{N} = 0$ in $\widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$. In fact, $i_\alpha : X \to J$ maps x to the section $\iota : S \to J$ corresponding to the line bundle $d\mathcal{O}(x) - \alpha$ on X. By the arithmetic Hodge index theorem in Theorem 2.8,

$$x^* i_\alpha^* \overline{\Theta} = \iota^* \overline{\Theta} = -\pi_* \langle d\overline{\mathcal{O}}(x)_a - \overline{\alpha}, \, d\overline{\mathcal{O}}(x)_a - \overline{\alpha} \rangle.$$

By the adjunction formula in Proposition 2.7,

$$x^* \overline{\alpha} = \langle \overline{\mathcal{O}}(x)_a, \, \overline{\alpha} \rangle$$

and

$$x^* \overline{\omega}_{X/S,a} = \langle \overline{\mathcal{O}}(x)_a, \, \overline{\omega}_{X/S,a} \rangle = -\langle \overline{\mathcal{O}}(x)_a, \, \overline{\mathcal{O}}(x)_a \rangle.$$

Then $x^* \overline{N} = 0$ by bi-linearity of the Deligne pairing.

Second, we extend the result to the statement that, for any flat and quasi-projective normal integral scheme S' over S, and for any S-morphism $y : S' \to X$, the pull-back $y^* \overline{N} = 0$ in $\widehat{\mathrm{Pic}}(S')_{\mathbb{Q}}$. In fact, denote by $\pi' : X' \to S'$ the base change of $\pi : X \to S$ by $S' \to S$. Then $y : S' \to X$ induces a section $y' : S' \to X'$ of $\pi' : X' \to S'$. Denote by $\overline{N}'$ the pull-back of $\overline{N}$ to $\widehat{\mathrm{Pic}}(X')_{\mathbb{Q}}$. It suffices to prove $y'^* \overline{N}' = 0$ in $\widehat{\mathrm{Pic}}(S')_{\mathbb{Q}}$. This is reduced to the above case.

Third, in above statement, take $S' = X$ and take $y : S' \to X$ to be the identity section. Then $y^* \overline{N} = 0$ in $\widehat{\mathrm{Pic}}(S')_{\mathbb{Q}}$ just means $\overline{N} = 0$ in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$. This finishes the proof. $\qquad\square$

2.3.3 Pull-back formula: explicit examples

The following special examples of the theorem will be very useful for our later considerations.

Theorem 2.10. *Assume that $g > 1$.*

(1) Let $\omega = \omega_{X/S}$ be the canonical bundle of X over S. Then

$$i_\omega^* \overline{\Theta} = 4g(g-1)\overline{\omega}_{X/S,a} - \pi^* \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle,$$

in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$, and

$$\pi_* \langle i_\omega^* \overline{\Theta}, i_\omega^* \overline{\Theta} \rangle = 16g(g-1)^3 \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$$

in $\widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$. Therefore, $\overline{\omega}_{X/S,a}$ is nef on X, and $\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$ is nef on S.

(2) Let j be the morphism

$$j : X \times_S X \longrightarrow J, \quad (x, y) \longmapsto y - x.$$

Then

$$j^* \overline{\Theta} = 2\overline{\mathcal{O}}(\Delta)_a + p_1^* \overline{\omega}_{X/S,a} + p_2^* \overline{\omega}_{X/S,a}$$

in $\widehat{\mathrm{Pic}}(X \times_S X)_{\mathbb{Q}}$, and

$$p_{1*} \langle j^* \overline{\Theta}, j^* \overline{\Theta} \rangle = 4g\, \overline{\omega}_{X/S,a} + \pi^* \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$$

in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$.

(3) Let τ be the morphism

$$\tau : J \times_S X \longrightarrow J \times_S J, \quad (y, x) \longmapsto (y, y + (2g - 2)x - \omega_{X/S}).$$

Then

$$q_{1*} \langle \tau^* (\overline{\Theta}_J), \tau^* (\overline{\Theta}_J) \rangle = 16(g-1)^3 \overline{\Theta} + 16g(g-1)^3 \pi_J^* \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$$

in $\widehat{\mathrm{Pic}}(J)_{\mathbb{Q}}$. Here $\pi_J : J \to S$ denotes the structure morphism.

Proof. Note that (1) is a direct consequence of Theorem 2.9(1)(2). The nefness of $\overline{\omega}_{X/S,a}$ comes from the nefness of $\overline{\Theta}$ by the formula

$$\overline{\omega}_{X/S,a} = \frac{1}{4g(g-1)} i_\omega^* \overline{\Theta} + \frac{1}{64g^2(g-1)^4} \pi^* \pi_* \langle i_\omega^* \overline{\Theta}, i_\omega^* \overline{\Theta} \rangle.$$

For (2), the key is to interpret j in the form of Theorem 2.9(3). Denote $X_X = X \times_S X$ and $J_X = X \times_S J$, viewed as X-schemes via the first projections $p_1 : X_X \to X$ and $p_1 : J_X \to X$. Then $p_1 : J_X \to X$ is canonically isomorphic to the Jacobian scheme of $p_1 : X_X \to X$. View $\Delta : X \to X \times_S X = X_X$ as a section of $p_1 : X_X \to X$, which defines an X-morphism

$$i_\Delta = i_{\mathcal{O}(\Delta)} : X_X \longrightarrow J_X, \quad x \longmapsto x - \Delta.$$

Denote by $\overline{\Theta}_X$ the pull-back of the adelic line bundle $\overline{\Theta}$ via the projection $J_X \to J$.

Apply Theorem 2.9(3) to the X-morphism $i_\Delta : X_X \to J_X$. We obtain

$$i_\Delta^* (\overline{\Theta}_X) = 2\overline{\mathcal{O}}(\Delta)_a + p_1^* \overline{\omega}_{X/S,a} + p_2^* \overline{\omega}_{X/S,a}$$

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in $\widehat{\mathrm{Pic}}(X_X)$, and

$$p_{1*}\langle i_\Delta^*(\overline{\Theta}_X), i_\Delta^*(\overline{\Theta}_X)\rangle = 4g\,\overline{\omega}_{X/S,a} + \pi^*\pi_*\langle\overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle$$

in $\widehat{\mathrm{Pic}}(X)$.

On the other hand, we can write i_Δ as

$$i_\Delta : X \times_S X \longrightarrow X \times_S J, \quad (x,y) \longmapsto (x, y-x).$$

Thus $j : X \times_S X \to J$ is equal to the composition of i_Δ with the projection $p_2 : X \times_S J \to J$. This gives $j^*(\overline{\Theta}) = i_\Delta^*(\overline{\Theta}_X)$. Then (2) is proved.

To prove (3), we first consider the case that $\pi : X \to S$ has a section $x : S \to X$. This gives an immersion

$$(\mathrm{id}, i_x) : J \times_S X \longrightarrow J \times_S J.$$

Then we have a universal line bundle $Q = (\mathrm{id}, i_x)^* P$ on $J \times_S X$, where P is the Poincaré line bundle on $J \times_S J$. As in [YZ2, §6.5.2] or [YZ2, Thm. 6.1.2], there is a canonical extension $\overline{Q}$ of Q in $\widehat{\mathrm{Pic}}(J \times_S X)_{\mathrm{int}}$ such that $[2]_X^* \overline{Q} = 2\overline{Q}$. Here $[2]_X : J \times_S X \to J \times_S X$ is the base change of $[2] : J \to J$.

Write $X_J = J \times_S X$ and $J_J = J \times_S J$, viewed as J-schemes by projections to the first factors. Then J_J is canonically isomorphic to the Jacobian scheme of X_J over J. We claim that the morphism

$$\tau : J \times_S X \longrightarrow J \times_S J, \quad (y,x) \longmapsto (y, y + (2g-2)x - \omega_{X/S})$$

is equal to

$$i_{\omega - Q} : X_J \longrightarrow J_J, \quad x \longmapsto (2g-2)x - (\omega_{X/S} - Q).$$

To prove the claim, we can assume that S is the spectrum of an algebraically closed field. Then for any point $\beta \in J(S)$, the fiber of $i_{\omega - Q} : X_J \to J_J$ above β is isomorphic to the morphism

$$i_{\omega - Q_\beta} : X \longrightarrow J, \quad x \longmapsto (2g-2)x - (\omega_{X_J/J} - Q_\beta).$$

By the universal property, $Q_\beta = Q|_{X \times_S \beta} = \beta$ as a line bundle on X. This proves the claim.

With the claim, apply Theorem 2.9(2) to $i_{\omega - Q_\beta} : X_J \to J_J$. It gives

$$q_{1*}\langle\tau^*(\overline{\Theta}_J),\, \tau^*(\overline{\Theta}_J)\rangle = 16g(g-1)^3\pi_J^*\pi_*\langle\overline{\omega}_{X/S,a},\, \overline{\omega}_{X/S,a}\rangle + 16(g-1)^3\iota^*\overline{\Theta}.$$

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Here $\iota : J \to J$ is the morphism corresponding to the line bundle Q on $J \times_S X$. Then ι is exactly the identity morphism. See also the universal Hodge index theorem in [YZ2, Cor. 6.5.5]. This proves the result assuming that $X \to S$ has a section.

As in the proof of Theorem 2.9, it is easy to extend this result to the general case. In fact, take the base change of everything by $X \to S$, which converts to the case with sections, and then we can recover the original case by Lemma 2.1(1). $\qquad\square$

3 Bigness of the admissible canonical bundle

The goal of this section is to study the positivity of the admissible canonical line bundle $\overline{\omega}_{X/S,a}$ in Theorem 2.3. The following is the main result of this section.

Theorem 3.1. *Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$ with maximal variation. Then the admissible canonical bundle $\overline{\omega}_{X/S,a}$ is nef and big on X.*

Recall that the relative curve $\pi : X \to S$ has *maximal variation* if the moduli morphism $S \to M_{g,k}$ is generically finite, where $M_{g,k}$ denotes the coarse moduli scheme of smooth curves of genus g over k. Note that we require $g > 1$ in this section, while we only require $g > 0$ in the previous section. An equivalent form of the theorem is as follows.

Theorem 3.2. *Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$ with maximal variation. Then the Deligne pairing $\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$ is nef and big on S.*

By Lemma 2.2(1), the Deligne pairing of nef and big adelic line bundles is still nef and big. Then Theorem 3.2 is an easy consequence of Theorem 3.1. However, in our treatment, we will first prove Theorem 3.2 and then use it to deduce Theorem 3.1 by the following argument.

Proof of Theorem 3.1 by Theorem 3.2. By Theorem 2.10(1), $\overline{\omega}_{X/S,a}$ is nef on X, and
$$4g(g-1)\overline{\omega}_{X/S,a} = i_\omega^* \overline{\Theta} + \pi^* \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle.$$

It suffices to estimate the self-intersection of the right-hand side. Denote by $d = \dim S$. Then the self-intersection of the right-hand side contains a term

$$i_\omega^* \overline{\Theta} \cdot \left(\pi^* \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle \right)^d = a \left(\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle \right)^d > 0.$$

Here $a = 4g(g-1)(2g-2)$ is the degree of $i_\omega^* \overline{\Theta}$ on the generic fiber of $X \to S$. This finishes the proof. $\qquad\square$

The idea of the proof of Theorem 3.2 is explained in the introduction. The first three subsections of this section are preparatory results for the proof, and then the last two subsections give the proofs of the geometric case and the arithmetic case respectively.

3.1 Basics on the moduli space of curves

In this subsection, we review some standard notions on moduli spaces of curves and stable curves.

3.1.1 Basics on the moduli space

We refer to [DM, §1] (or our §1.6) for quick definitions of stable curves.

Let k be either $\mathbb{Z}$ or a field. Fix an integer $g > 1$. Denote by $\mathcal{M}_g$ the moduli stack of smooth curves of genus g over k, and by $\overline{\mathcal{M}}_g$ the moduli stack of stable curves of genus g over k. Denote by $\pi_g : \mathcal{C}_g \to \mathcal{M}_g$ and $\pi_g : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ respectively the universal curves of the stacks.

By [DM, Thm. 5.2], $\mathcal{M}_g$ is smooth over k, and $\overline{\mathcal{M}}_g$ is proper and smooth over k. Denote by M_g and $\overline{M}_g$ the coarse moduli schemes of $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ respectively. We refer to [DM] for more details on the moduli stacks.

By the standard theory of stacks, line bundles and Cartier divisors on the stacks are defined in terms of descent. We refer to [AC, §1] and [CH, §4] for explanation on line bundles on $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$. Or one can always take a fine level structure to avoid stacks. Over $\overline{\mathcal{M}}_g$, there are a Hodge bundle λ and tautological divisors $\Delta_0, \cdots, \Delta_{[g/2]}$ defined as follows.

The Hodge bundle on $\overline{\mathcal{M}}_g$ is defined by

$$\lambda := \det \pi_{g*}(\omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}) = \wedge^g \pi_{g*}(\omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}),$$

where $\omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$ is the relative dualizing sheaf. It is a line bundle on $\overline{\mathcal{M}}_g$. Note that in the literature, the term "Hodge bundle" may also mean the vector

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bundle $\pi_{g*}(\omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g})$, but in this paper it always means the determinant of $\pi_{g*}(\omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g})$.

By [DM, Thm. 5.2],

$$\Delta := \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$$

is a divisor of normal crossing in $\overline{\mathcal{M}}_g$. In terms of "prime divisors", we can write

$$\Delta = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{[g/2]},$$

where Δ_i for $i > 0$ (resp. $i = 0$) is irreducible and parametrizes non-smooth stable curves C such that the partial normalization of C at one of its nodes consists of two connected components of arithmetic genera i and $g - i$ (resp. is connected).

The following universal Noether's formula is obtained by Mumford [Mum, p. 102] as an easy application of Grothendieck's Riemann–Roch theorem.

Theorem 3.3 (Noether's formula). *In* $\mathrm{Pic}(\overline{\mathcal{M}}_g)$,

$$12\lambda = \pi_{g*}\langle \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}, \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}\rangle + \mathcal{O}_{\overline{\mathcal{M}}_g}(\Delta).$$

3.1.2 Pull-back to arbitrary families

Let $\overline{S}$ be an integral noetherian scheme. Let $\pi : \overline{X} \to \overline{S}$ be a stable relative curve of genus g. Assume that the generic fiber of $\overline{X}$ is smooth. By the moduli property, there is a morphism $\iota : \overline{S} \to \overline{\mathcal{M}}_g$ such that $\overline{X} \to \overline{S}$ is isomorphic to the base change of the universal curve $\overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$. We have the Hodge bundle

$$\lambda_{\overline{S}} = \iota^*\lambda \simeq \det \pi_*\omega_{\overline{X}/\overline{S}} = \wedge^g \pi_*\omega_{\overline{X}/\overline{S}}.$$

We also have the boundary divisors

$$\Delta_{\overline{S}} = \iota^*\Delta, \quad \Delta_{\overline{S},i} = \iota^*\Delta_i, \quad i = 0, 1, \cdots, [g/2].$$

They are effective Cartier divisors on $\overline{S}$. We have

$$\Delta_{\overline{S}} = \Delta_{\overline{S},0} + \Delta_{\overline{S},1} + \cdots + \Delta_{\overline{S},[g/2]}.$$

Via pull-back, Noether's formula in Theorem 3.3 gives

$$12\lambda_{\overline{S}} = \pi_*\langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}}\rangle + \mathcal{O}(\Delta_{\overline{S}}).$$

in $\mathrm{Pic}(\overline{S})$.

If $\overline{S}$ is normal, the divisor Δ_i on $\overline{S}$ can be defined explicitly as follows. It suffices to define the multiplicity $\mathrm{ord}_v(\Delta_i)$ of Δ_i along a codimension one point v of $\overline{S}$. The problem is further reduced to the curve $\overline{X}_{\mathcal{O}_{\overline{S},v}}$ over $\mathcal{O}_{\overline{S},v}$. This is essentially contained in the example of [CH, p. 464]. In fact, by a finite unramified base change of $\mathcal{O}_{\overline{S},v}$, we can assume that all nodes of $\overline{X}_v$ are rational over the residue field $k(v)$. For any node x of $\overline{X}_v$, denote its type by $i(x) \in \{0, 1, \cdots, [g/2]\}$. The multiplicity $m(x)$ of x is defined to be a positive integer such that the local equation of $\overline{X}$ at x is of type $t_1 t_2 = \varpi^{m(x)}$, where $\varpi \in \mathcal{O}_{\overline{S},v}$ is a generator of the maximal ideal. With these data, we have the formula

$$\mathrm{ord}_v(\Delta_i) = \sum_{x \in \overline{X}_v,\, i(x)=i} m(x).$$

There is another equivalent definition of the Hodge bundle, which is essential in the definition of the Faltings height. Resume the above setting; i.e., let $\overline{S}$ be an integral noetherian scheme, and let $\pi : \overline{X} \to \overline{S}$ be a stable relative curve of genus g. By [BLR, §9.4, Thm. 1], the Picard functor $\mathrm{Pic}_{\overline{X}/\overline{S}}$ is representable by an algebraic space over $\overline{S}$, and more importantly the relative identity component $\overline{J} = \mathrm{Pic}^0_{\overline{X}/\overline{S}}$ is represented by a (separated and smooth) semi-abelian group scheme over $\overline{S}$. The *Hodge bundle* of $\overline{J}$ over $\overline{S}$ is defined as

$$\underline{\omega}_{\overline{S}} := \det(e^* \Omega^1_{\overline{J}/\overline{S}}),$$

where $e : \overline{S} \to \overline{J}$ is the identity section. Then we have the following result, which is well-known to the experts.

Lemma 3.4. *There is a canonical isomorphism*

$$\lambda_{\overline{S}} \longrightarrow \underline{\omega}_{\overline{S}}.$$

Proof. It suffices to establish a canonical isomorphism

$$\pi_* \omega_{\overline{X}/\overline{S}} \longrightarrow e^* \Omega^1_{\overline{J}/\overline{S}}.$$

By duality, it suffices to establish a canonical isomorphism

$$\mathrm{Lie}(\overline{J}/\overline{S}) \longrightarrow R^1 \pi_* \mathcal{O}_{\overline{X}}.$$

This is a consequence of the deformation theory (cf. [BLR, §8.4, Thm. 1]). □

3.1.3 Stable compactification

Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve of genus $g > 1$.

By a *stable compactification* of $\pi : X \to S$, we mean a projective integral scheme $\overline{S}$ over k with an open immersion $S \to \overline{S}$, a stable relative curve $\overline{\pi} : \overline{X} \to \overline{S}$ of genus g, and an open immersion $X \to \overline{X}$ compatible with the previous morphisms.

For any smooth relative curve $\pi : X \to S$ as above, there is an integral scheme S' with a finite and surjective morphism $S' \to S$ such that the base change $\pi' : X' \to S'$ of $\pi : X \to S$ by $S' \to S$ has a stable compactification. If $\dim S = 1$, this fact is a consequence of the stable reduction theorem (or the properness of the stack $\overline{\mathcal{M}}_g$). If $\dim S > 1$, we sketch a proof as follows.

We claim that there is a projective normal integral scheme $\overline{\mathcal{M}}'_g$ over $\mathbb{Z}$, together with a (representable) finite and surjective morphism $\overline{\mathcal{M}}'_g \to \overline{\mathcal{M}}_g$. In fact, by the abstract result about Deligne–Mumford stacks in [LMB, Thm. 16.6], there is a (representable) finite and surjective morphism $\overline{\mathcal{M}}'_g \to \overline{\mathcal{M}}_g$ from a scheme $\overline{\mathcal{M}}'_g$. Then $\overline{\mathcal{M}}'_g$ is a proper scheme over $\mathbb{Z}$, and we need to prove that it is actually projective over $\mathbb{Z}$. For that, the key is that the coarse moduli scheme $\overline{M}_g$ of $\overline{\mathcal{M}}_g$ is projective over $\mathbb{Z}$ (cf. [CLM, Thm. 7.2]). Then it suffices to prove that the composition $\overline{\mathcal{M}}'_g \to \overline{\mathcal{M}}_g \to \overline{M}_g$ is a finite morphism of schemes. Since the composition is a morphism of proper schemes over $\mathbb{Z}$, it suffices to check that it is quasi-finite, and thus it suffices to check that for any algebraically closed field F, the composition $\overline{\mathcal{M}}'_g(F) \to \overline{\mathcal{M}}_g(F) \to \overline{M}_g(F)$ as a map of sets has finite fibers. This holds since the first arrow has finite fibers and the second arrow is bijective. Thus $\overline{\mathcal{M}}'_g$ is projective, and we can assume that it is integral and normal by passing to the normalization of a suitable irreducible component.

The scheme $\overline{\mathcal{M}}'_g$ is equipped with a "universal" stable curve $\overline{\mathcal{C}}'_g \to \overline{\mathcal{M}}'_g$ defined as the base change of the universal curve $\overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ by $\overline{\mathcal{M}}'_g \to \overline{\mathcal{M}}_g$.

Return to the existence of the stable compactification. we can take $S' \to S$ to be base change of $\overline{\mathcal{M}}'_g \to \overline{\mathcal{M}}_g$ by the moduli map $S \to \overline{\mathcal{M}}_g$, and a stable compactification is given by the Zariski closure of S' in $\overline{\mathcal{M}}'_g$. In this process, we may also need to replace S' by a suitable irreducible component to make it integral.

As we have just seen, neither $\mathcal{M}_g$ nor $\overline{\mathcal{M}}_g$ is (representable by) a scheme brings lots of inconveniences in application. We can find two approaches for

the purpose of this paper. The first approach is to use the above scheme $\overline{\mathcal{M}}_g'$ with its universal family. Another approach is to add a full level-N structure (for $N \geq 3$) to the definition of $\mathcal{M}_g$ to get a (fine) moduli scheme $\mathcal{M}_{g,N}$. This approach is more common and has the advantage of being explicit, but it has other issues including that it requires the base scheme to avoid the prime factors of N, and that there is no natural level-N structure to add to $\overline{\mathcal{M}}_g$. The second issue has a weak solution by van Geemen–Oort [GO], which constructs a stable compactification of the universal family of $\mathcal{M}_{g,N}$.

In this paper, both approaches can solve our related issues. However, we will usually take the approach using level structures, since it is more common in the literature.

3.2   Passing to the relative dualizing sheaf

Our first step to treat $\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$ in Theorem 3.2 is to convert it to the relative dualizing sheaf of a stable compactification.

Let k be a field. Note that we exclude the case $k = \mathbb{Z}$ here. Let S be a quasi-projective normal integral scheme over k. Let $\pi : X \to S$ a smooth relative curve of genus $g > 1$ with a stable compactification $\overline{\pi} : \overline{X} \to \overline{S}$ over k.

Note that the relative dualizing sheaf $\omega_{\overline{X}/\overline{S}}$ is naturally viewed as an adelic line bundle on X/k by the functor $\mathcal{P}\mathrm{ic}(\overline{X}) \to \widehat{\mathcal{P}\mathrm{ic}}(X/k)$. Consider the difference

$$\overline{\pi}_* \langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}} \rangle - \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle \simeq \mathcal{O}(\overline{E})$$

in $\widehat{\mathrm{Pic}}(S)$. Here we explain the right-hand side as follows. The underlying line bundle (over S) of left-hand side is canonically isomorphic to the trivial bundle $\mathcal{O}_S$. The section $1 \in \mathcal{O}_S$ corresponds to a rational section t of the left-hand side, and the adelic divisor

$$\overline{E} := \widehat{\mathrm{div}}(t)$$

on S has underlying divisor 0 on S.

There is a reasonably explicit description of $\overline{E}$ by the graph-theoretic approach of [Zha1]. In fact, to describe $\overline{E}$, it suffices to describe its image under the analytification map

$$\widehat{\mathrm{Div}}(S/k) \longrightarrow \widehat{\mathrm{Div}}(S^{\mathrm{an}})$$

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in [YZ2, Prop. 3.3.1]. As the underlying divisor of $\overline{E}$ on S is 0, it suffices to describe its total Green's function

$$\widetilde{g}_{\overline{E}} : S^{\mathrm{an}} \longrightarrow \mathbb{R}.$$

By [YZ2, Lem. 3.1.1(1)], the set of $v \in S^{\mathrm{an}}$ corresponding to discrete valuations of $k(S)$ is dense in S^{an}. By continuity, it suffices to describe $\widetilde{g}_{\overline{E}}(v)$ for any $v \in S^{\mathrm{an}}$ corresponding to a discrete valuation. In this case, by [Zha1, Thm. 4.4],

$$\widetilde{g}_{\overline{E}}(v)/\log e_v = \epsilon(\Gamma_v) = \int_{\Gamma_v} g_\mu(x,x)((2g-2)\mu + \delta_{K_X,v}).$$

Here Γ_v is the reduction graph of X over the discrete valuation field H_v, and $e_v = |\varpi_v|^{-1}$, where ϖ_v is a generator of the maximal ideal of the valuation ring O_{H_v}. Note that the loc. cit. is about the case $e_v = e$, but the result can be transferred to general e_v by the norm-equivariance property in [YZ2, §3.2].

By this description, we have the following result. Recall that an adelic divisor is effective if it is the limit of effective model adelic divisors.

Lemma 3.5. *The difference* $(2g-2)\Delta_{\overline{S}} - \overline{E}$ *is an effective adelic divisor in* $\widehat{\mathrm{Div}}(S)$.

Proof. Note that both adelic divisors $\Delta_{\overline{S}}$ and $\overline{E}$ have trivial underlying divisors in $\mathrm{Div}(S)$. By [YZ2, Lem. 5.1.2], it suffices to prove that the corresponding total Green's function $(2g-2)\widetilde{g}_{\Delta_{\overline{S}}} - \widetilde{g}_{\overline{E}}$ is non-negative on S^{an}. By continuity and density, it suffices to prove that $(2g-2)\widetilde{g}_{\Delta_{\overline{S}}}(v) \geq \widetilde{g}_{\overline{E}}(v)$ for discrete valuations $v \in S^{\mathrm{an}}$. Note that $\widetilde{g}_{\Delta_{\overline{S}}}(v)/\log e_v = \ell(\Gamma_v)$ is exactly the total length of the reduction graph Γ_v. Thus it suffices to check $\epsilon(\Gamma_v) \leq (2g-2)\ell(\Gamma_v)$.

This is basic in graph theory. In fact, by [Zha3, eq. (4.1.4)],

$$\epsilon(\Gamma_v) = \int_{\Gamma_v \times \Gamma_v} r(x,y)\delta_{K_X,v}(x)\mu(y).$$

Here $r(x,y)$ is the resistance function defined right before [Zha1, Prop. 3.3] or equivalently in [BF, Def. 12]. The inequality follows from the bound $r(x,y) \leq \ell(\Gamma_v)$, which in turn follows from the bound $r(x,y) \leq \ell(L_{x,y})$, where $L_{x,y}$ is a path in Γ_v connecting x,y.

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The intuitive bound $r(x, y) \leq \ell(L_{x,y})$ is proved in [KR, Thm. D] as a consequence of Rayleigh's monotonicity law. To fit our terminology, we sketch a proof following the idea of [BF, Exercise 12]. In fact, set $y = x$ in [BF, Thm. 8], we have an equality

$$j_z(x, x) - j_z(w, x) = j_w(x, x) - j_w(z, x).$$

By [BF, Exercise 9], $j_z(w, x) \geq 0$ and $j_w(z, x) \leq j_w(z, z)$, so the equality implies

$$j_z(x, x) \geq j_w(x, x) - j_w(z, z).$$

This is just the triangle inequality

$$r(x, w) \leq r(x, z) + r(z, w).$$

With the triangle inequality, we can reduce the proof of $r(x, y) \leq \ell(L_{x,y})$ to the case that $L_{x,y}$ is an edge of the graph Γ_v with vertices x, y. Here we can assume that x, y lie in the vertex set of Γ_v by taking a subdivision. Then the equality follows from [BF, Thm. 10]. $\qquad \square$

It is reasonable to say that $\overline{E}$ is the *globalization* of the ϵ-invariant of [Zha1], and $\Delta_{\overline{S}}$ is the *globalization* of the ℓ-invariant (or the δ-invariant). In the following, we will see an adelic divisor $\overline{\Phi}$, the *globalization* of the φ-invariant of Zhang [Zha3].

3.3 Lower bound of the pairing

We also need a lower bound of $\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$, which is given by a globalization of the φ-invariant of Zhang [Zha3].

3.3.1 Zhang's φ-invariant

We first recall the φ-invariant of curves over local fields from [Zha3]. Let C be a smooth projective curve of genus $g > 1$ over a complete field K. The invariant $\varphi(C)$ is defined as the local intersection number

$$\varphi(C) = - \int_{(C \times C)^{\mathrm{an}}} g_{\Delta,a} \, c_1(\mathcal{O}(\overline{\Delta})_a)^2.$$

As in [Zha3, Thm 1.3.1], it has explicit expressions as follows.

If K is non-archimedean with a discrete valuation, then

$$\varphi(C) = -\frac{1}{4}\delta(C) + \frac{1}{4}\int_{\Gamma(C)} g_\mu(x, x)((10g + 2)\mu - \delta_{K_C}).$$

Here the terms in the integration are as in §A.5. The invariant $\delta(C)$ is the total length $\ell(\Gamma)$ of the reduction graph Γ of C over K. If C has split semi-stable reduction, $\delta(C)$ is also the total number of nodes on the minimal regular model of C over O_K; in general, if K' is a finite extension of K such that C has split semi-stable reduction over K', then $\delta(C) = \delta(C_{K'})/[K' : K]$. We refer to [Zha3, Lem 3.5.4] for more details on this expression.

If $K = \mathbb{R}$, define $\varphi(C) = \varphi(C_\mathbb{C})$. If $K = \mathbb{C}$, the computation of [Zha3, Prop 2.5.3] gives

$$\varphi(C) = \sum_\lambda \sum_{j=1}^g \sum_{k=1}^g \frac{2}{\lambda_n} \left| \int_C \varphi_\lambda \, \omega_j \wedge \overline{\omega}_k \right|^2.$$

Here $\omega_1, \cdots, \omega_g$ is an orthonormal basis of $\Gamma(C, \Omega_C^1)$ with respect to the inner product

$$\langle \alpha, \beta \rangle = \frac{i}{2} \int_C \alpha \wedge \overline{\beta},$$

the first summation goes over all positive eigenvalues λ of the Laplacian operator

$$\Delta_{d\mu} f = (\frac{1}{\pi i} \partial \overline{\partial} f)/d\mu$$

over $C^\infty(C_\mathbb{C})$, and φ_λ is an eigenvector of λ normalized such that $\{\varphi_\lambda\}_\lambda$ is orthonormal with respect to the inner product

$$\langle f_1, f_2 \rangle = \int_C f_1 \overline{f}_2 d\mu.$$

We refer to [Zha3, Prop 2.5.3] for more details on this expression.

3.3.2   Globalization $\overline{\Phi}$ of the φ-invariant

Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve of genus $g > 1$.

Consider the morphism $(\pi, \pi) : X \times_S X \to S$ and its diagonal divisor $\Delta : X \to X \times_S X$. There are canonical isomorphisms

$$(\pi, \pi)_* \langle \mathcal{O}(\Delta), \mathcal{O}(\Delta), \mathcal{O}(\Delta) \rangle \longrightarrow \pi_* \langle \Delta^* \mathcal{O}(\Delta), \Delta^* \mathcal{O}(\Delta) \rangle \longrightarrow \pi_* \langle \omega, \omega \rangle.$$

This defines a section s of the underlying line bundle of the adelic line bundle

$$\pi_* \langle \overline{\omega}, \overline{\omega} \rangle - (\pi, \pi)_* \langle \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a \rangle.$$

Via this adelic line bundle, define

$$\overline{\Phi} = \widehat{\mathrm{div}}(s),$$

viewed as an adelic divisor on S. The underlying divisor of $\overline{\Phi}$ is 0 by definition. By the integration formula, the total Green's function $\widetilde{g}_{\overline{\Phi}}$ on S^{an} at any discrete or archimedean valuation $v \in S^{\mathrm{an}}$ is given by

$$\widetilde{g}_{\overline{\Phi}}(v) = \int_{X_v^{\mathrm{an}}} \log \|1\|_{\Delta,a} c_1(\overline{\mathcal{O}}(\Delta)_a)^2.$$

By [Zha3, Prop. 2.5.3, Lem. 3.5.4], this is exactly the φ-invariant of X_{H_v} in the standard case $e_v = e$. It transfers to general e_v by the norm-equivariance property in [YZ2, §3.2].

3.3.3 The lower bound

Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve of genus $g > 1$. Denote by $(\pi, \pi) : X \times_S X \to S$ the structure morphism as before.

Let $J = \mathrm{Pic}^0_{X/S}$ be the Jacobian scheme. Recall that we have a symmetric and relatively ample line bundle Θ on J by Definition 2.4. Recall that $\overline{\Theta}$ is the nef adelic line bundle on J/k extending Θ such that $[2]^* \overline{\Theta} = 4 \overline{\Theta}$ in $\widehat{\mathrm{Pic}}(J)$.

Let j be the morphism

$$j : X \times_S X \longrightarrow J, \quad (x, y) \longmapsto y - x.$$

Recall that in Theorem 2.10(2), we have obtained a formula

$$j^* \overline{\Theta} = 2 \overline{\mathcal{O}}(\Delta)_a + p_1^* \overline{\omega}_{X/S,a} + p_2^* \overline{\omega}_{X/S,a}$$

in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$. The following result computes the Deligne pairing $(\pi, \pi)_* \langle j^* \overline{\Theta}, j^* \overline{\Theta}, j^* \overline{\Theta} \rangle$, which is nef by the nefness of $\overline{\Theta}$. It is a family version of and inspired by

de Jong [dJo2, Thm. 8.1], where the latter was in turn based on the main result of Zhang [Zha3].

As above, for a discrete valuation $v \in S^{\mathrm{an}}$, denote $e_v = |\varpi_v|_v^{-1}$, where ϖ_v is a generator of the maximal ideal of the valuation ring O_{H_v}; for an archimedean valuation $v \in S^{\mathrm{an}}$, denote $e_v = |e|_v$.

Theorem 3.6. *Let k be either $\mathbb{Z}$ or a field. There is an identity in $\widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$ given by*

$$(\pi, \pi)_* \langle j^* \overline{\Theta}, j^* \overline{\Theta}, j^* \overline{\Theta} \rangle = (12g - 4) \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle - 8\mathcal{O}(\overline{\Phi}).$$

Proof. The proof follows the framework of [dJo2, Thm. 8.1]. Denote $\overline{\omega} = \overline{\omega}_{X/S,a}$ and $\overline{\omega}_i = p_i^* \overline{\omega}_{X/S,a}$ for $i = 1, 2$. Expand $(\pi, \pi)_* \langle j^* \overline{\Theta}, j^* \overline{\Theta}, j^* \overline{\Theta} \rangle$ by the formula

$$j^* \overline{\Theta} = 2\overline{\mathcal{O}}(\Delta)_a + \overline{\omega}_1 + \overline{\omega}_2.$$

We have

$$\begin{aligned}
&\langle j^* \overline{\Theta}, \, j^* \overline{\Theta}, \, j^* \overline{\Theta} \rangle \\
=\ & 8\langle \overline{\mathcal{O}}(\Delta)_a, \, \overline{\mathcal{O}}(\Delta)_a, \, \overline{\mathcal{O}}(\Delta)_a \rangle + 12\langle \overline{\mathcal{O}}(\Delta)_a, \, \overline{\mathcal{O}}(\Delta)_a, \, \overline{\omega}_1 + \overline{\omega}_2 \rangle \\
& + 6\langle \overline{\mathcal{O}}(\Delta)_a, \, \overline{\omega}_1 + \overline{\omega}_2, \, \overline{\omega}_1 + \overline{\omega}_2 \rangle + \langle \overline{\omega}_1 + \overline{\omega}_2, \, \overline{\omega}_1 + \overline{\omega}_2, \, \overline{\omega}_1 + \overline{\omega}_2 \rangle.
\end{aligned}$$

For further simplifications, we need the following results:

(1) $\langle \overline{\omega}_1, \overline{\omega}_1, \overline{\omega}_1 \rangle = \langle \overline{\omega}_2, \overline{\omega}_2, \overline{\omega}_2 \rangle = 0.$

(2) $\langle \overline{\omega}_1, \overline{\omega}_1, \overline{\omega}_2 \rangle = \langle \overline{\omega}_2, \overline{\omega}_2, \overline{\omega}_1 \rangle = (2g - 2) \pi_* \langle \overline{\omega}, \overline{\omega} \rangle.$

(3) $\langle \overline{\mathcal{O}}(\Delta)_a, \overline{\omega}_i, \overline{\omega}_j \rangle = \pi_* \langle \overline{\omega}, \overline{\omega} \rangle, \quad i, j \in \{1, 2\}.$

(4) $\langle \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a, \overline{\omega}_1 \rangle = \langle \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a, \overline{\omega}_2 \rangle = -\pi_* \langle \overline{\omega}, \overline{\omega} \rangle.$

(5) $\langle \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a \rangle = \pi_* \langle \overline{\omega}, \overline{\omega} \rangle - \mathcal{O}(\overline{\Phi}).$

It is easy to derive the result from (1)-(5). In fact, (1)-(4) gives

$$\langle j^* \overline{\Theta}, \, j^* \overline{\Theta}, \, j^* \overline{\Theta} \rangle = 8\langle \overline{\mathcal{O}}(\Delta)_a, \, \overline{\mathcal{O}}(\Delta)_a, \, \overline{\mathcal{O}}(\Delta)_a \rangle + 6(2g - 2) \pi_* \langle \overline{\omega}, \overline{\omega} \rangle.$$

Then apply (5).

Note that (5) follows from the definition of $\overline{\Phi}$. The idea to prove the identities in (1)-(4) is to apply [YZ2, Lem. 4.6.1(2)] to

$$X \times_S X \xrightarrow{p_i} X \xrightarrow{\pi} S.$$

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This gives (for $i = 1$)

$$\langle \overline{\omega}_1, \overline{\alpha}, \overline{\beta} \rangle = \pi_* \langle \overline{\omega}, p_{1*} \langle \overline{\alpha}, \overline{\beta} \rangle \rangle.$$

If $\alpha = \omega_1$, then we can apply [YZ2, Lem. 4.6.1(2)] to compute $p_{1*} \langle \overline{\alpha}, \overline{\beta} \rangle$. This solves (1), (2) and (3) with $i = j$. The remaining case of (3) is a consequence of $p_{1*} \langle \overline{\omega}, \overline{\mathcal{O}}(\Delta)_a \rangle = \overline{\omega}$ from Theorem 2.3(2).

For (4), it suffices to check

$$p_{1*} \langle \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a \rangle = -\overline{\omega}.$$

There is a canonical isomorphism $\Delta^* \overline{\mathcal{O}}(\Delta)_a \to -\overline{\omega}$ by Theorem 2.3(1). It suffices to have a canonical isomorphism

$$p_{1*} \langle \overline{\mathcal{O}}(\Delta)_a, \overline{\mathcal{O}}(\Delta)_a \rangle \longrightarrow \Delta^* \overline{\mathcal{O}}(\Delta)_a.$$

This follows from Proposition 2.7(1) applied to the section $\Delta : X \to X \times_S X$ of the smooth relative curve $p_1 : X \times_S X \to X$. Alternatively, it is also easy to modify the proof of Proposition 2.7(1) to the current situation. In fact, by the analytification functor $\widehat{\mathcal{P}ic}(X) \to \widehat{\mathcal{P}ic}(X^{\mathrm{an}})$, it suffices to prove that the norm of the above morphism is 1 at any $v \in X^{\mathrm{an}}$. By the integration formulae in [YZ2, Thm. 4.6.2] and [YZ2, §4.2.2], the logarithm of the norm is equal to

$$- \int_{X^{\mathrm{an}}_{H_v}} g_{\Delta,a}(x, \cdot) c_1(x^* \overline{\mathcal{O}}(\Delta)_a).$$

Here H_v is the completed residue field of $v \in X^{\mathrm{an}}$, $x : \operatorname{Spec} H_v \to X$ is the point corresponding to v, and $g_{\Delta,a}$ is the admissible Green's function as in Theorem A.1. The integral is 0 by Theorem A.1(1)(2). This finishes the proof. □

Remark 3.7. For the sake of a lower bound of $\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$, the result could be strengthened by the method of [Wil1, Thm. 1.2] and [LSW, Prop. 6.1] combining with the Hodge index theorems of [YZ3, Car].

3.3.4 Cinkir's bound

The following result essentially follows from Cinkir [Cin1, Thm. 2.11], which was conjectured by Zhang [Zha3].

Theorem 3.8. *Let S be a quasi-projective normal integral scheme over a field k. Let $\pi : X \to S$ be a smooth relative curve of genus $g > 1$ with a stable compactification $\overline{\pi} : \overline{X} \to \overline{S}$ over k. Then $\overline{\Phi} - \dfrac{1}{39}\Delta_{\overline{S}}$ is an effective adelic divisor in $\widehat{\mathrm{Div}}(S)$.*

Proof. This is similar to Lemma 3.5. It suffices to prove $39\widetilde{g}_{\overline{\Phi}}(v) - \widetilde{g}_{\Delta_{\overline{S}}}(v) \geq 0$ for discrete valuations $v \in S^{\mathrm{an}}$. By [Cin1, Thm. 2.11],

$$39\varphi(\Gamma_v) \geq \sum_i \delta_i(\Gamma_v) = \ell(\Gamma_v).$$

The result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3.3.5 A special adelic line bundle

The φ-invariant was introduced by Zhang [Zha3] to study the Beilinson–Bloch height of the Gross–Schoen cycle. Then our adelic divisor $\overline{\Phi}$ can be used to study the variation of the height. The exposition here will not be used elsewhere of this paper.

Let K be either a number field, or the function field of one variable over a field k. Let C be a smooth projective curve over K of genus $g > 1$. Let ξ be a line bundle on C such that $(2g-2)\xi = \omega_{C/K}$ in $\mathrm{Pic}(C)$, which exists by replacing K by a suitable finite extension. By [Zha3, Thm. 1.3.1],

$$\langle \Delta_\xi(C), \Delta_\xi(C) \rangle = \frac{2g+1}{2g-2}\overline{\omega}_{C/K,a}^2 - \sum_v \varphi(C_v)\deg(v).$$

Here $\Delta_\xi(C)$ is the Gross–Schoen cycle, the modified diagonal cycle on C^3, and the left-hand side is its Beilinson–Bloch height.

Let S be a quasi-projective normal variety over K. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$. Define an adelic $\mathbb{Q}$-line bundle $\overline{L}$ on S by

$$\overline{L} = \frac{2g+1}{2g-2}\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle - \mathcal{O}(\overline{\Phi}).$$

Then the height function $h_{\overline{L}} : S(\overline{K}) \to \mathbb{R}$ exactly gives

$$h_{\overline{L}}(y) = \langle \Delta_\xi(X_y), \Delta_\xi(X_y)\rangle, \quad y \in S(\overline{K}).$$

This generalizes the first statement of [Zha3, Thm. 1.3.5] to the quasi-projective setting.

It is natural to ask wether $\overline{L}$ is nef on S. If $\pi : X \to S$ has maximal variation, we further ask whether $\overline{L}$ is big on S.

3.4 Bigness in the geometric case

In the geometric case that k is a field, Theorem 3.2 is a consequence of the following result.

Theorem 3.9. *Let k be a field. Let S be a quasi-projective normal integral scheme over k. Let $\pi : X \to S$ a smooth relative curve of genus $g > 1$ with a stable compactification $\overline{\pi} : \overline{X} \to \overline{S}$ over k. In $\widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$, we have*

$$\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle = \frac{3}{5(2g-1)(3g-1)} \lambda_{\overline{S}} + \overline{A} + \mathcal{O}(\overline{B})$$

for a nef adelic $\mathbb{Q}$-line bundle $\overline{A} \in \widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$ and an effective adelic divisor $\overline{B} \in \widehat{\mathrm{Div}}(S)$ with underlying divisor $B = 0$ in $\mathrm{Div}(S)$.

Proof. This is a combination of all the major results of this section. In fact, we first have in $\widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$,

$$\begin{aligned}
\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle &= \overline{\pi}_* \langle \omega_{\overline{X}/\overline{S}}, \omega_{\overline{X}/\overline{S}} \rangle - \mathcal{O}(\overline{E}) \\
&= 12 \lambda_{\overline{S}} - \mathcal{O}(\overline{E} + \Delta_{\overline{S}}) \\
&= 12 \lambda_{\overline{S}} - (2g-1)\mathcal{O}(\Delta_{\overline{S}}) + \mathrm{eff}.
\end{aligned}$$

Here "eff" denotes the adelic line bundle associated to an effective adelic divisor in $\widehat{\mathrm{Div}}(S)$ with underlying divisor 0 in $\mathrm{Div}(S)$. In fact, the first equality follows from the definition of $\overline{E}$ in §3.2, the second equality follows from Noether's formula in Theorem 3.3, and the last inequality follows from Lemma 3.5.

On the other hand, by Theorem 3.6 and Theorem 3.8,

$$\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle = \frac{2}{3g-1} \mathcal{O}(\overline{\Phi}) + \mathrm{nef} = \frac{2}{39(3g-1)} \mathcal{O}(\Delta_{\overline{S}}) + \mathrm{nef} + \mathrm{eff}.$$

Here "nef" denotes a nef element of $\widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$.

To cancel $\mathcal{O}(\Delta_{\overline{S}})$, a positive linear combination of these equalities gives

$$\left(1 + \frac{39(3g-1)}{2}(2g-1) \right) \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle = 12 \lambda_{\overline{S}} + \mathrm{nef} + \mathrm{eff}.$$

This implies the result after replacing the coefficient on the left-hand side by the slightly larger number $20(3g-1)(2g-1)$. $\qquad\square$

It is easy to see how Theorem 3.9 implies Theorem 3.2 in the geometric case. The nefness part is already proved in Theorem 2.10(1). For the bigness part, by passing to a finite extension of S, we can assume that $\pi : X \to S$ has a stable compactification $\overline{\pi} : \overline{X} \to \overline{S}$ over k. The normality of S can be kept by taking a further normalization. This process uses the fact that for any finite and surjective morphism $S' \to S$ from an integral scheme S', a nef adelic line bundle on S is big if and only if its pull-back to S' is big, which follows from the projection formula for top intersection numbers in [YZ2, Prop. 4.1.2].

Once we have the stable compactification, the key fact is that the Hodge bundle $\lambda_{\overline{S}}$ is nef and big on $\overline{S}$. If so, then

$$\frac{3}{5(2g-1)(3g-1)}\lambda_{\overline{S}} + \overline{A}$$

is nef and big on S, and

$$\frac{3}{5(2g-1)(3g-1)}\lambda_{\overline{S}} + \overline{A} + \mathcal{O}(\overline{B})$$

is big on S.

The nefness and bigness of the Hodge bundle $\lambda_{\overline{S}}$ on $\overline{S}$ are well known to the experts. For lack of a direct reference, we explain a proof by modifying the setting of [Knu] slightly. Denote by A_g the coarse moduli scheme of principally polarized abelian varieties of dimension g over k. Denote by $\overline{A}_g$ the minimal compactification of A_g. By [FC, V, Thm. 2.3], the Hodge $\mathbb{Q}$-line bundle $\omega_{\overline{A}_g}$ of $\overline{A}_g$ is ample. Denote by $t : \overline{M}_g \to \overline{A}_g$ the Torelli morphism induced by the functor of taking Jacobian schemes. The definition is similar to the map at the top of [Knu, p. 211]. The Hodge bundle λ_g on the stack $\overline{\mathcal{M}}_g$ descends to a $\mathbb{Q}$-line bundle $\lambda_{\overline{M}_g}$ on $\overline{M}_g$. As in [Knu, p. 211], we have $\lambda_{\overline{M}_g} = t^*\omega_{\overline{A}_g}$. Finally, let $\pi : \overline{X} \to \overline{S}$ be as in the theorem. This gives a morphism $\iota : \overline{S} \to \overline{M}_g$, whose restriction to S is generically finite. Note that $t : \overline{M}_g \to \overline{A}_g$ is finite on M_g, so the composition $t \circ \iota : \overline{S} \to \overline{A}_g$ is generically finite. Then $\lambda_{\overline{S}} = (t \circ \iota)^*\omega_{\overline{A}_g}$ on $\overline{S}$ is nef and big.

3.5 Bigness in the arithmetic case

In the arithmetic case $k = \mathbb{Z}$, Theorem 3.2 is consequence of the geometric case $k = \mathbb{Q}$ and the following result.

Theorem 3.10. *For any integer $g > 1$, there is a constant $c(g) > 0$ such that $\varphi(C) \geq c(g)$ for any compact Riemann surface C of genus g.*

Due to its arithmetic importance, asymptotic behavior of degeneration of the φ-invariant is widely studied in the literature. See [dJo1, Thm. 1.1] for a precise asymptotic formula for $g = 2$, which also gives a conjectural formula for $g > 2$. For $g \geq 2$ and degeneration to isolated singularities, the asymptotic formula was later proved by [JS2, Thm. 7.1] and [Wil2, Cor. 1.2]. These imply Theorem 3.10 in the case $g = 2$ and in the case of 1-parameter families of Riemann surfaces of genus $g \geq 2$. As we will see later, our proof of the theorem takes a different approach, which asserts that the φ-invariant goes to infinity under degeneration, but does not give a precise asymptotic behavior.

Let us first prove the arithmetic case of Theorem 3.2 assuming its geometric case and Theorem 3.10. Let $\pi : X \to S$ be as in Theorem 3.2. Let $c = c(g) > 0$ be as in Theorem 3.10. Denote by $\mathcal{O}(c)$ the trivial line bundle on $k = \mathbb{Z}$ with metric given by $\|1\| = e^{-c}$. View $\mathcal{O}(c)$ as an adelic line bundle on S by pull-back. By Theorem 3.6 and Theorem 3.10, we have

$$\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle = \frac{2}{3g-1}\mathcal{O}(\overline{\Phi}) + \text{nef} = \frac{2}{3g-1}\mathcal{O}(c) + \text{nef} + \text{eff}.$$

Denote $\overline{L} = \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$, and denote by $\widetilde{L}$ the image of $\overline{L}$ under the canonical map $\widehat{\text{Pic}}(S/\mathbb{Z}) \to \widehat{\text{Pic}}(S_{\mathbb{Q}}/\mathbb{Q})$. Recall that the map is defined in [YZ2, §2.5.5], as the limit of the canonical map $\widehat{\text{Div}}(\mathcal{X}) \to \text{Div}(\mathcal{X}_{\mathbb{Q}})$ for projective models $\mathcal{X}$ of S over $\mathbb{Z}$. Denote $d = \dim S$. Since $\overline{L}$ is nef, we have

$$\overline{L}^d \geq \overline{L}^{d-1} \cdot \frac{2}{3g-1}\mathcal{O}(c) = \frac{2c}{3g-1}\widetilde{L}^{d-1} > 0.$$

Here the last inequality uses Theorem 3.2 for the geometric case $X_{\mathbb{Q}} \to S_{\mathbb{Q}}$ over $\mathbb{Q}$. This proves the arithmetic case of Theorem 3.2.

It remains to prove Theorem 3.10. Our proof uses information of the arithmetic divisor $\overline{\Phi}$ in both the geometric case and the arithmetic case.

Proof of Theorem 3.10. Fix an integer $N \geq 3$, and denote by $\mathcal{M} = \mathcal{M}_{g,N,\mathbb{Q}}$ the (fine) moduli scheme of smooth curves of genus g over $\mathbb{Q}$ with a full level-N structure. Then $M_{g,N,\mathbb{Q}}$ is a smooth quasi-projective variety over $\mathbb{Q}$, which follows from the GIT construction in [MFK, §7.4]. By [GO, Thm.

2.1], there is a projective compactification $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,N,\mathbb{Q}}$ of $\mathcal{M}$ together with a tautological stable relative curve $\overline{\mathcal{C}} \to \overline{\mathcal{M}}$.

The φ-invariant defines a function $\varphi : \mathcal{M}(\mathbb{C}) \to \mathbb{R}$. Note the values of φ are strictly positive from Remark 1 after [Zha3, Prop 2.5.3]. We will see that φ is continuous on $\mathcal{M}(\mathbb{C})$ and tends to infinity around the boundary $\overline{\mathcal{M}}(\mathbb{C}) \setminus \mathcal{M}(\mathbb{C})$. These are sufficient to imply a positive lower bound of φ.

By Theorem 3.6, there is an adelic divisor $\overline{\Phi} \in \widehat{\mathrm{Div}}(\mathcal{M}/\mathbb{Z})$ with underlying divisor 0 on $\mathcal{M}$, such that the value $\widetilde{g}_{\overline{\Phi}}(v)$ of the total Green's function $\widetilde{g}_{\overline{\Phi}}$ at any discrete or archimedean valuation $v \in S_v^{\mathrm{an}}$ is equal to $\varphi(X_{H_v})$. Then $\varphi : \mathcal{M}(\mathbb{C}) \to \mathbb{R}$ is continuous by the continuity of $\widetilde{g}_{\overline{\Phi}}$.

Denote by $\widetilde{\Phi}$ the image of $\overline{\Phi}$ under the canonical composition

$$\widehat{\mathrm{Div}}(\mathcal{M}/\mathbb{Z}) \to \widehat{\mathrm{Div}}(\mathcal{M}/\mathbb{Q}) \to \widehat{\mathrm{Div}}(\mathcal{M}_{\mathbb{C}}/\mathbb{C}).$$

Recall that the map is defined in [YZ2, §2.5.5], as the limit of the canonical map $\widehat{\mathrm{Div}}(\mathcal{X}) \to \mathrm{Div}(\mathcal{X}_{\mathbb{C}})$ for projective models $\mathcal{X}$ of $\mathcal{M}$ over $\mathbb{Z}$.

Apply Theorem 3.8 to $k = \mathbb{C}$. We have that $\widetilde{\Phi} - \frac{1}{39}\Delta_{\overline{\mathcal{M}}_{\mathbb{C}}}$ is an effective adelic divisor in $\widehat{\mathrm{Div}}(\mathcal{M}_{\mathbb{C}}/\mathbb{C})$. This is sufficient to imply that φ tends to infinity around the boundary $\Delta_{\overline{\mathcal{M}}_{\mathbb{C}}} = \overline{\mathcal{M}}(\mathbb{C}) \setminus \mathcal{M}(\mathbb{C})$. To see that, we will use the theory of adelic divisors on the pair $\overline{B} = (\mathbb{C}, |\cdot|)$ in [YZ2, §2.7], which can be avoided, but its use makes the situation clear.

Let us first recall the theory of adelic divisors on $\overline{B} = (\mathbb{C}, |\cdot|)$. First, the group of model divisors are defined by

$$\widehat{\mathrm{Div}}(\mathcal{M}_{\mathbb{C}}/\overline{B})_{\mathrm{mod},\mathbb{Q}} = \varinjlim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X}/\overline{B})_{\mathbb{Q}},$$

where the limit is over the system of projective models $\mathcal{X}$ of $\mathcal{M}_{\mathbb{C}}$ over $\mathbb{C}$. Here $\widehat{\mathrm{Div}}(\mathcal{X}/\overline{B})$ is the group of pairs (D, g_D), where D is a Cartier divisor on $\mathcal{X}$ and g_D is a Green's function of D on $\mathcal{X}$.

Second, take a boundary divisor $(\overline{\mathcal{M}}_{\mathbb{C}}, \overline{D}_0)$ with $\overline{D}_0 = (D_0, g_0)$, where $D_0 = \Delta_{\overline{\mathcal{M}}_{\mathbb{C}}}$, and g_0 is a strictly positive Green's function of D_0 on $\overline{\mathcal{M}}_{\mathbb{C}}$.

Third, $\widehat{\mathrm{Div}}(\mathcal{M}_{\mathbb{C}}/\overline{B})_{\mathbb{Q}}$ is defined to be the completion of $\widehat{\mathrm{Div}}(\mathcal{M}_{\mathbb{C}}/\overline{B})_{\mathrm{mod},\mathbb{Q}}$ with respect to the topology induced by $(\overline{\mathcal{M}}_{\mathbb{C}}, \overline{D}_0)$.

Return to φ. The image of $\overline{\Phi}$ under the canonical map $\widehat{\mathrm{Div}}(\mathcal{M}/\mathbb{Z}) \to \widehat{\mathrm{Div}}(\mathcal{M}_{\mathbb{C}}/\overline{B})_{\mathbb{Q}}$ is represented by the pair $(\widetilde{\Phi}, \varphi)$. By the process of the completion, there is a sequence $\{(E_i, g_i)\}_{i \geq 1}$ in $\widehat{\mathrm{Div}}(\mathcal{M}_{\mathbb{C}}/\overline{B})_{\mathrm{mod},\mathbb{Q}}$ such that

$$-a_i(D_0, g_0) \leq (\widetilde{\Phi}, \varphi) - (E_i, g_i) \leq a_i(D_0, g_0)$$

for a sequence of rational numbers $\{a_i\}_{i \geq 1}$ converging to 0. This gives a bound

$$\varphi \geq g_i - a_i g_0.$$

Fix an i with $a_i < 1/78$. Then

$$E_i \geq \widetilde{\Phi} - a_i D_0 \geq (1/39 - a_i) D_0 > 0$$

is effective. Note that $g_i - a_i g_0$ is a Green's function of

$$E_i - a_i D_0 \geq (1/39 - a_i) D_0 - a_i D_0 = (1/39 - 2a_i) D_0 > 0.$$

As a consequence, $g_i - a_i g_0$ goes to infinity around D_0, and so does φ. This finishes the proof. $\qquad\square$

4 Uniform Bogomolov-type theorem

The main goal of this section is to prove the uniform Bogomolov-type theorem. In §4.1, we introduce a notion of potential bigness, prove a key criterion for it, and consider a quick consequence of this notion on small points. In §4.2, we introduce three examples of potential bigness, as consequences of the bigness of the admissible canonical bundle in Theorem 3.1. The exposition of the remaining subsections are based on these examples. In §4.3 and §4.4, we state and prove our uniform Bogomolov-type theorem. In §4.5, we prove a uniform fiberwise bigness of the admissible canonical bundle. Finally, in §4.6, we deduce some consequences on non-degenerate subvarieties and on the relative Bogomolov conjecture.

4.1 Potentially big line bundles

Let k be either $\mathbb{Z}$ or a field. Let S be a quasi-projective and flat normal integral scheme over k. Let Y be an integral scheme with a projective and flat morphism $\pi : Y \to S$ over k. Assume that the generic fiber of $\pi : Y \to S$ is geometrically integral. Let $\overline{L}$ be an adelic line bundle on Y/k.

Recall that $\overline{L}$ is *big* on Y if $\widehat{\mathrm{vol}}(\overline{L}) > 0$ (cf. [YZ2, §5.2]). If $\overline{L}$ is nef, then we have the adelic Hilbert–Samuel formula $\widehat{\mathrm{vol}}(\overline{L}) = \overline{L}^{\dim Y}$ (cf. [YZ2, Thm. 5.2.2]). In this case, $\overline{L}$ is big if and only if $\overline{L}^{\dim Y} > 0$.

We say that $\overline{L}$ is *potentially big on Y/S* if there is a positive integer m such that the adelic line bundle

$$\overline{L}^{\boxtimes m} := (p_1^*\overline{L}) \otimes (p_2^*\overline{L}) \otimes \cdots \otimes (p_m^*\overline{L})$$

is big on the m-fold fiber product

$$Y_{/S}^m := Y \times_S Y \times_S \cdots \times_S Y.$$

In additive notations, the above is written as

$$m_{\boxtimes}\overline{L} = p_1^*\overline{L} + p_2^*\overline{L} + \cdots + p_m^*\overline{L}.$$

We remark that $Y_{/S}^m$ is always integral under the assumptions that Y and S are integral, and that $\pi : Y \to S$ is projective and flat with a geometrically integral generic fiber. In fact, the generic fiber of $Y_{/S}^m \to S$ is geometrically integral. By flatness, $Y_{/S}^m \to S$ is equi-dimensional. Then $Y_{/S}^m$ is irreducible since its generic fiber is irreducible. To prove that $Y_{/S}^m$ is reduced, note that its generic fiber over S is reduced, it suffices to check that it has no embedded components. Then it suffices to check that $Y_{/S}^m$ satisfies property S_1 (cf. [Liu, §8.2, 2.19, 2.20]). This follows from flatness. In fact, for any $y \in Y_{/S}^m$ with image $s \in S$, if s is not the generic point of S, then the flatness implies that $\mathrm{depth}_y(Y_{/S}^m) \geq \mathrm{depth}_s(S) \geq 1$. If s is the generic point of S, y lies in the generic fiber of $Y_{/S}^m \to S$, which is integral and thus satisfies S_1.

4.1.1   A quick criterion

Concerning potential bigness, we have the following key criterion.

Lemma 4.1. *Denote $d = \dim S$. Assume that $\overline{L}$ is nef on Y, and the degree of L on fibers of $\pi : Y \to S$ is strictly positive. Then the following are equivalent:*

(1) $\pi_*\langle \overline{L}, \cdots, \overline{L}\rangle$ is big on S;

(2) $m_{\boxtimes}\overline{L}$ is big on $Y_{/S}^m$ for all $m \geq d$;

(3) $\overline{L}$ is potentially big on Y/S; i.e., $m_{\boxtimes}\overline{L}$ is big on $Y_{/S}^m$ for some $m \geq 1$.

Proof. We first prove that (1) implies (2). Since $\overline{L}$ is nef, it is immediate that $m_{\boxtimes}\overline{L}$ is nef. Its bigness is equivalent to the positivity of the intersection number

$$(m_{\boxtimes}\overline{L})^{d+em} = (p_1^*\overline{L} + p_2^*\overline{L} + \cdots + p_m^*\overline{L})^{d+em}.$$

Here $e = \dim Y - \dim S$ and $\dim Y_{/S}^m = d + em$. Expanding the right-hand side, we obtain a term

$$b_m = (p_1^*\overline{L})^{e+1} \cdots (p_d^*\overline{L})^{e+1} \cdot (p_{d+1}^*\overline{L})^e \cdots (p_m^*\overline{L})^e.$$

We claim that $b_m = a^{m-d}(\overline{M}^d)$, where a denotes the degree of L on fibers of $\pi : Y \to S$, and $\overline{M}^d$ denotes the top self-intersection number of $\overline{M} = \pi_*\langle \overline{L}, \cdots, \overline{L}\rangle$ on S. The claim implies $b_m > 0$ and thus proves that (1) implies (2).

To prove the claim, we first treat the arithmetic case $k = \mathbb{Z}$. By a limit process, we can assume that S is projective and flat over $S_0 = \operatorname{Spec}\mathbb{Z}$, and $Y \to S$ is projective and flat. We can further assume that $\overline{L}$ is a nef hermitian line bundle on Y, so that

$$\overline{M} = \pi_*\langle \overline{L}, \cdots, \overline{L}\rangle$$

is a nef hermitian line bundle on S. Consider the projection $\psi_m : Y_{/S}^m \to S_0$. By the compatibility of the Deligne pairing with the intersection number, b_m is the arithmetic degree of

$$\overline{B}_m = \psi_{m*}\langle (p_1^*\overline{L})^{e+1}, \cdots, (p_d^*\overline{L})^{e+1}, (p_{d+1}^*\overline{L})^e, \cdots, (p_m^*\overline{L})^e\rangle.$$

Here the notation $(p_i^*\overline{L})^t$ means that $p_i^*\overline{L}$ appears t times in the pairing. Apply [YZ2, Lem. 4.6.1(1)] to the composition

$$Y_{/S}^m \xrightarrow{\ q\ } Y_{/S}^{m-1} \xrightarrow{\ \psi_{m-1}\ } S_0.$$

Here q is the projection by forgetting the first component of $Y_{/S}^m$. In the line bundles defining $\overline{B}_m$, view the first $e + 1$ terms, indicated by $(p_1^*\overline{L})^{e+1}$, as line bundles on $Y_{/S}^m$, and view the remaining $(d + me) - (e + 1)$ terms as the pull-back of line bundles via q. Then the lemma implies that

$$\overline{B}_m = \psi_{m-1,*}\langle \overline{M}_{Y_{/S}^{m-1}}, (p_2^*\overline{L})^{e+1}, \cdots, (p_d^*\overline{L})^{e+1}, (p_{d+1}^*\overline{L})^e, \cdots, (p_m^*\overline{L})^e\rangle.$$

Here by abuse of notations, $p_2, \cdots, p_m : Y_{/S}^{m-1} \to Y$ denotes the projections to the $m - 1$ components. Repeat the process on $(p_2^* \overline{L})^{e+1}, \cdots, (p_d^* \overline{L})^{e+1}$. We obtain

$$\overline{B}_m = \psi_{m-d,*} \langle (\overline{M}_{Y_{/S}^{m-d}})^d, (p_{d+1}^* \overline{L})^e, \cdots, (p_m^* \overline{L})^e \rangle.$$

Apply [YZ2, Lem. 4.6.1(3)] to the composition

$$Y_{/S}^{m-d} \longrightarrow S \xrightarrow{\psi_0} S_0.$$

Here we view the first d terms as the pull-back of line bundles from S. Then the lemma implies that

$$\overline{B}_m = a^{m-d} \psi_{0*} \langle \overline{M}^d \rangle.$$

This proves that (1) implies (2) in the arithmetic case.

For the geometric case that k is a field, we can first reduce it to the case that S is projective over k. By blowing-up S if necessary, we can find a fibration $S \to S_0$ with $S_0 = \mathbb{P}_k^1$. Then the proof is similar to the arithmetic case by taking Deligne pairings to S_0.

Note that (2) implies (3) trivially. It remains to prove that (3) implies (1). Assuming that for some $m \geq 1$, $m_{\boxtimes} \overline{L}$ is big on $Y_{/S}^m$. By Lemma 2.2(1), the Deligne pairing

$$\overline{N} = (\pi_m)_* \langle (m_{\boxtimes} \overline{L})^{em+1} \rangle$$

is nef and big on S. Here $\pi_m : Y_{/S}^m \to S$ is the structure morphism. It suffices to prove that $\overline{N}$ is a positive multiple of $\overline{M} = \pi_* \langle \overline{L}^{e+1} \rangle$.

In fact, consider the expansion of

$$\overline{N} = (\pi_m)_* \langle (p_1^* \overline{L} + p_2^* \overline{L} + \cdots + p_m^* \overline{L})^{em+1} \rangle$$

by linearity. It is a positive linear combination of

$$\overline{N}(r_1, \cdots, r_m) = (\pi_m)_* \langle (p_1^* \overline{L})^{r_1}, (p_2^* \overline{L})^{r_2}, \cdots, (p_m^* \overline{L})^{r_m} \rangle$$

with $r_1 + \cdots + r_m = em + 1$. We claim that $\overline{N}(r_1, \cdots, r_m) \neq 0$ only if $(r_1, \cdots, r_m)$ is a permutation of $(e+1, e, \cdots, e)$; in that case, $\overline{N}(r_1, \cdots, r_m) = a^{m-1} \overline{M}$. Here a denotes the degree of L on fibers of $\pi : Y \to S$ as above.

In fact, by $r_1 + \cdots + r_m = em + 1$, some term $r_i \geq e + 1$. By symmetry, we assume that $r_1 \geq e + 1$. Write

$$\overline{N}(r_1, \cdots, r_m) = (\pi_m)_* \langle (p_1^* \overline{L})^{e+1}, (p_1^* \overline{L})^{r_1 - e - 1}, (p_2^* \overline{L})^{r_2}, \cdots, (p_m^* \overline{L})^{r_m} \rangle$$

Apply [YZ2, Lem. 4.6.1(3)] to the composition

$$Y_{/S}^m \xrightarrow{p_1} Y \xrightarrow{\pi} S.$$

We see that

$$\overline{N}(r_1, \cdots, r_m) = c(r_1, \cdots, r_m) \pi_* \langle \overline{L}^{e+1} \rangle,$$

where $c(r_1, \cdots, r_m)$ is the intersection number of

$$(p_1^* L)^{r_1 - e - 1}, (p_2^* L)^{r_2}, \cdots, (p_m^* L)^{r_m}$$

over a fiber of $p_1 : Y_{/S}^m \to Y$. Then $c(r_1, \cdots, r_m) \neq 0$ only if $r_1 - e - 1 = 0$, and $r_i = e$ for $i > 1$; in that case $c(r_1, \cdots, r_m) = a^{m-1}$. This finishes the proof. $\qquad\square$

4.1.2   Consequence on small points

Let k be either $\mathbb{Z}$ or a field. By a *global field over* k, we mean a field K as follows. If $k = \mathbb{Z}$, then K is a number field; if k is a field, then K is a function field of one variable over k.

In the terminology of [YZ2, §2.3], a quasi-projective variety V over K is essentially quasi-projective over k. In [YZ2, §2.4, §2.5], we can talk about adelic divisors and adelic line bundles on V with base k, which are defined by direct limits over quasi-projective models of V over k.

Let V be a quasi-projective variety over K. Let $\overline{L}$ be an adelic line bundle on V over k. We define a height function

$$h_{\overline{L}} : V(\overline{K}) \longrightarrow \mathbb{R}$$

as follows. For any $x \in V(\overline{K})$, denote by $\tilde{x}$ the closed point of V corresponding to x. By the construction in [YZ2, §2.5.5], the pull-back $\overline{L}|_{\tilde{x}}$ gives an adelic line bundle on $\tilde{x}/k$. Since $\tilde{x}$ is the spectrum of a global function field over k, we can take the arithmetic degree $\widehat{\deg}(\overline{L}|_{\tilde{x}})$. Here if k is a field, the degree is normalized by multiplicity functions given by degrees over k.

If $k = \mathbb{Z}$, we define

$$h_{\overline{L}}(x) = \frac{1}{\deg(\tilde{x}/\mathbb{Q})} \widehat{\deg}(\overline{L}|_{\tilde{x}}).$$

Note that the normalizing factor has the degree of $\tilde{x}$ over $\mathbb{Q}$ (instead of K), which is different from that in [YZ2, §5.3].

If k is a field, we define

$$h_{\overline{L}}(x) = \frac{1}{\deg(\tilde{x}/K)} \widehat{\deg}(\overline{L}|_{\tilde{x}}).$$

The normalizing factor depends on K and agrees with that in [YZ2, §5.3].

The following is our key result relating potential bigness to distribution of small points. Due to different normalization of heights and arithmetic degrees, we state the result for the arithmetic case and for the geometric case separately.

Theorem 4.2. *The following statements hold.*

(1) *Let S be a quasi-projective variety over a number field K. Let $\pi : X \to S$ be a smooth relative curve. Let $\overline{L}$ be a nef adelic line bundle on $X/\mathbb{Z}$, and $\overline{M}$ be an adelic line bundle on $S/\mathbb{Z}$. If $\overline{L}$ is potentially big on X/S, then there are a non-empty Zariski open subvariety $U \subset S$, and constants $c_1, c_2 > 0$, such that for any $y \in U(\overline{K})$,*

$$\#\{x \in X(\overline{K}) : \pi(x) = y,\ h_{\overline{L}}(x) \le c_1\, h_{\overline{M}}(y)\} \le c_2.$$

(2) *Let S be a quasi-projective variety over a field k with $\dim S > 0$. Let $\pi : X \to S$ be a smooth relative curve. Let $\overline{L}$ be a nef adelic line bundle on X/k, and $\overline{M}$ be an adelic line bundle on S/k. If $\overline{L}$ is potentially big on X/S, then there are a non-empty Zariski open subvariety $U \subset S$, and constants $c_1, c_2 > 0$, such that for any 1-dimensional point $y \in U$,*

$$\#\{x \in X_y(\overline{k(y)}) : h_{\overline{L}}(x) \le c_1\, \widehat{\deg}(\overline{M}|_y)\} \le c_2.$$

Here $h_{\overline{L}} : X_y(\overline{k(y)}) \to \mathbb{R}$ is the height function associated to $\overline{L}|_{X_y}$ and normalized by degrees over $k(y)$, and $\widehat{\deg} : \widehat{\mathrm{Pic}}(k(y)/k) \to \mathbb{R}$ is normalized by multiplicity functions given by degrees over k.

Proof. We first prove (1). By assumption, $\overline{L}_m = m_{\boxtimes}\overline{L}$ is big on $X_m = X^m_{/S}$ for some $m \geq 1$. Fix such an m. By [YZ2, Thm. 5.3.5(1)], there are a closed subset Z of codimension one in X_m and $\epsilon > 0$ such that

$$\{x \in X_m(\overline{K}) : h_{\overline{L}_m}(x) \leq \epsilon h_{\overline{M}}(\pi_m(x))\} \subset Z(\overline{K}).$$

For any $y \in S(\overline{K})$, denote

$$\Sigma(y) := \{x \in X(\overline{K}) : \pi(x) = y, h_{\overline{L}}(x) \leq \frac{\epsilon}{m} h_{\overline{M}}(y)\}.$$

The key is the inclusion
$$\Sigma(y)^m \subset Z(\overline{K}).$$

This follows from the height identity

$$h_{\overline{L}_m}(x) = h_{\overline{L}}(x_1) + \cdots + h_{\overline{L}}(x_m)$$

for any $x \in X_m(\overline{K})$ represented by $(x_1, \cdots, x_m)$ under the expression

$$X_m(\overline{K}) = \{(x_1, \cdots, x_m) \in X(\overline{K})^m : \pi_m(x_1) = \cdots = \pi_m(x_m)\}.$$

Let U be a non-empty open subscheme of S such that Z is flat over U. By Lemma 4.3 below, the inclusion $\Sigma(y)^m \subset Z_y(\overline{K})$ forces $\Sigma(y)$ to be finite with

$$\#\Sigma(y) \leq \deg_{m_{\boxtimes}N}(Z_\eta), \quad \forall y \in U(\overline{K}).$$

Here Z_η is the generic fiber of $Z \to S$, N is a line bundle of degree 1 on the generic fiber X_η of $X \to S$, so that $m_{\boxtimes}N$ is a line bundle on X^m_η. This proves (1).

The proof of (2) is similar to that of (1), so we will only emphasize the difference. We still have that $\overline{L}_m = m_{\boxtimes}\overline{L}$ is big on $X_m = X^m_{/S}$ for some $m \geq 1$. We are not able to apply [YZ2, Thm. 5.3.5(1)] directly, but we can modify its proof slightly. In fact, as in the proof of the loc. cit., $\overline{L}_m - \epsilon\pi^*\overline{M}$ is big for some positive rational number $\epsilon > 0$. Then some multiple $n(\overline{L}_m - \epsilon\overline{M})$ of $\overline{L}_m - \epsilon\overline{M}$ is an (integral) adelic line bundle with a nonzero effective section s.

Denote $Z = \mathrm{div}(s)$. We claim that for any 1-dimensional point y of S,

$$\{x \in (X_m)_y(\overline{k(y)}) : h_{\overline{L}_m}(x) \leq \epsilon \widehat{\deg}(\overline{M}|_y)\} \subset Z_y(\overline{k(y)}).$$

In fact, denote by $\tilde{x}$ the closed point of $(X_m)_y$ corresponding to x, which is a 1-dimensional point of X_m. If $x \notin Z_y(\overline{K})$ or equivalently $\tilde{x} \notin Z_y$, then the restriction of s gives a nonzero effective section of

$$n(\overline{L}_m - \epsilon \pi^* \overline{M})|_{\tilde{x}} = n(\overline{L}_m|_{\tilde{x}} - \epsilon \pi_x^*(\overline{M}|_y)),$$

where $\pi_x : \tilde{x} \to y$ is the finite morphism. Taking arithmetic degrees, we have

$$0 \leq \widehat{\deg}(\overline{L}_m|_{\tilde{x}}) - \epsilon \widehat{\deg}(\pi_x^*(\overline{M}|_y)) = \deg(\pi_x)\big(h_{\overline{L}_m}(x) - \epsilon \widehat{\deg}(\overline{M}|_y)\big).$$

This prove the claim. The remaining part of the proof is similar to that of (1). □

In the above proof, we have used the following basic result, which improves [DGH1, Lem. 6.3] and [Gao2, Lem. 7.3].

Lemma 4.3. *Let C be a smooth projective curve over an algebraically closed field F. Let $Z \subsetneq C^m$ be a Zariski closed subset of codimension $d > 0$. Let Σ be a finite subset of $C(F)$ such that $\Sigma^m \subset Z(F)$. Then*

$$\#\Sigma \leq (\deg_{m \boxtimes N}(Z))^{1/d}.$$

Here N is a line bundle on C of degree 1, and

$$\deg_{m \boxtimes N}(Z) = \sum_{i=1}^{r} \deg_{m \boxtimes N}(Z_i),$$

where $Z_1, \cdots, Z_r$ are all the irreducible components of Z.

Proof. Denote $M = m \boxtimes N$ and $n = \#\Sigma$. We claim that for every $i = 1, \cdots, m$

$$\deg_{nM}(Z) \geq \deg_{nM}(Z \cap p_i^{-1}\Sigma).$$

Here $p_i : C^m \to C$ is the projection to the i-th component, and $Z \cap p_i^{-1}\Sigma$ is endowed with the reduced structure.

If the claim holds, then successively applying the inequality gives

$$\deg_{nM}(Z) \geq \deg_{nM}(Z \cap p_1^{-1}\Sigma \cap \cdots \cap p_m^{-1}\Sigma) = \#(\Sigma^m) = n^m.$$

This implies the result since

$$\deg_{nM}(Z) \leq n^{m-d} \deg_M(Z)$$

by $\dim Z_i \leq m - d$.

It remains to prove the claim. By writing in terms of irreducible components, we can assume that Z is irreducible. If Z is contained in $p_i^{-1}\Sigma$, the degree does not change. If Z is not contained in $p_i^{-1}\Sigma$, then it intersects $p_i^{-1}\Sigma$ properly, and thus

$$\deg_{nM}(Z) = (nM)^{\dim Z} \cdot Z \geq (nM)^{\dim Z - 1} \cdot (np_i^*N) \cdot Z \geq \deg_{nM}(Z \cap p_i^{-1}\Sigma).$$

This proves the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.4. Here we describe a path to find effective constants (c_1, c_2) in Theorem 4.2, which depends on many numerical invariants of $(X/S, \overline{L}, \overline{M})$. We focus on the arithmetic case. Instead of applying [YZ2, Thm. 5.3.5(1)], we take the idea sketched right after Theorem 1.2. First, by [YZ2, Lem. 5.1.6], we can find a nef adelic line bundle $\overline{M}'$ on S such that $\overline{M}' - \overline{M}$ is effective. Then we can assume that $\overline{M}$ is nef by replacing $\overline{M}$ by $\overline{M}'$ everywhere in the problem. Second, we have

$$\widehat{\mathrm{vol}}(\overline{L}_m - \epsilon\overline{M}) \geq \overline{L}_m^{d+m} - (d+m)\overline{L}_m^{d+m-1} \cdot \epsilon\overline{M} > 0.$$

Here d is the dimension of a quasi-projective model of S over k, m is any fixed integer with $m \geq d$, and ϵ is chosen to be any positive rational number satisfying the second inequality. Then the constant $c_1 = \epsilon/m$. Third, some multiple $n(\overline{L}_m - \epsilon\overline{M})$ of $\overline{L}_m - \epsilon\overline{M}$ has a nonzero effective section s. By Lemma 4.3, we can express c_2 in terms of intersection numbers involving $Z = \mathrm{div}(s)$. Note that Z is linearly equivalent to $n(\overline{L}_m - \epsilon\overline{M})$. To find an effective c_2, it suffices to find an effective positive integer n such that $n(\overline{L}_m - \epsilon\overline{M})$ is an (integral) adelic line bundle with a nonzero effective section.

4.2   Three potentially big examples

The goal of this subsection is to introduce three potentially big examples built on fibered powers of family of curves.

Let k be either $\mathbb{Z}$ or a field. Let S be a flat and quasi-projective normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$. Let $\pi_J : J \to S$ be the relative Jacobian scheme of X over S.

Here we review three canonical morphisms from X to J (up to base changes) treated in Theorem 2.9 and Theorem 2.10. Only the third morphism will be crucially used in our treatments later, but we include the other two

for their own value. For example, the second morphism is essentially the one used in the proofs of [DGH1, Kuh].

The first morphism is the basic one. Let α be a line bundle on X with degree $d > 0$ on the fibers of $X \to S$. This gives a finite S-morphism

$$i_\alpha : X \longrightarrow J, \quad x \longmapsto dx - \alpha.$$

The second one is the morphism

$$i_\Delta : X \times_S X \longrightarrow X \times_S J, \quad (x, y) \longmapsto (x, y - x).$$

This agrees with the X-morphism

$$i_\Delta : X_X \longrightarrow J_X, \quad x \longmapsto x - \Delta$$

defined in the proof of Theorem 2.10(2). Namely, denote $X_X = X \times_S X$ and $J_X = X \times_S J$, viewed as X-schemes via the first projections $p_1 : X_X \to X$ and $q_1 : J_X \to X$. Then $q_1 : J_X \to X$ is canonically isomorphic to the Jacobian scheme of $p_1 : X_X \to X$. View $\Delta : X \to X \times_S X = X_X$ as a section of $p_1 : X_X \to X$. This defines an X-morphism

$$i_\Delta = i_{\mathcal{O}(\Delta)} : X_X \longrightarrow J_X, \quad x \longmapsto x - \Delta.$$

The third one is the morphism

$$\tau : J \times_S X \longrightarrow J \times_S J, \quad (y, x) \longmapsto (y, y + (2g - 2)x - \omega_{X/S}).$$

As in the proof of Theorem 2.10, if $X \to S$ has a section, then $J \times_S X$ has a universal line bundle Q, and τ agrees with

$$i_{\omega - Q} : X_J \longrightarrow J_J, \quad x \longmapsto (2g - 2)x - (\omega_{X_J/J} - Q).$$

Recall that from Definition 2.4 we have $\Theta = \Delta_J^*(P^\vee)$ on J. We also have a line bundle $\Theta_X = p_2^* \Theta$ on J_X. Then we have the canonical adelic extensions $\overline{\Theta}$ on J and its base changes $\overline{\Theta}_X$ to J_X and $\overline{\Theta}_J$ to J_J. Finally, we have the following bigness result.

Theorem 4.5. *Let k be either $\mathbb{Z}$ or a field. Let S be a flat and quasi-projective normal integral scheme over k. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$ with maximal variation. Then the following hold:*

(1) *The adelic line bundle $\pi_* \langle i_\alpha^* \overline{\Theta}, i_\alpha^* \overline{\Theta} \rangle$ is nef and big on S. Therefore, $i_\alpha^* \overline{\Theta}$ is potentially big on X/S.*

(2) *The adelic line bundle $p_{1*} \langle i_\Delta^* (\overline{\Theta}_X), i_\Delta^* (\overline{\Theta}_X) \rangle$ is nef and big on X. Therefore, $i_\Delta^* (\overline{\Theta}_X)$ is potentially big on X_X / X.*

(3) *The adelic line bundle $q_{1*} \langle \tau^* (\overline{\Theta}_J), \tau^* (\overline{\Theta}_J) \rangle$ is nef and big on J. Therefore, $\tau^* (\overline{\Theta}_J)$ is potentially big on X_J / J.*

(4) *Let Y be a quasi-projective variety over k with a generically finite morphism $Y \to J$. Denote by T the image of $Y \to S$. Assume that the composition $T \to S \to M_{g,k}$ is generically finite. Then the adelic line bundle $q_{1*} \langle \tau^* (\overline{\Theta}_J), \tau^* (\overline{\Theta}_J) \rangle|_Y$ is nef and big on Y. Therefore, $(\tau^* (\overline{\Theta}_J))|_{Y \times_S X}$ is potentially big on $Y \times_S X / Y$.*

Proof. For (1), by Theorem 2.9(2),

$$\pi_* \langle i_\alpha^* \overline{\Theta}, i_\alpha^* \overline{\Theta} \rangle = \frac{g d^4}{g-1} \pi_* \langle \overline{\omega}_{X/S,a} \, \overline{\omega}_{X/S,a} \rangle + \frac{d^2}{g-1} \iota_\alpha^* \overline{\Theta}$$

in $\widehat{\mathrm{Pic}}(S)_{\mathbb{Q}}$. By Theorem 3.2, $\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$ is nef and big on S. Note that $\iota_\alpha^* \overline{\Theta}$ is nef on S, since $\overline{\Theta}$ is nef on J. It follows that $\pi_* \langle i_\alpha^* \overline{\Theta}, i_\alpha^* \overline{\Theta} \rangle$ is nef and big on S. The second statement follows from Proposition 4.14.

For (2), by Theorem 2.10(2),

$$p_{1*} \langle i_\Delta^* (\overline{\Theta}_X), i_\Delta^* (\overline{\Theta}_X) \rangle = 4g \, \overline{\omega}_{X/S,a} + \pi^* \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$$

in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$. By Theorem 3.1, $\overline{\omega}_{X/S,a}$ is nef and big on X. Then $p_{1*} \langle i_\Delta^* (\overline{\Theta}_X), i_\Delta^* (\overline{\Theta}_X) \rangle$ is nef and big on X.

For (3), by Theorem 2.10(3),

$$q_{1*} \langle \tau^* (\overline{\Theta}_J), \tau^* (\overline{\Theta}_J) \rangle = 16(g-1)^3 \overline{\Theta} + 16g(g-1)^3 \pi_J^* \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$$

in $\widehat{\mathrm{Pic}}(J)_{\mathbb{Q}}$. Then $q_{1*} \langle \tau^* (\overline{\Theta}_J), \tau^* (\overline{\Theta}_J) \rangle$ is nef on J. For the bigness, it suffices to note that the binomial expansion of the self-intersection of $q_{1*} \langle \tau^* (\overline{\Theta}_J), \tau^* (\overline{\Theta}_J) \rangle$ on J has a term

$$\left(\pi_J^* \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle \right)^{\dim S} \cdot \overline{\Theta}^{\dim J - \dim S} = n \left(\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle \right)^{\dim S} > 0.$$

Here n is the degree of Θ on a fiber of $J \to S$, and the identity is the limit version of that on projective varieties over k.

For (4), it suffices to check that

$$q_{1*}\langle \tau^*(\overline{\Theta}_J), \tau^*(\overline{\Theta}_J)\rangle|_Y = 16(g-1)^3\overline{\Theta}|_Y + 16g(g-1)^3\pi_J^*\pi_*\langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a}\rangle|_Y$$

is big in $\widehat{\mathrm{Pic}}(Y)_{\mathbb{Q}}$. The proof is similar to (3), as $\Theta|_Y$ is ample on fibers of $Y \to S$. $\qquad\square$

Remark 4.6. Only (3) and (4) will be used in the proof of our uniform Bogomolov-type theorem. Note that (4) essentially implies (1)-(3), but we include all of them here for their own simplicity. There are also counterparts of (4) for (1) and (2), but we omit them in this paper.

4.3 The uniform Bogomolov-type theorem: statement

Now we state the following uniform version of the Bogomolov conjecture, which generalizes the new gap principle proved in [DGH1, Prop. 7.1] and [Kuh, Thm. 3] and summarized in [Gao2, Thm. 4.1].

Theorem 4.7 (Theorem 1.1). *Let $g > 1$ be an integer. Then there are constants $c_1, c_2 > 0$ depending only on g satisfying the following properties. Let K be either a number field or a function field of one variable over a field k. Then for any geometrically integral, smooth and projective curve C of genus g over K, and for any line bundle $\alpha \in \mathrm{Pic}(C_{\overline{K}})$ of degree 1, with the extra assumption that $(C_{\overline{K}}, \alpha)$ is non-isotrivial over $\overline{k}$ in the case that K is a function field of one variable over a field k, one has*

$$\#\left\{x \in C(\overline{K}) : \widehat{h}(x - \alpha) \le c_1\big(\max\{h_{\mathrm{Fal}}(C), 1\} + \widehat{h}((2g-2)\alpha - \omega_{C/K})\big)\right\} \le c_2.$$

The theorem will be proved in §4.4. In the following, we explain the precise definition of the heights in the theorem, and explore the non-isotriviality condition in terms of moduli spaces.

4.3.1 Normalization of the heights

Since Theorem 4.7 asserts uniformity when varying K, it is necessary to explain how the heights are normalized in terms of K. The normalization here is compatible with that in Theorem 4.2.

Let us first explain the Néron–Tate height. Let C be a geometrically integral, smooth and projective curve of genus $g > 1$ over K. Take a point

$x_0 \in C(\overline{K})$. Denote by J the Jacobian variety of C over K. The theta divisor θ_{x_0} is defined as the image of the morphism

$$C_{\overline{K}}^{g-1} \longrightarrow J_{\overline{K}}, \quad (x_1, \cdots, x_{g-1}) \longmapsto x_1 + \cdots + x_{g-1} - (g-1)x_0.$$

Then $\Theta = \mathcal{O}(\theta_{x_0} + [-1]^* \theta_{x_0})$ is a symmetric and ample line bundle on $J_{\overline{K}}$. By Lemma A.3(3), the isomorphism class of Θ is independent of the choice of x_0 actually descends to a line bundle on J. Moreover, the current definition of Θ is also compatible with that of Definition 2.4.

The Néron–Tate height $\widehat{h} : J(\overline{K}) \to \mathbb{R}$ is defined by $\widehat{h} = \dfrac{1}{2}\widehat{h}_\Theta$; the canonical height $\widehat{h}_\Theta : J(\overline{K}) \to \mathbb{R}$ is defined by Tate's limiting argument from a Weil height of Θ; the Weil height is induced by the classical height $h : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$ as follows.

If K is a number field, the classical height $h : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$ is given by

$$h(x_0, \cdots, x_n) = \frac{1}{[K' : \mathbb{Q}]} \sum_{v \in M_{K'}} \log \max\{\|x_0\|_v, \cdots, \|x_n\|_v\}.$$

Here K' is a finite extension of K containing the coordinates $x_0, \cdots, x_n$, $M_{K'}$ is the set of places of K', and $\| \cdot \|_v$ is normalized as follows:

(1) If v is real, it is the usual absolute value on $\mathbb{R}$.

(2) If v is complex, it is the square of the usual absolute value on $\mathbb{C}$.

(3) If v is non-archimedean, then $\|\cdot\|_v = N_v^{-\mathrm{ord}_v(\cdot)}$. Here N_v is the cardinality of the residue field of v in $O_{K'}$.

If K is a function field of one variable over a field k, the classical height $h : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$ is given by

$$h(x_0, \cdots, x_n) = \frac{1}{[K' : K]} \sum_{v \in M_{K'}} \log \max\{\|x_0\|_v, \cdots, \|x_n\|_v\}.$$

Here K' is a finite extension of K containing the coordinates $x_0, \cdots, x_n$, $M_{K'}$ is the set of places of K', and $\| \cdot \|_v$ is normalized by $\| \cdot \|_v = e^{-[k(v):k]\,\mathrm{ord}_v(\cdot)}$. Here $k(v)$ is the residue field of the place v of K'. Note that even if K' (or even K) contains a bigger constant field k', the above factor is still $[k(v) : k]$ instead of $[k(v) : k']$.

Now we explain the Faltings height, which is actually the stable Faltings height. Let C and J be as above. Let K' be a finite extension of K such that C (or equivalently J) has *semi-stable reduction* everywhere over K'. Denote by $C' = C_{K'}$ and $J' = J_{K'}$ the base change to K'.

In the number field case, denote $B' = \operatorname{Spec} O_{K'}$; in the function field case, denote by B' the unique projective and regular curve over k with function field K'. Denote by $\mathcal{J}'$ the Néron model of J' over B', and denote by $e : B' \to \mathcal{J}'$ the identity section. Denote the Hodge bundle

$$\underline{\omega}_{B'} = e^*\Omega^g_{\mathcal{J}'/B'},$$

which is a line bundle on B'.

In the function field case, define the Faltings height

$$h_{\mathrm{Fal}}(C) = h_{\mathrm{Fal}}(J) = \frac{1}{[K':K]}\deg_{B'/k}(\underline{\omega}_{B'}).$$

Here the degree $\deg_{B'/k}(\underline{\omega}_{B'})$ comes from the map $\deg_{B'/k} : \operatorname{Div}(C') \to \mathbb{Z}$ given by $\deg_{B'/k}(v) = [k(v) : k]$, where $v \in B'$ is a closed point and $k(v)$ is the residue field of v in B'.

Via the stable relative curve $\mathcal{C}' \to B'$ extending $C_{K'}$, we obtain a Hodge bundle $\lambda_{B'}$ over B' defined in §3.1.2. We claim that

$$h_{\mathrm{Fal}}(C) = \frac{1}{[K':K]}\deg_{B'/k}(\lambda_{B'}).$$

This follows from a special case of Lemma 3.4. In fact, by [BLR, §7.4, Prop. 4], the group scheme $\operatorname{Pic}^0_{\mathcal{C}'/B'}$ is the relative identity component of the Néron model of $J_{K'}$ over B'. Then Lemma 3.4 gives a canonical isomorphism $\lambda_{B'} \to \underline{\omega}_{B'}$.

In the number field case, endow $\underline{\omega}_{B'}$ with a hermitian metric such that

$$\|\rho\|^2 = \frac{i^{g^2}}{(2\pi)^g}\int_{J'_\sigma(\mathbb{C})}\rho \wedge \bar{\rho}$$

for any embedding $\sigma : K' \to \mathbb{C}$ and any element ρ of

$$(\underline{\omega}_{B'})_\sigma(\mathbb{C}) = e^*\Omega^g_{J'_\sigma(\mathbb{C})/\mathbb{C}} \simeq \Gamma(J'_\sigma(\mathbb{C}), \Omega^g_{J'_\sigma(\mathbb{C})/\mathbb{C}}).$$

Define the Faltings height

$$h_{\mathrm{Fal}}(C) = h_{\mathrm{Fal}}(J) = \frac{1}{[K':\mathbb{Q}]}\widehat{\deg}(\underline{\omega}_{B'}, \|\cdot\|).$$

Here the degree $\widehat{\deg}(\omega_{B'}, \|\cdot\|)$ comes from the map $\widehat{\deg} : \widehat{\mathrm{Div}}(O_{K'}) \to \mathbb{R}$ given by $\widehat{\deg}(v) = \log N_v$, where N_v is the order of the residue field of v.

4.3.2   The non-isotriviality condition

Now let us explain the non-isotriviality condition and interpret it in terms of moduli spaces of curves. For convenience, we focus on "isotrivial", and view "non-isotrivial" as the negation of "isotrivial".

Fix $g > 1$ and a function field K of one variable over k in the following. Fix an embedding $\bar{k} \hookrightarrow \overline{K}$ extending $k \hookrightarrow K$. For the sake of level structures, fix an integer $N \geq 3$ non-divisible by $\mathrm{char}(k)$.

Consider the pair (C, α) consisting of a smooth projective curve C of genus g over $\overline{K}$, and a line bundle $\alpha \in \mathrm{Pic}(C)$ of degree 1. We say that (C, α) is *isotrivial* over $\bar{k}$ if (C, α) is isomorphic to the base change from $\bar{k}$ to $\overline{K}$ of some pair (C_0, α_0) consisting of a smooth projective curve C_0 over $\bar{k}$ and a line bundle $\alpha_0 \in \mathrm{Pic}(C_0)$ of degree 1.

By a *triple of degree d and level N over a field F*, we mean a triple (C, η, α) consisting of the following data:

(1) a geometrically integral, smooth and projective curve C of genus g over F,

(2) an isomorphism $\eta : (\mathbb{Z}/N\mathbb{Z})^{2g} \to \mathrm{Pic}(C)[N]$,

(3) an element $\alpha \in \mathrm{Pic}(C)$ of degree d on C.

By a *pair of degree d over F*, we mean a pair (C, α) of C and α described in (1) and (3).

We say that a triple of degree d and level N (resp. a pair of degree d) over $\overline{K}$ is *isotrivial* if it is isomorphic to the base change of from $\bar{k}$ to $\overline{K}$ of a triple of degree d and level N (resp. a pair of degree d) over $\bar{k}$. We have the following easy result.

Lemma 4.8. *Let (C, η, α) be a triple of degree d and level N over $\overline{K}$. Then the following are equivalent:*

(a) (C, α) is isotrivial as a pair of degree d;

(b) (C, η, α) is isotrivial as a triple of degree d and level N,

(c) $(C, \eta, (2g-2)\alpha - d\omega_{C/\overline{K}})$ is isotrivial as a triple of degree 0 and level N.

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Proof. It is trivial to have (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c). For (a) $\Rightarrow$ (b), note that if (C, α) is the base change from $\overline{k}$ to $\overline{K}$ of some pair (C_0, α_0), then $\text{Pic}(C)[N] \simeq \text{Pic}(C_0)[N]$, which can be seen in terms of torsion points of Jacobian varieties.

For (c) $\Rightarrow$ (b), assume that $(C, \eta, (2g-2)\alpha - d\omega_{C/\overline{K}})$ is the base change from $\overline{k}$ to $\overline{K}$ of some triple $(C_0, \eta_0, (2g-2)\alpha_0 - d\omega_{C_0/\overline{k}})$. Then $(2g-2)\alpha = (2g-2)\alpha_0$ in $\text{Pic}(C)$. Then $\alpha \in \alpha_0 + \text{Pic}(C)[2g-2]$. Similar to the above, we have $\text{Pic}(C)[2g-2] = \text{Pic}(C_0)[2g-2]$ and thus $\alpha \in \text{Pic}(C_0)$. This proves (c) $\Rightarrow$ (b). $\qquad\square$

As before, denote by $M_{g,N,k}$ the moduli scheme of smooth curves of genus g with a level-N structure over k, which is exactly the moduli space of pairs (C, η) over k. Denote by $\mathcal{C} \to M_{g,N,k}$ the universal curve. Denote by $\text{Pic}_{\mathcal{C}/M_{g,N,k}}$ the relative Picard functor, and denote by $\text{Pic}^d_{\mathcal{C}/M_{g,N,k}}$ the subfunctor of line bundles of degree d on fibers. Then both $\text{Pic}_{\mathcal{C}/M_{g,N,k}}$ and $\text{Pic}^d_{\mathcal{C}/M_{g,N,k}}$ are representable, and $\mathcal{J} = \text{Pic}^0_{\mathcal{C}/M_{g,N,k}}$ is the relative Jacobian scheme of $\mathcal{C} \to M_{g,N,k}$. We refer to [BLR, §8.1,§8.2] for these standard results.

Consider the moduli space of triples (C, α, η) of degree d and level N over k. Then it is represented by $\text{Pic}^d_{\mathcal{C}/M_{g,N,k}}$ over k, which can be checked by considering relative representability of the functor $(C, \alpha, \eta) \mapsto (C, \eta)$. For our purpose, it suffices to note that $\text{Pic}^d_{\mathcal{C}/M_{g,N,k}}(F)$ is bijective to the isomorphism classes of triples (C, α, η) of degree d and level N over F for any algebraically closed field F over k.

Finally, our key result in terms of the Picard functor is as follows. A triple (C, α, η) of degree d and level N over $\overline{K}$ is isotrivial if and only the point of $\text{Pic}^d_{\mathcal{C}/M_{g,N,k}}(\overline{K})$ representing the triple (C, α, η) lies in the image of the natural map
$$\text{Pic}^d_{\mathcal{C}/M_{g,N,k}}(\overline{k}) \longrightarrow \text{Pic}^d_{\mathcal{C}/M_{g,N,k}}(\overline{K}).$$

This is further equivalent to the property that the point of $\text{Pic}^d_{\mathcal{C}/M_{g,N,k}}(\overline{K})$ representing the triple (C, α, η) corresponds to a closed point of $\text{Pic}^d_{\mathcal{C}/M_{g,N,k}}$. The proof is trivial.

4.3.3   An easy lower bound

Here we present an easy uniform lower bound of the stable Faltings height in the function field case.

Lemma 4.9. *Let K be a function field of one variable over a field k. Let C be a geometrically integral, smooth and projective curve over K of genus $g > 1$. Assume that $C_{\overline{K}}$ is non-isotrivial over $\overline{k}$. Then*

$$h_{\mathrm{Fal}}(C) > \begin{cases} 3^{-4g^2} & \text{if } \mathrm{char}(K) \neq 3, \\ 4^{-4g^2} & \text{if } \mathrm{char}(K) = 3. \end{cases}$$

Proof. Let K' be a finite extension of K such that C has semistable reduction over K', and let B' be the unique projective regular curve over k with function field K'. Then we have

$$h_{\mathrm{Fal}}(C) = \frac{1}{[K':K]} \deg_{B'/k}(\omega_{B'}) = \frac{1}{[K':K]} \deg_{B'/k}(\lambda_{B'}) \geq \frac{1}{[K':K]}.$$

The inequality is based on two facts. First, the non-isotriviality condition implies that $h_{\mathrm{Fal}}(C) > 0$. This well-known result is a consequence of the bigness of $\lambda_{\overline{S}}$ (for $\overline{S} = B'$) described at the end of §3.4. Alternatively, it can be derived from [MB, XI, Thm. 4.5] and Torelli's theorem. Second, $\deg_{B'/k}(\omega_{B'}) > 0$ implies $\deg_{B'/k}(\omega_{B'}) \geq 1$ because it is an integer.

With the claim, it suffices to find K' such that $[K':K]$ is small enough for the bound. Let $N \geq 3$ be an integer non-divisible by $\mathrm{char}(K)$, and set K' to be the smallest subfield of $\overline{K}$ such that all points of $J(\overline{K})[N]$ are defined over K'. Here J is the Jacobian variety of X over K. By Raynaud's theorem (cf. [Gro, Prop. 4.7]), J has semistable reduction over K'. By [DM, Thm. 2.4], C has semistable reduction over K'. So K' satisfies the requirement.

It remains to bound $[K':K]$. Note that $\mathrm{Gal}(K'/K)$ has a natural injection to $\mathrm{GSp}(J(\overline{K})[N]) \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. This has a very rough upper bound N^{4g^2}. $\qquad\square$

4.4 The uniform Bogomolov-type theorem: proof

Now we prove Theorem 4.7. Our proof consists of three parts, which treat respectively the number field case, the function field case, and the uniformity of the constants c_1, c_2 on K. It is possible to merge them into a single part, but we introduce them case by case for the sake of readability.

4.4.1 Part 1: number field

In this part, we treat the number field case. Fix an integer $N \geq 3$, and denote by $M_{g,N,\mathbb{Q}}$ the (fine) moduli space of smooth curves of genus g over $\mathbb{Q}$

with a full level-N structure. As before, $M_{g,N,\mathbb{Q}}$ is a smooth quasi-projective variety over $\mathbb{Q}$, which follows from the GIT construction in [MFK, §7.4]. Set $S = M_{g,N,\mathbb{Q}}$ and let $\pi : X \to S$ be the universal curve.

As above, consider the situation of Theorem 2.10(3). Then we have a morphism

$$\tau : J \times_S X \longrightarrow J \times_S J, \quad (y, x) \longmapsto (y, y + (2g - 2)x - \omega_{X/S}).$$

Rewrite it as a J-morphism

$$\tau : X_J \longrightarrow J_J.$$

Here we write $X_J = J \times_S X$ and $J_J = J \times_S J$, viewed as J-schemes via the first projections $q_1 : J \times_S X \to J$ and $p_1 : J \times_S J \to J$.

Denote

$$\overline{L} = \tau^*(\overline{\Theta}_J), \quad \overline{\Theta}_J = q_2^* \overline{\Theta}.$$

We claim that for any adelic line bundle $\overline{M}$ on J, there are constants $c_1, c_2 > 0$ such that for any $y \in J(\overline{K})$,

$$\#\{z \in X_J(\overline{K}) : q_1(z) = y,\ h_{\overline{L}}(z) \le c_1\, h_{\overline{M}}(y)\} \le c_2.$$

To prove the claim, by Theorem 4.5(3), $\overline{L} = \tau^*(\overline{\Theta}_J)$ is potentially big on X_J/J. Apply Theorem 4.2 to the family $q_1 : X_J \to J$. Then there are a non-empty Zariski open subvariety $U \subset J$ and constants $c_1, c_2 > 0$ such that the claim holds for all $y \in U(\overline{K})$. This is already very close to the claim, except that it restricts y to be in the open subset U of J. To cover all algebraic points of J, we we need to apply a similar result to every irreducible component of $J \setminus U$, and repeat the process finitely many times. For that, it suffices to prove that, for any (non-empty) closed subvariety Y of J, there is a non-empty open subvariety V of Y, and constants $c_1, c_2 > 0$ such that for any $y \in V(\overline{K})$,

$$\#\{z \in X_J(\overline{K}) : q_1(z) = y,\ h_{\overline{L}}(z) \le c_1\, h_{\overline{M}}(y)\} \le c_2.$$

This is again a consequence of Theorem 4.2, applied to the base change $X_Y \to Y$ of $q_1 : X_J \to J$ by $Y \to J$. Here the potential bigness condition is obtained by Theorem 4.5(4). This proves the claim.

Now we verify that the claim implies the theorem for number fields. In fact, denote by $\tau_1 : X_J \to J$ the composition $X_J \xrightarrow{\tau} J_J \xrightarrow{q_2} J$. This gives $\tau_1^* \overline{\Theta} = \overline{L}$. Under the identification

$$X_J(\overline{K}) = (J \times_S X)(\overline{K}) = \{(y, x) \in J(\overline{K}) \times X(\overline{K}) : \pi_J(y) = \pi(x)\},$$

$\tau_1 : X_J \to J$ maps a point $z = (y, x) \in X_J(\overline{K})$ with $s = \pi(x) \in S(\overline{K})$ to the point of $J(\overline{K})$ corresponding to the divisor $y + (2g - 2)x - \omega_{X_s}$ on X_s. The projection formula for $\tau_1 : X_J \to J$ gives

$$h_{\overline{L}}(z) = h_{\overline{\Theta}}((2g - 2)x - (\omega_{X_s} - y)) = 2\,\widehat{h}((2g - 2)x - (\omega_{X_s} - y)).$$

Then the claim implies for any $y \in J(\overline{K})$ and $s = \pi_J(y) \in S(\overline{K})$,

$$\#\{x \in X_s(\overline{K}) : 2\,\widehat{h}((2g - 2)x - (\omega_{X_s} - y)) \leq c_1\, h_{\overline{M}}(y)\} \leq c_2.$$

We claim that there is an adelic line bundle $\overline{M}$ on J such that for any $y \in J(\overline{K})$ with $s = \pi_J(y) \in S(\overline{K})$,

$$h_{\overline{M}}(y) \geq c_3\big(\max\{h_{\mathrm{Fal}}(X_s), 1\} + \widehat{h}(y)\big)$$

for some constant $c_3 > 0$.

If this holds, then for any $s \in S(\overline{K})$ and $y \in J_s(\overline{K})$,

$$\#\left\{x \in X_s(\overline{K}) : 2\,\widehat{h}((2g - 2)x - (\omega_{X_s} - y)) \leq c_1 c_3\big(\max\{h_{\mathrm{Fal}}(X_s), 1\} + \widehat{h}(y)\big)\right\}$$

is less than or equal to c_2. For any $\alpha \in \mathrm{Pic}(X_s)$ of degree 1, set

$$y = \omega_{X_s} - (2g - 2)\alpha.$$

It gives the theorem for number fields.

For the construction of $\overline{M}$, set

$$\overline{M} = \overline{\Theta} + \pi_J^* \overline{\lambda}_S + \mathcal{O}(c).$$

Here $c > 0$ and $\mathcal{O}(c)$ is the trivial line bundle on $\mathbb{Z}$ with metric given by $\|1\| = e^{-c}$, and viewed as an adelic line bundle on J by pull-back, and $\overline{\lambda}_S$ is the Hodge bundle associated to $J \to S$, endowed with the Faltings metric. Note that $\overline{\lambda}_S$ is an adelic line bundle on S by [YZ2, §2.6.2], and the current notation is justified by Lemma 3.4. This gives

$$h_{\overline{M}}(y) = 2\,\widehat{h}(y) + h_{\mathrm{Fal}}(X_s) + c.$$

By a result of Bost [Bos] (cf. [GR, App.] or [JS1, §1.3]), we have an explicit uniform lower bound

$$h_{\mathrm{Fal}}(X_s) \geq -\frac{g}{2}\log(2\pi).$$

Note that the constant is different from those in some references due to different normalization of the metric defining the Faltings height. Then it suffices to take

$$c = \frac{g}{2}\log(2\pi) + 1.$$

This proves the theorem for number fields.

4.4.2 Part 2: function fields

In this part, we prove Theorem 4.7 in the case that K is a function field of one variable over k. The proof is similar to the number field case, so we sketch it here and emphasize on the difference.

Fix an integer $N \geq 3$ invertible in k, and denote by $M_{g,N,k}$ the (fine) moduli space of smooth curves of genus g over k with a full level-N structure. Set $S = M_{g,N,k}$ and let $\pi : X \to S$ be the universal curve. By [GO, Thm. 2.1] again, $\pi : X \to S$ has a stable compactification $\overline{\pi} : \overline{X} \to \overline{S}$.

As in the number field case, we still have a J-morphism

$$\tau : X_J \longrightarrow J_J, \quad (y, x) \longmapsto (y, y + (2g-2)x - \omega_{X/S}).$$

Denote $\overline{L} = \tau^*(\overline{\Theta}_J)$ as an element of $\widehat{\mathrm{Pic}}(X_J/k)$.

We claim that for any adelic line bundle $\overline{M}$ on J/k, there are constants $c_1, c_2 > 0$ such that for any 1-dimensional point $y \in J$,

$$\#\{z \in (X_J)_y(\overline{k(y)}) : h_{\overline{L}}(z) \leq c_1 \widehat{\deg}(\overline{M}|_y)\} \leq c_2.$$

Here $(X_J)_y$ is the fiber of $q_1 : X_J \to J$ above y. This is similar to the number field case and is obtained by successively applying Theorem 4.5(4) and Theorem 4.2 to the family $q_1 : X_J \to J$ and the adelic line bundle $\overline{L} = \tau^*(\overline{\Theta}_J)$.

Denote $s = \pi_J(y) \in S$, viewed as a schematic point. Denote by X_s and J_s the fibers of $\pi : X \to S$ and $\pi_J : J \to S$ above s. As z lies in the fiber $J \times_S X \to S$ of above $s \in S$, we can write $z = (y, x)$ with $y \in J_s(\overline{k(y)})$ and $x \in X_s(\overline{k(y)})$. Similarly to the number field case, the claim implies that for any 1-dimensional point $y \in J$ with $s = \pi_J(y) \in S$,

$$\#\{x \in X_{s,k(y)}(\overline{k(y)}) : 2\widehat{h}((2g-2)x - (\omega_{X_s} - y)) \leq c_1 \widehat{\deg}(\overline{M}|_y)\} \leq c_2.$$

Note that $s = \pi_J(y)$ is either 1-dimensional or 0-dimensional in S, but $X_{s,k(y)} = X_s \times_s y$ is viewed as a curve over the function field $k(y)$ such that the Néron–Tate height is defined with respect to $k(y)/k$.

Now we take the adelic line bundle $\overline{M}$ on J by

$$\overline{M} = \overline{\Theta} + \pi_J^* \overline{\lambda}_S.$$

Here $\overline{\lambda}_S$ is the Hodge bundle $\lambda_{\overline{S}}$ on $\overline{S}$ defined in in §3.1.2, viewed as an adelic line bundle on S/k. It remains to compute

$$\widehat{\deg}(\overline{M}|_y) = \widehat{\deg}(\overline{\Theta}|_y) + \widehat{\deg}((\pi_J^* \overline{\lambda}_S)|_y).$$

For this, resume the above notations $s = \pi_J(y)$ and $z = (y, x)$ with $y \in J_s(\overline{k(y)})$ and $x \in X_s(\overline{k(y)})$.

First, $\overline{\Theta}|_y = (\overline{\Theta}|_{J_s})|_y$ is the pull-back of the adelic line bundle $\overline{\Theta}|_{J_s}$ to $y \in J_s$, and its arithmetic degree is just $2\widehat{h}(y)$ for $y \in J_{s,k(y)}(k(y))$ over the function field $k(y)$.

Second, $(\pi_J^*\overline{\lambda}_S)|_y = \pi_y^*((\overline{\lambda}_S)|_s)$. Here $\pi_y : y \to s$ denotes the natural morphism. If s is a 0-dimensional point of S, then $\pi_y^*((\overline{\lambda}_S)|_s)$ has arithmetic degree 0. Otherwise, s is a 1-dimensional point of S, and thus

$$\widehat{\deg}((\pi_J^*\overline{\lambda}_S)|_y). = \deg(\pi_y)\widehat{\deg}((\overline{\lambda}_S)|_s) = \deg(\pi_y)h_{\mathrm{Fal}}(X_s) = h_{\mathrm{Fal}}(X_{s,k(y)}).$$

Hence, for any 1-dimensional point $y \in J$ with $s = \pi_J(y) \in S$,

$$\widehat{\deg}(\overline{M}|_y) = \widehat{\deg}(\overline{\Theta}|_y) + \widehat{\deg}((\pi_J^*\overline{\lambda}_S)|_y) = 2\widehat{h}(y) + h_{\mathrm{Fal}}(X_{s,k(y)}).$$

Then we have proved that there are constants $c_1, c_2 > 0$ such that for any 1-dimensional point $y \in J$ with $s = \pi_J(y) \in S$,

$$\#\{x \in X_{s,k(y)}(\overline{k(y)}) : \widehat{h}((2g-2)x - (\omega_{X_s} - y)) \leq c_1\big(h_{\mathrm{Fal}}(X_{s,k(y)}) + \widehat{h}(y)\big)\} \leq c_2.$$

We can further modify the statement as follows. For any function field K of one variable over k, and for any non-isotrivial k-morphism $y' : \operatorname{Spec} K \to J$ with image $s' = \pi \circ y' : \operatorname{Spec} K \to S$, we have

$$\#\{x \in X_{s'}(\overline{K}) : \widehat{h}((2g-2)x - (\omega_{X_{s'}} - y')) \leq c_1\big(h_{\mathrm{Fal}}(X_{s'}) + \widehat{h}(y')\big)\} \leq c_2.$$

Here $y' : \operatorname{Spec} K \to J$ is said to be *non-isotrivial* if its image is a 1-dimensional point of J. The heights are normalized over K/k, and this statement is a consequence of the original one by the relation $y = \operatorname{Im}(y')$. It allows K to be any arbitrary finite extension of $k(y)$ (via y'), which does not change the truth of the inequality, since a finite extension replaces the heights by multiplying the same constant.

By the moduli property, the above result further implies that for any function field K of one variable over k, and for any non-isotrivial triple (C, η, y'') of degree 0 and level N over K,

$$\#\{x \in C(\overline{K}) : \widehat{h}((2g-2)x - (\omega_C - y'')) \leq c_1\big(h_{\mathrm{Fal}}(C) + \widehat{h}(y'')\big)\} \leq c_2.$$

We refer to §4.3.2 for the equivalence of the non-isotriviality conditions. By Lemma 4.8, we can remove η from the non-isotriviality condition, so the result

holds for any non-isotrivial pair (C, y'') of degree 0 over K. Moreover, by the relation $y'' = \omega_C - (2g - 2)\alpha$, the result implies that for any non-isotrivial pair (C, α) of degree 1 over K,

$$\#\{x \in C(\overline{K}) : (2g - 2)^2 \widehat{h}(x - \alpha) \leq c_1\big(h_{\mathrm{Fal}}(C) + \widehat{h}(\omega_C - (2g - 2)\alpha)\big)\} \leq c_2.$$

We can further allow $\alpha \in \mathrm{Pic}^1(C_{\overline{K}})$ (instead of $\alpha \in \mathrm{Pic}^1(C)$) by extending K.

Finally, by Lemma 4.9,

$$h_{\mathrm{Fal}}(C) > 4^{-2g} \max\{h_{\mathrm{Fal}}(C), 1\}.$$

Thus we can replace the term $h_{\mathrm{Fal}}(C)$ by $\max\{h_{\mathrm{Fal}}(C), 1\}$ in the above result. This proves the theorem for function fields.

4.4.3 Part 3: uniformity of constants

The constants c_1, c_2 obtained above a priori depend on (g, K), but we are going to prove that they can be chosen uniformly for all K. It suffices to treat the uniformity for K varying as a function field. The key is the following uniform version of Theorem 4.2.

Theorem 4.10. *Let S be a flat and quasi-projective integral scheme over $\mathbb{Z}$. Let $\pi : X \to S$ be a smooth relative curve of genus $g > 1$. Let $\overline{L}$ be a nef adelic line bundle on $X/\mathbb{Z}$, and $\overline{M}$ be an adelic line bundle on $S/\mathbb{Z}$. If $\overline{L}$ is potentially big on X/S, then there is a non-empty open subscheme U of S, and constants $c_1, c_2 > 0$, such that for any function field K of one variable over a field k, and for any non-isotrivial morphism $y : \mathrm{Spec}\, K \to U$,*

$$\#\{x \in X_y(\overline{K}) : h_{\overline{L}|_{X_y}}(x) \leq c_1 \widehat{\deg}(\overline{M}|_y)\} \leq c_2.$$

We first explain many terms of the statement. Let K be a function field of one variable over a field k as in the theorem. Denote by k' the algebraic closure of k in K. The morphism $y : \mathrm{Spec}\, K \to U$ is said to be *non-isotrivial* if it does not factor through the natural morphism $\mathrm{Spec}\, K \to \mathrm{Spec}\, k'$.

In the following, we refer to [YZ2, §2.5.5] for various functorial maps on the groups (or categories) of adelic line bundles. Denote $p = \mathrm{char}(k)$. If $p > 0$, the restriction $\overline{M}|_y$ denotes the image via the composition

$$\widehat{\mathrm{Pic}}(S/\mathbb{Z}) \longrightarrow \widehat{\mathrm{Pic}}(S_{\mathbb{F}_p}/\mathbb{F}_p) \longrightarrow \widehat{\mathrm{Pic}}(S_k/k) \xrightarrow{y'^*} \widehat{\mathrm{Pic}}(K/k).$$

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Here $y' : \operatorname{Spec} K \to S_k = S \times_{\mathbb{Z}} k$ is induced by $y : \operatorname{Spec} K \to S$ via the fiber product. Similarly, if $p = 0$, the restriction $\overline{M}|_y$ denotes the image via the composition

$$\widehat{\operatorname{Pic}}(S/\mathbb{Z}) \longrightarrow \widehat{\operatorname{Pic}}(S_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \widehat{\operatorname{Pic}}(S_k/k) \xrightarrow{y'^*} \widehat{\operatorname{Pic}}(K/k).$$

In both cases, $\widehat{\operatorname{deg}}(\overline{M}|_y)$ is the image of the degree map $\widehat{\operatorname{deg}} : \widehat{\operatorname{Pic}}(K/k) \to \mathbb{R}$ normalized by multiplicity functions given by degrees over k.

The term $X_y = X \times_S (\operatorname{Spec} K, y)$ is a smooth projective curve over K. If $p > 0$, then $\overline{L}|_{X_y}$ denotes the image via the composition

$$\widehat{\operatorname{Pic}}(X/\mathbb{Z}) \longrightarrow \widehat{\operatorname{Pic}}(X_{\mathbb{F}_p}/\mathbb{F}_p) \longrightarrow \widehat{\operatorname{Pic}}(X_k/k) \longrightarrow \widehat{\operatorname{Pic}}(X_y/k).$$

If $= 0$, then $\overline{L}|_{X_y}$ denotes the image via the composition

$$\widehat{\operatorname{Pic}}(X/\mathbb{Z}) \longrightarrow \widehat{\operatorname{Pic}}(X_{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \widehat{\operatorname{Pic}}(X_k/k) \longrightarrow \widehat{\operatorname{Pic}}(X_y/k).$$

In both cases, the height function $h_{\overline{L}|_{X_y}} : X_y(\overline{K}) \to \mathbb{R}$ is normalized by K/k.

Proof of Theorem 4.10. The proof is similar to that of Theorem 4.2(2). In fact, we still have that $\overline{L}_m = m_{\boxtimes}\overline{L}$ is big on $X_m = X_{/S}^m$ for some $m \geq 1$. Then $\overline{L}_m - \epsilon\pi^*\overline{M}$ is big for some positive rational number $\epsilon > 0$, and thus some multiple $n(\overline{L}_m - \epsilon\overline{M})$ of $\overline{L}_m - \epsilon\overline{M}$ is an (integral) adelic line bundle on $X_m/\mathbb{Z}$ with a nonzero effective section s. Denote $Z = \operatorname{div}(s)$. Similarly, for any non-isotrivial point $y : \operatorname{Spec} K \to S$,

$$\{x \in (X_m)_y(\overline{K}) : h_{\overline{L}_m|_{(X_m)_y}}(x) \leq \epsilon\, \widehat{\operatorname{deg}}(\overline{M}|_y)\} \subset Z_y(\overline{K}).$$

The remaining part of the proof is also similar. $\qquad\square$

Now we prove that we can choose (c_1, c_2) uniformly for all K/k in Theorem 4.2. Fix an integer $N \geq 3$, and denote by $M_{g,N}$ the (fine) moduli space of smooth curves of genus g over $\mathbb{Z}[1/N]$ with a full level-N structure. Set $S = M_{g,N}$ and let $\pi : X \to S$ be the universal curve. As in the proof of the number field case, we have data $(X, S, J, X_J, J_J, \overline{\Theta}, \overline{\Theta}_J, \overline{L})$ over $\mathbb{Z}[1/N]$. For example, $\overline{L} = \tau^*(\overline{\Theta}_J)$ now lies in $\widehat{\operatorname{Pic}}(X_J/\mathbb{Z})$.

Let $\overline{M}$ be the adelic line bundle over $J/\mathbb{Z}$ by

$$\overline{M} = \overline{\Theta} + \pi_J^*\overline{\lambda}_S.$$

Here $\overline{\lambda}_S$ is the Hodge bundle associated to $J \to S$ over $\mathbb{Z}$, endowed with the Faltings metric. This is essentially the one used in the number field case, but the metrics at archimedean places will not play an essential role here.

Apply Theorem 4.5(4) and Theorem 4.10 successively to the family $q_1 : X_J \to J$ and the adelic line bundles $\overline{L}$ and $\overline{M}$. As in the proof in the function field case, we conclude that there are a closed subscheme Z of S which is disjoint with $S_{\mathbb{Q}}$, and constants $c_1, c_2 > 0$, such that for any function field K of one variable over k, and for any non-isotrivial point $y : \operatorname{Spec} K \to J$ whose image $s = \pi \circ y : \operatorname{Spec} K \to S$ lies in $S \setminus Z$, we have

$$\#\{x \in X_s(\overline{K}) : \widehat{h}((2g-2)x - (\omega_{X_s} - y)) \le c_1\big(h_{\mathrm{Fal}}(X_s) + \widehat{h}(y)\big)\} \le c_2.$$

The heights are normalized over K/k.

Note that Z lies in finitely many fibers of S over $\operatorname{Spec} \mathbb{Z}$. Then we conclude that there is a positive integer N' such that the constants (c_1, c_2) in Theorem 4.2 are uniform for all K/k with $\operatorname{char}(k) \nmid N'$.

It remains to treat the finitely many characteristics given by $\operatorname{char}(k) \mid N'$. Then we can fix a prime number p, and treat all K/k with $\operatorname{char}(k) = p$. This is proved similarly, except that we work over a fine moduli space $S = M_{g,N'',\mathbb{F}_p}$ over $\mathbb{F}_p$ (with $p \nmid N''$). Accordingly, Theorem 4.10 can be modified to fit the setting that S is quasi-projective over $\mathbb{F}_p$ and $\overline{L}$ is adelic over $X/\mathbb{F}_p$. We omit the details here.

4.5 Uniform fiberwise bigness

Now we prove Theorem 1.4. Due to different settings, we divide the treatment into the number field case (Theorem 4.11) and the function field case (Theorem 4.12). We start with the number field case.

Theorem 4.11 (Theorem 1.4: number field case)**.** *Let $g > 1$ be an integer. Then there is a constants $c_3 > 0$ depending only on g such that for any geometrically connected, smooth and projective curve C of genus g over a number field K, one has*

$$[\overline{\omega}^2_{C/K,a}] > c_3 \cdot \max\{h_{\mathrm{Fal}}(C), 1\}.$$

Proof. As mentioned after Theorem 1.4, it is still a consequence of Theorem 1.3 (or Theorem 3.1, Theorem 3.2), and the proof is similar to but much easier than that of Theorem 1.1 (or equivalently Theorem 4.7).

As in §4.4.1, take S to be a fine moduli space $M_{g,N,\mathbb{Q}}$ over $\mathbb{Q}$ with $N \geq 3$, and take $\pi : X \to S$ to be the universal curve. By Theorem 3.2,

$$\overline{N} := \pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle$$

is big on S. By the height inequality in [YZ2, Thm. 5.3.5], for any adelic line bundle $\overline{M}$ on $S/\mathbb{Z}$, there are a Zariski closed subset $Z \subsetneq S$ and a constant $\epsilon > 0$ such that

$$h_{\overline{N}}(s) \geq \epsilon\, h_{\overline{M}}(s), \quad \forall s \in (S \setminus Z)(\overline{K}).$$

Applying the height inequality to $(\overline{M}|_Z, \overline{N}|_Z)$ repeatedly, and decrease ϵ if necessary, we can assume that

$$h_{\overline{N}}(s) \geq \epsilon\, h_{\overline{M}}(s), \quad \forall s \in S(\overline{K}).$$

Now we convert the inequality to the inequality of the theorem. For the left-hand side of the inequality, we claim that

$$h_{\overline{N}}(s) = [\overline{\omega}^2_{X_s,a}], \quad \forall s \in S(\overline{K}).$$

In fact, denote by $s' \in S$ the closed point corresponding to s. Then the functoriality of the Deligne pairing (cf. [YZ2, Thm. 4.1.3]) gives

$$
\begin{aligned}
h_{\overline{N}}(s) &= \frac{1}{[\mathbb{Q}(s') : \mathbb{Q}]} \widehat{\deg}(\pi_* \langle \overline{\omega}_{X/S,a}, \overline{\omega}_{X/S,a} \rangle|_{s'}) \\
&= \frac{1}{[\mathbb{Q}(s') : \mathbb{Q}]} \widehat{\deg}(\langle \overline{\omega}_{X_{s'}/s',a}, \overline{\omega}_{X_{s'}/s',a} \rangle) \\
&= \frac{1}{[\mathbb{Q}(s') : \mathbb{Q}]} \overline{\omega}^2_{X_{s'}/s',a} \\
&= [\overline{\omega}^2_{X_s,a}].
\end{aligned}
$$

For the right-hand side, similarly to §4.4.1, choose the adelic line bundle $\overline{M}$ on S by

$$\overline{M} = \overline{\lambda}_S + \mathcal{O}(c), \quad c = \frac{g}{2}\log(2\pi) + 1.$$

Then for any point $t \in S(\overline{K})$, we have

$$h_{\overline{M}}(t) = h_{\mathrm{Fal}}(X_t) + c \geq \max\{h_{\mathrm{Fal}}(X_t), 1\}.$$

This finishes the proof of the arithmetic case. $\qquad\square$

Now we prove the following explicit version of the function field case of Theorem 1.4.

Theorem 4.12 (Theorem 1.4: function fields)**.** *Let K be a function field of one variable over a field k. Let C be a geometrically integral, smooth and projective curve of genus $g > 1$ over K. Assume that $C_{\overline{K}}$ is non-isotrivial over $\overline{k}$. Then*

$$\overline{\omega}^2_{C/K,a} \geq \frac{3}{5(2g-1)(3g-1)}\, h_{\mathrm{Fal}}(C) > \frac{3}{5(2g-1)(3g-1)\cdot 4^{2g}}\, \max\{h_{\mathrm{Fal}}(C), 1\}.$$

Proof. The second inequality follows from Lemma 4.9. The first inequality follows from the case $\dim S = 1$ of Theorem 3.9. In this situation, the proof only requires Zhang's theory of adelic line bundles on projective curves over function fields of one variable, and is a direct consequence of the inequalities in [Zha1, Zha3, Cin1, dJo2]. $\square$

Remark 4.13. The constants in the theorem can be improved by a more careful analysis of Lemma 4.9 and by applying [LSW, Prop. 6.1] and [Cin2].

4.6 Non-degeneracy and relative Bogomolov conjecture

The goal of this subsection is to introduce the consequences of our potential bigness on the non-degeneracy problem and relative Bogomolov conjecture related to the three examples in §4.2. These results are key ingredients in the approaches of [Gao1, GH, DGH1, DGH2, Kuh] to uniform Bogomolov-type results, but we do not use them directly in our approach.

4.6.1 Generality on non-degeneracy

Let k be a *field*. Let S be a quasi-projective normal variety over k. Let $\psi : A \to S$ be an abelian scheme over S of relative dimension g, with identity section $e : S \to A$. Let Y be a quasi-projective variety over k with a generically finite morphism $\iota : Y \to A$ over k. Let us first review the notion of non-degeneracy of Y in A. For convenience, we do not assume that ι is a closed immersion.

Let L be a symmetric and ψ-ample line bundle on A, rigidified by an isomorphism $e^*L \simeq \mathcal{O}_S$. Let $\overline{L}$ be the unique adelic line bundle on A extending L and satisfying $[2]^*\overline{L} \simeq 4\overline{L}$ as constructed in [YZ2, Thm. 6.1.1].

Following [YZ2, §6.2.2], we say that the morphism $\iota : Y \to A$ is *non-degenerate* if $\overline{L}|_Y = \iota^*\overline{L}$ is a big adelic line bundle on Y. The definition is independent of the choice of L, as any two ample line bundles can bound each other up to positive multiples. As explained in the loc. cit., if there is an embedding $k \to \mathbb{C}$, the definition is equivalent to the definition of [DGH1, Def. 1.5], which requires the Betti map $Y(\mathbb{C})_V \to (\mathbb{R}/\mathbb{Z})^{2g}$ to have a full rank at some point of $Y(\mathbb{C})_V$ for some simply connected open subset V of $S(\mathbb{C})$.

As an easy consequence of Lemma 4.1, we have the following quick criterion.

Proposition 4.14. *Assume that $Y \to S$ is projective and flat with a geometrically integral generic fiber. If $L|Y$ is big on the generic fiber of $Y \to S$, and $(\pi|_Y)_*\langle \overline{L}|_Y, \cdots, \overline{L}|_Y \rangle$ is big on S/k, then $\iota^m : Y^m_{/S} \to A^m_{/S}$ is non-degenerate for all $m \geq \dim S$.*

4.6.2   Three non-degeneracy examples

Let k be a *field*. Let S be a quasi-projective normal variety over k. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$. Let α be a line bundle on X of degree $d > 0$ on fibers of $\pi : X \to S$.

Recall from §4.2 that we have the following three natural morphisms:

(1) the S-morphism $i_\alpha : X \to J$,

(2) the X-morphism $i_\Delta : X_X \to J_X$,

(3) the J-morphism $\tau = i_{\omega-Q} : X_J \to J_J$.

Note that the definition of Q requires a section of $X \to S$, but the definition of the morphism τ does not. We explain this discrepancy slightly. In fact, we usually require a section of $X \to S$ to rigidify Q and thus determine Q as a unique class in $\mathrm{Pic}(J \times_S X)$. However, without the section, Q can be considered a unique class in $\mathrm{Pic}(J \times_S X)/\mathrm{Pic}(J)$, which still gives a well-defined $i_{\omega-Q}$.

The morphisms induce for $m \geq 1$ three natural morphisms:

(a) the S-morphism

$$i_\alpha^m = (i_\alpha, \cdots, i_\alpha) : X^m_{/S} \longrightarrow J^m_{/S},$$

(b) the X-morphism

$$i_\Delta^m = (i_\Delta, \cdots, i_\Delta) : (X_X)_{/X}^m \longrightarrow (J_X)_{/X}^m,$$

(c) the J-morphism

$$i_{\omega-Q}^m = (i_{\omega-Q}, \cdots, i_{\omega-Q}) : (X_J)_{/J}^m \longrightarrow (J_J)_{/J}^m.$$

The morphism in (b) is closely related to the Faltings–Zhang morphism

$$i_{\mathrm{FZ},m} : X_{/S}^{m+1} \longrightarrow J_{/S}^m, \quad (x_0, \cdots, x_m) \longmapsto (x_1 - x_0, \cdots, x_m - x_0).$$

The morphism in (c) is closely related to the morphism

$$
\begin{aligned}
\tau_m : (X_{/S}^m) \times_S J &\longrightarrow J_{/S}^m, \\
(x_1, \cdots, x_m, y) &\longmapsto ((2g-2)x_1 - \omega_{X/S} + y, x_2 - x_1, \cdots, x_m - x_1).
\end{aligned}
$$

Again, the definition of $i_{\omega-Q}^m$ does not require a section of $X \to S$.

Now we have the following theorem, whose parts (1), (2), (4) in the case $\mathrm{char}(k) = 0$ were proved in [Gao1, Thm. 1.2(i), Thm 1.2'].

Theorem 4.15. *Let k be a field. Let S be a quasi-projective variety over k. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$ with maximal variation. Let α be a line bundle on X of positive degree on fibers of $\pi : X \to S$. Then the following morphisms are non-degenerate:*

(1) the S-morphism $i_\alpha^m : X_{/S}^m \to J_{/S}^m$ for any $m \geq \dim S$,

(2) the X-morphism $i_\Delta^m : (X_X)_{/X}^m \to (J_X)_{/X}^m$ for any $m \geq \dim S + 1$,

(3) the J-morphism $i_{\omega-Q}^m : (X_J)_{/J}^m \to (J_J)_{/J}^m$ for any $m \geq \dim S + g$,

(4) the S-morphism $i_{\mathrm{FZ},m} : X_{/S}^{m+1} \to J_{/S}^m$ for any $m \geq \dim S + 1$,

(5) the S-morphism $\tau_m : (X_{/S}^m) \times_S J \to J_{/S}^m$ for any $m \geq \dim S + g$.

Proof. We can assume that S is normal by replacing it by an open subscheme. Parts (1)-(3) follow from Theorem 4.5 and Proposition 4.14. We will see that (2) and (4) are equivalent, and (3) and (5) are equivalent.

For (4), we need to interpret $i_{\mathrm{FZ},m} : X_{/S}^{m+1} \to J_{/S}^m$ in terms of the morphism in (2). For this purpose, let us recall the X-morphism

$$i_\Delta : X_X \longrightarrow J_X, \quad (x_0, x_1) \longmapsto (x_0, x_1 - x_0).$$

Under the identifications $(X_X)_{/X}^m = X_{/S}^{m+1}$ and $(J_X)_{/X}^m = X \times_S J_{/S}^m$, the morphism $i_\Delta^m : (X_X)_{/X}^m \to (J_X)_{/X}^m$ becomes

$$i_\Delta^m : X_{/S}^{m+1} \longrightarrow X \times_S J_{/S}^m, \quad (x_0, \cdots, x_m) \longmapsto (x_0, x_1 - x_0, \cdots, x_m - x_0).$$

In the notation, $X \times_S (J_{/S}^m)$ is viewed as an abelian scheme over X via projection to the first component. By forgetting the first component, we obtain the non-degeneracy of $i_{\mathrm{FZ},m}$.

For (5), taking a generically finite base change $S' \to S$ if necessary, we can assume that $X \to S$ has a section. Then we have a universal line bundle Q on $J \times_S X$. Consider the morphism

$$i_{\omega - Q} : X_J \longrightarrow J_J, \quad x \longmapsto (2g - 2)x - (\omega_{X_J/J} - Q).$$

The morphism $i_{\omega - Q}^m : (X_J)_{/J}^m \to (J_J)_{/J}^m$ is just

$$i_{\omega - Q}^m : J \times_S (X_{/S}^m) \longrightarrow J \times_S (J_{/S}^m),$$

$$(y, x_1, \cdots, x_m) \longmapsto (y, (2g - 2)x_1 - \omega_{X/S} + y, \cdots, (2g - 2)x_m - \omega_{X/S} + y).$$

In the notation, $J \times_S (J_{/S}^m)$ is viewed as an abelian scheme over J via projection to the first component. By forgetting the first component, we obtain a non-degenerate morphism

$$(i_{\omega - Q}^m)' : J \times_S (X_{/S}^m) \longrightarrow J_{/S}^m,$$

$$(y, x_1, \cdots, x_m) \longmapsto ((2g - 2)x_1 - \omega_{X/S} + y, \cdots, (2g - 2)x_m - \omega_{X/S} + y).$$

Compose it further with the endomorphism

$$\iota : J_{/S}^m \to J_{/S}^m, \quad (y_1, \cdots, y_m) \longmapsto (y_1, y_2 - y_1, \cdots, y_m - y_1).$$

We obtain a non-degenerate morphism

$$(i_{\omega - Q}^m)'' : J \times_S (X_{/S}^m) \longrightarrow J_{/S}^m,$$

$$(y, x_1, \cdots, x_m) \longmapsto ((2g-2)x_1 - \omega_{X/S} + y, (2g-2)(x_2 - x_1) \cdots, (2g-2)(x_m - x_1)).$$

Removing the factors $2g - 2$ in the last $m - 1$ components, we end up with the morphism τ_m. This finishes the proof. $\qquad\square$

4.6.3 Generality on the relative Bogomolov conjecture

Let k be either $\mathbb{Z}$ or a field. Let K be a global field over k. Namely, K is a number field if $k = \mathbb{Z}$; K is a function field of one variable over k if k is a field.

Let S be a quasi-projective normal variety over K. Let $\psi : A \to S$ be an abelian scheme over S of relative dimension g, with identity section $e : S \to A$. Let Y be a quasi-projective variety over K with a generically finite morphism $\iota : Y \to A$ over K. Assume that the composition $Y \to S$ is surjective, $Y_{\bar{\eta}}$ is irreducible and generates the algebraic group $A_{\bar{\eta}}$, where $\bar{\eta}$ is the geometric generic point of S. Recall that the relative Bogomolov conjecture of [DGH2, Conj.1.2] asserts that, if $\dim Y < g$, then there is a constant $\epsilon > 0$ such that

$$Y(L, \epsilon) = \{y \in Y(\overline{K}) : \widehat{h}_L(\iota(y)) \leq \epsilon\}$$

is not Zariski dense in Y.

Here L is a symmetric and ψ-ample line bundle on A, rigidified by an isomorphism $e^*L \simeq \mathcal{O}_S$. The canonical height $\widehat{h}_L : A(\overline{K}) \to \mathbb{R}$ is defined fiberwise over S, which is also the height function associated to the adelic line bundle $\overline{L}$ in $\widehat{\mathrm{Pic}}(A/k)$ (instead of $\widehat{\mathrm{Pic}}(A/K)$) extending L and satisfying $[2]^*\overline{L} \simeq 4\overline{L}$ as constructed in [YZ2, Thm. 6.1.1].

The validity of the relative Bogomolov conjecture is independent of the choice of L. In fact, if L' is another such line bundle on A, then there is a positive integer n such that $nL' - L$ and $nL - L'$ are again such line bundles on A. Then $\widehat{h}_{nL'-L} \geq 0$ and $\widehat{h}_{nL-L'} \geq 0$, and thus $n^{-1}\widehat{h}_L \leq \widehat{h}_{L'} \leq n\widehat{h}_L$.

By the height inequality in [YZ2, Thm. 5.3.5(1)], if $\overline{L}|_Y$ is big on Y, then Y satisfies the relative Bogomolov conjecture. As an easy consequence of Lemma 4.1, we have the following result on the relative Bogomolov conjecture.

Proposition 4.16. *Assume that $Y \to S$ is projective and flat with a geometrically integral generic fiber. If $L|Y$ is big on the generic fiber of $Y \to S$, and $(\pi|_Y)_* \langle \overline{L}|_Y, \cdots, \overline{L}|_Y \rangle$ is big on S/k, then $\iota^m : Y^m_{/S} \to A^m_{/S}$ satisfies the relative Bogomolov conjecture for all $m \geq \dim S + 1$.*

The proposition needs the bigness of adelic line bundles over k instead of over K. Projective models of S over k have dimension $\dim S + 1$, which explains the bound of m in the proposition.

4.6.4 Three examples of the relative Bogomolov conjecture

The following theorem is analogous to Theorem 4.15.

Theorem 4.17. *Let K be either a number field or a function field of one variable. Let S be a quasi-projective variety over K. Let $\pi : X \to S$ be a smooth relative curve over S of genus $g > 1$ with maximal variation. Let α be a line bundle on X of positive degree on fibers of $\pi : X \to S$. Then the following morphisms satisfy the relative Bogomolov conjecture:*

(1) the S-morphism $i_\alpha^m : X_{/S}^m \to J_{/S}^m$ for any $m \geq \dim S + 1$.

(2) the X-morphism $i_\Delta^m : (X_X)_{/X}^m \to (J_X)_{/X}^m$ for any $m \geq \dim S + 2$.

(3) the J-morphism $i_{\omega - Q}^m : (X_J)_{/J}^m \to (J_J)_{/J}^m$ for any $m \geq \dim S + g + 1$.

(4) the S-morphism $i_{\mathrm{FZ},m} : X_{/S}^{m+1} \to J_{/S}^m$ for any $m \geq \dim S + 2$.

(5) the S-morphism $\tau_m : (X_{/S}^m) \times_S J \to J_{/S}^m$ for any $m \geq \dim S + g + 1$.

Proof. The proof is very similar to Theorem 4.15. In fact, parts (1)-(3) follow from Theorem 4.5 and Proposition 4.16. We also have that (2) and (4) are equivalent, and (3) and (5) are equivalent. Similar to Proposition 4.16, we need the bigness of the adelic line bundles over k instead of over K, where $k = \mathbb{Z}$ if K is a number field and k is the field of constants if K is a function field of one variable. As a consequence, the bounds for m are increased by 1 from those in Theorem 4.15. $\qed$

A   Admissible metrized line bundles

The goal of this appendix is to review the theory of Zhang [Zha1] on admissible pairings on curves, and we will use the terminology of adelic line bundles and metrized line bundles introduced in [Zha2]. Note that the treatment of [Zha1] was written before [Zha2] and was based on graph theory, so we think it is necessary to survey a detailed transfer of the terminology. We will use Berkovich spaces of [Ber] for metrics on line bundles, and our approach of the main result (Theorem A.1) works for both the archimedean case and the non-archimedean case.

A.1 Arakelov metrics in the complex case

We first recall the complex case introduced in Arakelov's original work [Ara].
Let C be a compact Riemann surface of genus $g > 0$.

Define a natural hermitian pairing on $\Gamma(C, \omega_C)$ by

$$\langle \alpha, \beta \rangle = i \int_C \alpha \wedge \overline{\beta}.$$

Let $\alpha_1, \cdots, \alpha_g$ be an orthonormal basis of this pairing. The *Arakelov Kähler form* on C is defined by

$$d\mu_{\mathrm{Ar}} = \frac{i}{g} \sum_{j=1}^{g} \alpha_j \wedge \overline{\alpha}_j.$$

The definition is independent of the choice of the orthonormal basis.

A smooth hermitian metric $\|\cdot\|$ on a line bundle L on C is called *admissible* if the Chern form

$$c_1(L, \| \cdot \|) = \deg(L)\, d\mu_{\mathrm{Ar}}.$$

A smooth Green's function $g_D : C \setminus |D| \to \mathbb{R}$ of a divisor D on C is called *admissible* if the hermitian metric on $\mathcal{O}(D)$ defined by $\|1\| = \exp(-g_D)$ is admissible. The condition is equivalent to

$$\frac{i}{\pi} \partial \overline{\partial} g_D = d\mu_{\mathrm{Ar}} - \delta_D$$

as currents on C.

A smooth hermitian metric on a line bundle on $C^2 = C \times C$ is called *admissible* if for any point $x \in C$, the pull-back metrics of the metric to $C \times x$ and $x \times C$ are both admissible.

Any line bundle on C (resp. C^2) admits an admissible metric, which is unique up to multiplicative constants.

The *Arakelov Green's function* on C^2 is the unique symmetric and smooth Green's function

$$g_{\mathrm{Ar}} : C^2 \setminus \Delta \longrightarrow \mathbb{R}$$

of the diagonal Δ in C^2 satisfying the following properties:

(1) For any point $x_0 \in C$, the Green's function $g_{\mathrm{Ar}}(x_0, \cdot)$ of x_0 on C is admissible.

(2) For any point $x_0 \in C$, the total integral

$$\int_C g_{\mathrm{Ar}}(x_0, \cdot) d\mu_{\mathrm{Ar}} = 0.$$

The Green's function gives a hermitian metric $\| \cdot \|_{\Delta, \mathrm{Ar}}$ of $\mathcal{O}(\Delta)$ on C^2 by $\|1\| = \exp(-g_{\mathrm{Ar}})$. It is actually admissible.

Finally, the *Arakelov metric* $\|\cdot\|_{\mathrm{Ar}}$ of ω_C on C is the unique smooth metric such that the residue map

$$(\omega_C \otimes_{\mathcal{O}_C} \mathcal{O}(x_0))|_{x_0} \longrightarrow \mathbb{C}$$

is an isometry. Here $\mathbb{C}$ is endowed with the usual absolute value, and $\mathcal{O}(x_0)$ is endowed with the metric given by $\|1\| = \exp(-g_{\mathrm{Ar}}(x_0, x))$.

The choice of $d\mu_{\mathrm{Ar}}$ implies that the metric $\| \cdot \|_{\mathrm{Ar}}$ is admissible. This is the reason to choose $d\mu_{\mathrm{Ar}}$, and determines $d\mu_{\mathrm{Ar}}$ uniquely.

A.2 Zhang metrics in the non-archimedean case

Let K be a non-archimedean field, i.e. a complete field with a non-trivial non-archimedean valuation $|\cdot|$.

For any projective variety X over K, denote by X^{an} the Berkovich analytic space of X over K. We refer to [YZ1, App. 1] for the notion of continuous (resp. semipositive, integrable) *metrics* of line bundles of X over K.

Let C be a smooth projective curve of genus $g > 0$ over K. As a convention, all curves are assumed to be geometrically integral in this appendix.

We start with some terminology of metrics on C^2. Denote by Δ the diagonal divisor of C^2. A metric $\|\cdot\|_\Delta$ of $\mathcal{O}_{C^2}(\Delta)$ on $(C^2)^{\mathrm{an}}$ is called *symmetric* if the Green's function

$$g_\Delta = -\log \|1\|_\Delta : (C^2)^{\mathrm{an}} \setminus \Delta^{\mathrm{an}} \to \mathbb{R}$$

is symmetric in the sense that it is invariant under the action on $(C^2)^{\mathrm{an}}$ induced by the transposition action on C^2 by switching the two components.

The Green's function induces a function

$$g_\Delta : C(\overline{K})^2 \setminus \Delta(\overline{K}) \longrightarrow \mathbb{R}$$

by the natural maps

$$C(\overline{K})^2 \longrightarrow |C^2|_0 \longrightarrow (C^2)^{\mathrm{an}}.$$

Here $|C^2|_0$ denotes the set of closed points of C^2. Then the symmetry can also be understood in the usual sense.

For any finite extension K' of K and any point $x \in C(K')$, denote

$$(\mathcal{O}(x), \|\cdot\|_x) := i_x^*(\mathcal{O}(\Delta), \|\cdot\|_\Delta)$$

as metrized line bundles on $C_{K'}$. Here we view x as a closed point of $X_{K'}$, $\mathcal{O}(x)$ is the line bundle on $X_{K'}$ corresponding to $x \in X_{K'}$, and

$$i_x = (x, \mathrm{id}) : \operatorname{Spec} K' \times C \longrightarrow C \times C$$

is the natural morphism. It follows that

$$g_x = -\log\|1\|_x : (C_{K'})^{\mathrm{an}} \setminus \{x\} \longrightarrow \mathbb{R}$$

is equal to the pull-back of g_Δ via the map $i_x^{\mathrm{an}} : (C_{K'})^{\mathrm{an}} \to (C^2)^{\mathrm{an}}$. We may also write $g_x = g_\Delta(x, \cdot)$ by abuse of notations. Finally, we re-organize the construction of [Zha1] as the following statement.

Theorem A.1. *Let K be a non-archimedean field, and C be a smooth projective curve of genus $g > 0$ over K. There is a unique pair $(\|\cdot\|_{\Delta,a}, \|\cdot\|_a)$, where*

(a) $\|\cdot\|_a$ is an integrable metric of $\omega_{C/K}$ on C^{an},

(b) $\|\cdot\|_{\Delta,a}$ is a symmetric integrable metric of $\mathcal{O}_{C^2}(\Delta)$ on $(C^2)^{\mathrm{an}}$,

satisfying the following properties for all finite extensions K'/K and all points $x, y \in C(K')$:

(1) the equalities
$$c_1(\mathcal{O}(x), \|\cdot\|_x) = c_1(\mathcal{O}(y), \|\cdot\|_y)$$

and
$$(2g-2)c_1(\mathcal{O}(x), \|\cdot\|_x) = c_1(\omega_{C_{K'}/K'}, \|\cdot\|_a),$$

hold as Chambert-Loir measures on $(C_{K'})^{\mathrm{an}}$;

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(2) the integral
$$\int_{(C_{K'})^{\mathrm{an}}} g_{\Delta,a}(x,\cdot)\, c_1(\mathcal{O}(x), \|\cdot\|_x) = 0.$$

(3) the residue map
$$(\omega_{C/K} \otimes_{\mathcal{O}_C} \mathcal{O}(x))|_x \longrightarrow K'$$

is an isometry, where K' is endowed with the absolute value extending that of K.

Here the Chambert-Loir measure $c_1(\omega_{C_{K'}/K'}, \|\cdot\|_a)$ uses the base change of the metric $\|\cdot\|_a$ from to $\omega_{C/K}$ to $\omega_{C_{K'}/K'}$. Note that if $g > 1$, the first two properties are equivalent to the following:

(1) the equality
$$(2g - 2)c_1(\mathcal{O}(x), \|\cdot\|_x) = c_1(\omega_{C_{K'}/K'}, \|\cdot\|_a)$$

holds as Chambert-Loir measures on $(C_{K'})^{\mathrm{an}}$;

(2) the integral
$$\int_{(C_{K'})^{\mathrm{an}}} g_{\Delta,a}(x,\cdot)c_1(\omega_{C_{K'}/K'}, \|\cdot\|_a) = 0.$$

We call $\|\cdot\|_a$ the *Zhang metric* of $\omega_{C/K}$ on C^{an}, and call $\|\cdot\|_{\Delta,a}$ the *Zhang metric* of $\mathcal{O}(\Delta)$ on $(C^2)^{\mathrm{an}}$. We also call them the *canonical admissible metrics.*

Note that the statement of the theorem also works for archimedean K, which concerns the Arakelov metrics recalled in the previous subsection. Moreover, our proof of the theorem in the following works in both the archimedean case and the non-archimedean case. For simplicity, we will still assume that K is non-archimedean in the following.

We provide two approaches of the uniqueness in the theorem. For the first approach, by base change, we can assume $C(K) \neq \emptyset$ and take a point $x_0 \in X(K)$. Assume that there is another pair $(\|\cdot\|'_{\Delta,a}, \|\cdot\|'_a)$ of metrics satisfying the properties, and denote $\|\cdot\|'_x$ similarly. Write $\|\cdot\|'_{x_0} = \|\cdot\|_{x_0} e^{\varphi}$ for a continuous function $\varphi : C^{\mathrm{an}} \to \mathbb{R}$. By property (1),

$$c_1(\mathcal{O}_C, \|\cdot\|'_a/\|\cdot\|_a) = (2g - 2)c_1(\mathcal{O}_C, \|\cdot\|'_{x_0}/\|\cdot\|_{x_0}) = c_1(\mathcal{O}_C, e^{(2g-2)\varphi})$$

and

$$c_1(\mathcal{O}_{C_{K'}}, \| \cdot \|'_x / \| \cdot \|_x) = c_1(\mathcal{O}_{C_{K'}}, \| \cdot \|'_{x_0} / \| \cdot \|_{x_0}) = c_1(\mathcal{O}_{C_{K'}}, e^\varphi).$$

By the non-archimedean Calabi theorem in [YZ1, Cor. 2.2],

$$\| \cdot \|'_a / \| \cdot \|_a = e^{(2g-2)\varphi + c}$$

over C^{an} for some constant $c \in \mathbb{R}$, and

$$\| \cdot \|'_x / \| \cdot \|_x = e^{\psi(x)} e^\varphi$$

over $C_{K'}^{\mathrm{an}}$ for some function $\psi : C(\overline{K}) \to \mathbb{R}$. This gives

$$g_\Delta(x, y) - g'_\Delta(x, y) = \psi(x) + \varphi(y), \quad x, y \in C(\overline{K}).$$

Since g'_Δ and g_Δ are symmetric, we have $\psi = \varphi + c'$ for a constant $c' \in \mathbb{R}$. By property (3),

$$\frac{\| \cdot \|'_a(x)}{\| \cdot \|_a(x)} \cdot \frac{\| \cdot \|'_x(x)}{\| \cdot \|_x(x)} = 1, \qquad \frac{\| \cdot \|'_a(x)}{\| \cdot \|_a(x)} \cdot e^{\psi(x)} e^{\varphi(x)} = 1,$$

and thus

$$\| \cdot \|'_a = \| \cdot \|_a e^{-2\varphi - c'}.$$

Compare with $\| \cdot \|'_a / \| \cdot \|_a = e^{(2g-2)\varphi + c}$ (for $g > 0$). We see that φ is a constant on C^{an}. It follows that $\| \cdot \|'_x / \| \cdot \|_x$ is a constant. Then

$$c_1(\mathcal{O}(x), \| \cdot \|'_x) = c_1(\mathcal{O}(x), \| \cdot \|_x).$$

Property (2) implies $\| \cdot \|'_x = \| \cdot \|_x$, and property (3) implies $\| \cdot \|'_a = \| \cdot \|_a$. This proves the uniqueness.

Our second proof of the uniqueness in the theorem follows from [Zha1, Thm. 4.6]. This also gives the connection to the construction in the loc. cit. In fact, for any $D, E \in \mathrm{Div}(C_{\overline{K}})$ with disjoint supports, define

$$(D, E)_a = g_{\Delta, a}(D, E),$$

by bi-linearity. We can check that this pairing satisfies all properties of [Zha1, Thm. 4.6], and thus the pairing $g_{\Delta, a} : C(\overline{K})^2 \setminus \Delta(\overline{K}) \to \mathbb{R}$ is unique up to additive constants. By continuity, the function $g_{\Delta, a} : (C^2)^{\mathrm{an}} \setminus \Delta^{\mathrm{an}} \to \mathbb{R}$ is unique to additive constants. This determines $c_1(\omega_{C/K}, \| \cdot \|_a)$ uniquely by

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property (1). Then g_x is uniquely determined by property (2), and $\|\cdot\|_a$ is uniquely determined by property (3). Note that [Zha1, Thm. 4.6] assumes that K has a discrete valuation, but it also holds for non-discrete valuations.

There are also two approaches of the existence in the literature. The first one is in terms of reduction graphs, which is worked out in details by Zhang [Zha1]. The second one is in terms of admissible metrics of line bundles on the Jacobian variety, which is outlined in [Zha1], and worked out partially by Heinz [Hei]. The first approach has the advantage of being explicit, while the second approach has the advantage of being functorial and includes all (non-classical) points of the Berkovich spaces. We will prove the theorem by the second approach, and then relate it to the first approach.

A.3 Admissible metrics

Let K be a non-archimedean field. As in [Hei, §3], we introduce admissible metrics of line bundles on curves (resp. square of curves) by means of Jacobian varieties.

Let A be an abelian variety over K, and M be a line bundle on A. Let $\|\cdot\|$ be a continuous metric of M on A^{an}, and denote $\overline{M} = (M, \|\cdot\|)$. Define admissibility in the following three cases:

(1) If M is even in the sense that $[-1]^*M \simeq M$, there is an isomorphism $[2]^*M \otimes M^{\otimes(-4)} \simeq \mathcal{O}_A$. The metric $\|\cdot\|$ is called *admissible* if there is a constant metric on $\mathcal{O}_A$ such that the isomorphism is an isometry.

(2) If M is odd in the sense that $[-1]^*M \simeq M^{\otimes(-1)}$, there is an isomorphism $[2]^*M \otimes M^{\otimes(-2)} \simeq \mathcal{O}_A$. The metric $\|\cdot\|$ is called *admissible* if there is a constant metric on $\mathcal{O}_A$ such that the isomorphism is an isometry.

(3) In general, the metric $\|\cdot\|$ of M is called *admissible* if the induced metrics on the even line bundle $M \otimes [-1]^*M$ and the odd line bundle $M \otimes [-1]^*M^{\otimes(-1)}$ are both admissible.

By Tate's limiting argument in [Zha2], any line bundle on A has an admissible metric, which is unique up to multiplicative constants. Note that all these arguments were originally written for metrics on $A(\overline{K})$ instead of on A^{an}, but they can be modified to A^{an} without essential difficulty. Moreover, admissible metrics are integrable for general L and semipositive for ample L.

Let C be a smooth projective curve of genus $g > 0$ over K. Let $\alpha \in \mathrm{Pic}^1(C)$ be a line bundle of degree 1 on C. If α does not exist, we will

need to pass to a finite extension of K, and we will come back to this issue later. Denote by $J = \mathrm{Pic}^0_{C/K}$ the Jacobian variety of C over K. Denote the canonical embedding

$$i_\alpha : C \longrightarrow J, \quad x \longmapsto (x) - \alpha.$$

The theta divisor $\theta_\alpha \subset J$ is the image of the morphism

$$C^{g-1} \longrightarrow J, \quad (x_1, \cdots, x_{g-1}) \longmapsto i_\alpha(x_1) + \cdots + i_\alpha(x_{g-1}).$$

Let L be a line bundle on C. Following [Hei, §4], a continuous metric $\| \cdot \|$ of L on C^{an} is called *admissible* if there exist a line bundle M on J, and an admissible metric $\| \cdot \|_M$ of M on J^{an}, such that M is algebraically equivalent to an integer multiple of θ_α on J, and such that $(L, \| \cdot \|)^{\otimes m}$ is isometric to $i_\alpha^*(M, \| \cdot \|_M)$ for some positive integer m.

This definition looks very random, but any line bundle L on C has an admissible metric, which is unique up to multiplicative constants. Moreover, the definition is independent of the choice of α and stable under base change.

In the case $\mathrm{Pic}^1(C) = \emptyset$, we say that a metric $\| \cdot \|$ of L on C^{an} is *admissible* if the induced metric of $L_{K'}$ on $(C_{K'})^{\mathrm{an}}$ is admissible for some finite extension K'/K over K with $\mathrm{Pic}^1(C_{K'}) \neq \emptyset$. Then in this general case, any line bundle L on C still has an admissible metric, unique up to multiplicative constants.

We have the following interpretation in terms of Monge–Ampère equations, where an explicit form of $d\mu_a$ will be given in Proposition A.5.

Proposition A.2. *There exists a unique probability measure $d\mu_a$ on C^{an} such that for any metrized line bundle $(L, \| \cdot \|)$ on C, the metric $\| \cdot \|$ is admissible if and only if it is integrable and satisfies*

$$c_1(L, \| \cdot \|) = \deg(L)\, d\mu_a$$

for the Chambert-Loir measure on C^{an}.

Proof. It is easy to reduce the problem to the case $\mathrm{Pic}^1(C) \neq \emptyset$. Then we take $\alpha \in \mathrm{Pic}^1(C)$.

Denote $(L, \| \cdot \|) = i_\alpha^*(\mathcal{O}(\theta_\alpha), \| \cdot \|_\alpha)$, where $\| \cdot \|_\alpha$ is an admissible metric of $\mathcal{O}(\theta_\alpha)$ on J^{an}. Define the measure $d\mu_a$ by the equation

$$c_1(L, \| \cdot \|) = \deg(L)\, d\mu_a.$$

To prove the equation for any admissible $(L, \|\cdot\|)$, it suffices to prove that if M is a line bundle on J algebraically equivalent to 0, then the measure $c_1(i_\alpha^*(M, \|\cdot\|_M)) = 0$ for any admissible metric $\|\cdot\|_M$ of M on J^{an}. This is exactly [YZ1, Thm. 5.16], which is essentially a result of Gubler.

It remains to prove that any metric satisfying the equation is admissible. Note that admissible metrics already satisfy the equation, so it suffices to prove that the equation determines the metric up to a constant multiple. This follows from the non-archimedean Calabi theorem in [YZ1, Cor. 2.2]. We only need the theorem for curves, in which case the ampleness assumption can be removed by additivity. □

Let L be a line bundle on C^2. A continuous metric of L on $(C^2)^{\mathrm{an}}$ is called *admissible* if its pull-back metrics to $(C_{K'})^{\mathrm{an}}$ via all morphisms $(x, \mathrm{id}) : C_{K'} \to C \times C$ and $(\mathrm{id}, x) : C_{K'} \to C \times C$, for all points $x \in C(K')$ over all finite extensions K'/K, are admissible. This definition follows from [Hei, Def. 4.2], except that the loc. cit. used the term "bi-admissible" instead of "admissible".

Any line bundle on C^2 admits an admissible metric, unique up to multiplicative constants. The uniqueness is an easy consequence of the definition. For the existence, it suffices to assume that $C(K)$ contains an element x_0 by extending K if necessary. Note that admissible metrics of line bundles in $p_1^*\mathrm{Pic}(C)$ and $p_2^*\mathrm{Pic}(C)$ can be obtained as pull-back of admissible metrics. One is reduced to treat the subgroup $\mathrm{Pic}^-(C^2)$ of $\mathrm{Pic}(C^2)$ consisting of line bundles on C^2 whose restrictions to $C \times \{x_0\}$ and $\{x_0\} \times C$ are both trivial. Denote by $\mathrm{Pic}^-(J^2)$ the subgroup of $\mathrm{Pic}(J^2)$ consisting of line bundles on J^2 whose restrictions to $J \times \{0\}$ and $\{0\} \times J$ are both trivial. It turns out that the canonical map $\mathrm{Pic}^-(J^2) \to \mathrm{Pic}^-(C^2)$ is an isomorphism, and thus elements of $\mathrm{Pic}^-(C^2)$ have admissible metrics by pull-back of admissible metrics from J^2. See [Zha3, Lem. 2.2.1, Lem. 2.2.2, Lem. 2.2.3] for all details in this construction.

By our construction below, we will see that in Theorem A.1, the metrics $\|\cdot\|_a$ and $\|\cdot\|_{\Delta,a}$ are both admissible.

A.4 Construction of the Zhang metrics

The goal is to construct the Zhang metrics in Theorem A.1 in terms of Jacobian varieties.

For the existence in the theorem, we remark that for any finite Galois extension K'/K, if a pair $(\|\cdot\|_{\Delta,a}, \|\cdot\|_a)$ exists over K', then it is Galois invariant by the uniqueness, and thus descends to K. Therefore, by replacing K by a finite Galois extension, we can assume that C has some convenient properties. For example, we can assume that $C(K) \neq \emptyset$.

We will need the following lemma here, and it will also be used in the main part of this article. For clarity, we will state all the notations and conditions of this lemma separately.

Lemma A.3. *Let K be any field, and C be a smooth projective curve of genus $g > 0$ over K. Denote by J the Jacobian variety of C over K. Let α be a divisor on C of degree 1. Let i_α be the canonical embedding*

$$i_\alpha : C \longrightarrow J, \quad x \longmapsto (x) - \alpha.$$

Let $\theta_\alpha \subset J$ be the theta divisor, i.e., the image of the morphism

$$C^{g-1} \longrightarrow J, \quad (x_1, \cdots, x_{g-1}) \longmapsto i_\alpha(x_1) + \cdots + i_\alpha(x_{g-1}).$$

Denote by P the Poincaré line bundle on $J \times J$. Then the following are true:

(1) There are isomorphisms

$$i_\alpha^* \mathcal{O}([-1]^* \theta_\alpha) \longrightarrow g\alpha,$$

$$i_\alpha^* \mathcal{O}(\theta_\alpha) \longrightarrow \omega_{C/K} + (2 - g)\alpha.$$

(2) Denote by $(i_\alpha, i_\alpha) : C \times C \to J \times J$ the natural morphism. Then there is an isomorphism

$$(i_\alpha, i_\alpha)^* P \longrightarrow \mathcal{O}(\Delta) - p_1^* \alpha - p_2^* \alpha.$$

(3) Denote by $\Delta_J : J \to J \times J$ the diagonal morphism. Then there is an isomorphism

$$\Delta_J^* P \longrightarrow \mathcal{O}(-(\theta_\alpha + [-1]^* \theta_\alpha)).$$

Proof. The first isomorphism of (1) is a special case of the first equation in [Ser, p. 75], and the second isomorphism is derived from the first one by [Ser, p. 74, eq. (1)]. Note that our $(\theta_\alpha, [-1]^* \theta_\alpha)$ is (Θ, Θ') in [Ser]. The isomorphism in (2) is [Ser, p. 76, eq. (4)], which assumes $(2g - 2)\alpha = \omega_{C/K}$,

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but the proof works without this assumption by some extra computation using the first equation in [Ser, p. 75]. The result can also be checked using the universal property of P. For (3), by [Ser, p. 76, eq. (3)], there is an isomorphism

$$P \simeq m^* \mathcal{O}(-\theta_\alpha) \otimes p_1^* \mathcal{O}(\theta_\alpha) \otimes p_2^* \mathcal{O}(\theta_\alpha).$$

It follows that

$$\Delta_J^* P \simeq [2]^* \mathcal{O}(-\theta_\alpha) \otimes \mathcal{O}(2\theta_\alpha).$$

The result follows from the basic formula

$$[2]^* \mathcal{O}(\theta_\alpha) \simeq \mathcal{O}(3\theta_\alpha + [-1]^* \theta_\alpha).$$

See the corollary in [Ser, p.33]. $\square$

Part (1) of the lemma gives a concrete way to construct admissible metrics of α and $\omega_{C/K}$ on C^{an}. Part (2) gives a concrete way to construct admissible metrics of $\mathcal{O}(\Delta)$ on $(C^2)^{\mathrm{an}}$.

Now we are ready to prove the existence in Theorem A.1. We will prove that there is a unique admissible metric $\|\cdot\|_a$ of $\omega_{C/K}$ and a unique admissible metric $\|\cdot\|_{\Delta,a}$ of $\mathcal{O}(\Delta)$ satisfying the requirements. For that purpose, fix an admissible metric $\|\cdot\|_a'$ of $\omega_{C/K}$ and an admissible metric $\|\cdot\|_\Delta'$ of $\mathcal{O}(\Delta)$. We will find scalar multiples of these two metrics to satisfy the conditions. By Lemma A.2, part (1) of the theorem holds for $(\|\cdot\|_\Delta', \|\cdot\|_a')$. It suffices to prove that (2) holds for $(\|\cdot\|_\Delta', \|\cdot\|_a')$ up to an additive constant independent of x, and that (3) holds for $(\|\cdot\|_\Delta', \|\cdot\|_a')$ up to a multiplicative constant independent of x. Then we can easily modify $\|\cdot\|_\Delta'$ to satisfy (2) and modify $\|\cdot\|_a'$ to satisfy (3).

We first prove that (3) holds for $(\|\cdot\|_\Delta', \|\cdot\|_a')$ up to a constant. By definition, there is a canonical isomorphism

$$\omega_{C/K} \otimes_{\mathcal{O}_C} (\mathcal{O}(\Delta)|_\Delta) \longrightarrow \mathcal{O}_C.$$

We claim that the isomorphism is an isometry up to constants under the metrics $(\|\cdot\|_\Delta', \|\cdot\|_a')$ and the trivial metric of $\mathcal{O}_C$. By uniqueness of admissible metrics, it suffices to prove that $(\mathcal{O}(\Delta)|_\Delta, \|\cdot\|_\Delta')$ is admissible on C. This is [Hei, Prop. 4.3]. It is also easy to prove it in our setting. In fact, construct an admissible metric on $\mathcal{O}(\Delta)$ by Lemma A.3(2). Note that the composition

$$C \xrightarrow{\Delta} C^2 \xrightarrow{(i_\alpha, i_\alpha)} J^2$$

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is equal to the composition

$$C \xrightarrow{i_\alpha} J \xrightarrow{\Delta_J} J^2,$$

where $\Delta_J : J \to J^2$ is the diagonal morphism. Thus it suffices to check that $\Delta_J^* P$ is algebraically equivalent to a multiple of θ_α, but this follows from Lemma A.3(3).

Therefore, the isomorphism

$$\omega_{C/K} \otimes_{\mathcal{O}_C} (\mathcal{O}(\Delta)|_\Delta) \longrightarrow \mathcal{O}_C$$

is an isometry up to a constant multiple. Take the pull-back of this isometry via $x : \operatorname{Spec} K' \to C$, we get an isomorphism

$$x^*\omega_{C/K} \otimes_{K'} x^*(\mathcal{O}(\Delta)|_\Delta) \longrightarrow K',$$

which is an isometry up to a constant independent of x. Note that the composition

$$\operatorname{Spec} K' \xrightarrow{x} C \xrightarrow{\Delta} C^2$$

is equal to the composition

$$\operatorname{Spec} K' \xrightarrow{x} C_{K'} \xrightarrow{(x,\mathrm{id})} C^2,$$

which induces canonical isomorphisms

$$x^*(\mathcal{O}(\Delta)|_\Delta) \longrightarrow x^*\mathcal{O}((x,\mathrm{id})^*\Delta) \longrightarrow x^*\mathcal{O}(x).$$

Under this composition, the above isomorphism becomes exactly the residue map. This proves that (3) holds for $(\|\cdot\|'_\Delta, \|\cdot\|'_a)$ up to a constant.

It remains to prove that (2) holds for $(\|\cdot\|'_\Delta, \|\cdot\|'_a)$ up to constants. Let L be a line bundle on C endowed with an admissible metric. There are canonical isomorphisms

$$p_{1*}\langle \mathcal{O}(\Delta), p_2^*L \rangle \longrightarrow \Delta^* p_2^* L \longrightarrow L,$$

where the left-hand side is the Deligne pairing with respect to the morphism $p_1 : C^2 \to C$. The metrics of $\mathcal{O}(\Delta)$ and p_2^*L induce a canonical continuous metric of the Deligne pairing. In the complex case, this process is explicitly treated in [Del, Elk] and [YZ2, §4.2]. The non-archimedean case is written in [YZ2, §4.6.2].

With the natural metric of the Deligne pairing, we claim that the isomorphism

$$p_{1*}\langle \mathcal{O}(\Delta), p_2^*L \rangle \longrightarrow L$$

is an isometry up to a constant. If this holds, taking the base change of $p_1 : C^2 \to C$ by the morphism $x : \operatorname{Spec} K' \to C$, we have the structure morphism $\pi' : C_{K'} \to \operatorname{Spec} K'$. The Deligne pairing under this base change gives an isomorphism

$$\pi'_*\langle \mathcal{O}(x), L_{K'} \rangle \longrightarrow L(x),$$

which is an isometry up to a constant. By definition of the metrics, the logarithm of the norm of this isomorphism is actually

$$-\int_{(C_{K'})^{\mathrm{an}}} g'_x c_1(L, \| \cdot \|_L).$$

The constancy of this integral with $L = \omega_{X/K}$ is exactly what we need for (2) of Theorem A.1.

It remains to prove the claim that the isomorphism

$$p_{1*}\langle \mathcal{O}(\Delta), p_2^*L \rangle \longrightarrow L$$

is an isometry on C up to a constant. By the uniqueness of admissible metrics, it suffices to prove that the metric of the left-hand side is admissible.

As a consequence of [YZ2, Thm. 4.6.2], the Deligne pairings and the metrics are compatible with base change. To apply this compatibility, denote by $q_1 : J \times C \to J$ and $q_2 : J \times C \to C$ the projections. View $p_1 : C^2 \to C$ as the base change of $q_1 : J \times C \to J$ by $i_\alpha : C \to J$. Denote by $Q = (\mathrm{id}, i_\alpha)^*P$ the pullback of the Poincaré bundle P via $(\mathrm{id}, i_\alpha) : J \times C \to J \times J$. By Lemma A.3(2), there are isomorphisms

$$(i_\alpha, \mathrm{id})^*Q \longrightarrow (i_\alpha, i_\alpha)^*P \longrightarrow \mathcal{O}(\Delta) - p_1^*\alpha - p_2^*\alpha.$$

Then the compatibility with base change gives an isometry (up to a constant)

$$p_{1*}\langle \mathcal{O}(\Delta) - p_1^*\alpha - p_2^*\alpha, \ p_2^*L \rangle \longrightarrow i_\alpha^*\big(q_{1*}\langle Q, q_2^*L \rangle\big),$$

where α is endowed with an admissible metric, and $Q = (\mathrm{id}, i_\alpha)^*P$ is endowed with the pull-back of an admissible metric of P.

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On the other hand, by a local version of [YZ2, Lem. 4.6.1(2)], $p_{1*}\langle p_1^*\alpha,\ p_2^*L\rangle$ is isometric to $\alpha^{\otimes \deg(L)}$, and thus admissible. Moreover, $p_{1*}\langle p_2^*\alpha,\ p_2^*L\rangle$ is isometric to $\mathcal{O}_C$ with a constant metric. To see this, by additivity and base change, we can assume that $\alpha = \mathcal{O}(x_0)$ for a point $x_0 \in C(K)$. Then we can apply the integration formula in [YZ2, §4.6.2] to get the result.

Therefore, it suffices to prove that $M = q_{1*}\langle Q, q_2^*L\rangle$ is algebraically equivalent to 0, and its metric is admissible on J. Consider the base change of $q_1 : J \times C \to J$ by $[2] : J \to J$. This gives an isometry of Deligne pairings

$$[2]^*q_{1*}\langle Q, q_2^*L\rangle \longrightarrow q_{1*}\langle [2]_C^*Q, [2]_C^*q_2^*L\rangle.$$

Here $[2]_C : J \times C \to J \times C$ is the base change of $[2] : J \to J$. Note that there is a natural isometry $[2]_C^*q_2^*L \simeq q_2^*L$. We also have an isometry (up to constant) $[2]_C^*Q \simeq Q^{\otimes 2}$, which can be obtained by the argument right before [YZ2, Lem. 6.5.3]. It follows that there is an isometry (up to constant) $[2]^*M \simeq M^{\otimes 2}$ on J. This proves that M is algebraically equivalent to 0, and its metric is admissible on J.

Therefore, part (2) of Theorem A.1 also holds up to multiplicative constants. Then the proof of the theorem is complete.

A.5 Zhang's construction by graph theory

Now we review the construction of [Zha1] to give explicit descriptions of the metrics in Theorem A.1.

Let K be a non-archimedean field with a discrete valuation. Denote $e_K = |\varpi|^{-1}$, where ϖ is a generator of the maximal ideal of the valuation ring O_K.

Let C be a smooth projective curve of genus $g > 0$ over K. As before, we can replace K by a finite Galois extension in the definitions. Then we assume that $C(K) \neq \emptyset$, and C has *split semistable reduction* over O_K, i.e., the minimal regular model $\mathcal{C}$ of C over O_K has a semistable special fiber $\mathcal{C}_s$ over the residue field κ of O_K, every node x of $\mathcal{C}_s$ is defined over κ, and the two tangent lines of $\mathcal{C}_s$ at every node x are also defined over κ.

Denote by $\|\cdot\|_{\mathrm{Ar}}$ the metric of $\omega_{C/K}$ on C^{an} induced by the integral model $(\mathcal{C}, \omega_{\mathcal{C}/O_K})$. In the following, we will modify $\|\cdot\|_{\mathrm{Ar}}$ to get $\|\cdot\|_a$.

Denote by $\Gamma = \Gamma(C)$ the *reduction graph* of C, which is the dual graph of $\mathcal{C}_s$. Then every irreducible component D of $\mathcal{C}_s$ is represented by a vertex $v(D)$ of $\Gamma(C)$; every node N of $\mathcal{C}_s$ is represented by an edge $e(N)$ of length 1

in $\Gamma(C)$ connecting the vertices $v(D_1), v(D_2)$, where D_1, D_2 are the (possible equal) irreducible components of $\mathcal{C}_s$ containing N. This is a metrized graph. We refer to [BF] and [Bak, §4.1] for some basics of metrized graphs.

The *canonical divisor* K_C of Γ is the formal linear combination

$$K_C = \sum_{\xi \in V(\Gamma)} \deg(\omega_{\mathcal{C}/O_K}|_{F_\xi})\, \xi,$$

where $V(\Gamma)$ denotes the vertex set of Γ, and F_ξ denotes the irreducible component of $\mathcal{C}_s$ corresponding to ξ. It is easy to see that $\deg(K_C) = 2g - 2$.

Following [Zha1, Def. a.3], denote by $F(\Gamma)$ the space of continuous and piecewise smooth functions $f : \Gamma \to \mathbb{R}$ such that all the one-sided directional derivatives $d_{\vec{v}}f(P)$ of f exist. The Laplacian operator Δ sends each $f \in F(\Gamma)$ to a signed measure on Γ given by

$$\Delta f = -f''(x)dx - \sum_{P \in \Gamma} \sum_{\vec{v} \in T_P(\Gamma)} d_{\vec{v}}f(P)\, \delta_P.$$

Here x represents a canonical coordinate on each edge of Γ, which is uniquely defined up to a sign, and $T_P(\Gamma)$ denotes the (finite) set of tangent directions of Γ at P. The summation for $P \in \Gamma$ on the right-hand side has only finitely many nonzero terms. See also [BF, Def. 5] and [Bak, Prop. 4.2.6].

By [Zha1, Thm. 3.2, Lem. 3.7], there is a unique positive measure $\mu = \mu_{K_C}$ on Γ of total volume 1, which is a finite linear combination of the uniform measures on the edges of Γ and the Dirac measures supported on the vertices of Γ, such that the function $g_\mu : \Gamma^2 \to \mathbb{R}$ uniquely defined by

$$g_\mu(x, \cdot) \in F(\Gamma), \quad \forall x \in \Gamma,$$

$$\Delta g_\mu(x, \cdot) = \delta_x - \mu,$$

$$\int_\Gamma g_\mu(x, \cdot)\mu_{K_C} = 0,$$

satisfies the condition that

$$c + g_\mu(K_C, x) + g_\mu(x, x) = 0$$

for a constant c independent of $x \in \Gamma$. The function is $g_\mu : \Gamma^2 \to \mathbb{R}$ is actually continuous and symmetric.

By [Ber, §4.3], there is a canonical injection

$$i : \Gamma(C) \longrightarrow C^{\mathrm{an}}$$

and a canonical retraction map

$$r : C^{\mathrm{an}} \longrightarrow \Gamma(C).$$

By composition, it induces a retraction map

$$r : C(\overline{K}) \longrightarrow \Gamma(C).$$

In terms of this map, we can view $g_\mu(K_C, x)$ and $g_\mu(x, x)$ as functions of $x \in C^{\mathrm{an}}$.

Finally, we have explicit formulas for the metrics $\|\cdot\|_a$ and $\|\cdot\|_{\Delta,a}$ in Theorem A.1. Following [Zha1, (4.1)], we have

$$\|\cdot\|_a(x) = \|\cdot\|_{\mathrm{Ar}}(x) \cdot e_K^{-c-g_\mu(K_C, r(x))} = \|\cdot\|_{\mathrm{Ar}}(x) \cdot e_K^{g_\mu(r(x), r(x))}, \quad x \in C^{\mathrm{an}}$$

and

$$g_{\Delta,a}(x, y) = i(x, y) \log e_K + g_\mu(r(x), r(y)) \log e_K, \quad x, y \in C(\overline{K}), \ x \neq y.$$

Here for $x, y \in C(\overline{K})$, if K' is a finite extension of K such that $x, y \in C(K')$, then $i(x, y)$ is the intersection number of the corresponding sections in the minimal regular model of C over $O_{K'}$, divided by the normalizing factor $[K' : K]$.

Note that the above expression for $\|\cdot\|_a$ is already defined over C^{an}, but the expression for $g_{\Delta,a}$ is defined over the $\overline{K}$-points, which determines its value over $(C^2)^{\mathrm{an}}$ by continuity.

To verify the above expressions, we first reduce to the case that $e_K = |\varpi|^{-1}$ is equal to e (the base of the natural logarithm). In fact, if we replace the absolute value $|\cdot|$ of K by $|\cdot|^a$ for some $a > 0$, then the canonical admissible metrics are changed to their powers of exponent a by the uniqueness of the metrics.

For the standard case $e_K = e$, it suffices to apply the last statement of [Zha1, Thm. 4.6], which has also been used to prove the uniqueness part of Theorem A.1. In translating the measure between the graph and the Berkovich space, it is helpful to have the following result, which is compatible with [Zha1, Thm. 2.3].

Proposition A.4. *For any $f \in F(\Gamma)$, denote by $\mathcal{O}_C(f)$ the trivial line bundle $\mathcal{O}_C$ endowed with the metric on C^{an} given by $\|1\| = e_K^{-r^*f}$. Then*

$$c_1(\mathcal{O}_C(f)) = -i_*(\Delta f)$$

as measures on C^{an}.

Proof. We only sketch the idea here. The base case is when $\mathcal{O}(f)$ is induced by an integral model $\mathcal{O}_{\mathcal{C}}(V)$ of $\mathcal{O}_C$ for an irreducible component V of the special fiber of the minimal regular model $\mathcal{C}$. In this case, both the Chambert-Loir measure and Laplacian operator are easy to compute. By linear combination, this gives the case that f is linear on edges of Γ. Note that the edges of Γ have rational lengths, and the division points of the edges are called rational points of Γ. By taking finite extensions of K, we can realizes all rational points of Γ as irreducible components of the special fibers of the minimal regular models. Then the equality holds for all piecewise linear function f on Γ whose critical points are rational points. Such functions are uniformly dense in $F(\Gamma)$. Then the result is extended by approximation. $\square$

Finally, the measure $d\mu_a$ on C^{an} defined in Proposition A.2 has the following explicit expression.

Proposition A.5. *The measure*

$$d\mu_a = \frac{1}{g} i_* \left(\sum_{v \in V(\Gamma(C))} g_v \delta_v + \sum_{e \in E(\Gamma(C))} \frac{1}{r_e + 1} \delta_e \right),$$

where

- $V(\Gamma(C))$ *is the vertex set of $\Gamma(C)$ (corresponding to irreducible components of the special fiber $\mathcal{C}_s$);*

- g_v *is the genus of the normalization of the irreducible component of $\mathcal{C}_s$ corresponding to v;*

- δ_v *is the Dirac measure supported at v;*

- $E(\Gamma(C))$ *is the edge set of $\Gamma(C)$ (corresponding to nodes of the special fiber $\mathcal{C}_s$);*

- r_e is the resistance of the endpoints p, q of e in $\Gamma(C) \setminus e^0$ with $e^0 = e \setminus \{p, q\}$;

- δ_e is the Lebesgue measure on the edge e of total integral 1.

Proof. It suffices to compute

$$c_1(\omega_{C/K}, \| \cdot \|_a) = c_1(\omega_{C/K}, \| \cdot \|_{\mathrm{Ar}}) + c_1(\mathcal{O}(c + g_\mu(K_C, \cdot))).$$

By the definition of the Chambert-Loir measure in [CL],

$$c_1(\omega_{C/K}, \| \cdot \|_{\mathrm{Ar}}) = i_* \delta_{K_C}.$$

By Lemma A.4,

$$c_1(\mathcal{O}(c + g_\mu(K_C, \cdot))) = -i_* \Delta g_\mu(K_C, \cdot) = i_*((2g - 2)\mu - \delta_{K_C}).$$

It follows that

$$c_1(\omega_{C/K}, \| \cdot \|_a) = i_*((2g - 2)\mu).$$

Apply the explicit formula for $\mu = \mu_{K_C}$ and $D = K_C$ in [Zha1, Lem. 3.7]. $\quad\square$

Remark A.6. The result is compatible with the abstract result in [Gub, Thm. 1.1].

A.6 General valuation fields

Let C be a smooth projective curve of genus $g > 0$ over a complete valuation field K (instead of a non-archimedean field). In this generality, we will still have a *canonical admissible metric* $\| \cdot \|_a$ of $\omega_{C/K}$ on C^{an}, and a *canonical admissible metric* $\|\cdot\|_{\Delta,a}$ of $\mathcal{O}(\Delta)$ on $(C^2)^{\mathrm{an}}$ satisfying Theorem A.1 in suitable senses.

In fact, if K is a non-archimedean field, then $\| \cdot \|_a$ and $\| \cdot \|_{\Delta,a}$ are the Zhang metrics introduced in Theorem A.1.

If K is an archimedean field, then $\| \cdot \|_a = \| \cdot \|_{\mathrm{Ar}}$ and $\| \cdot \|_{\Delta,a} = \| \cdot \|_{\Delta,\mathrm{Ar}}$ are the Arakelov metrics in §A.1. Note that if K is real, there should be a minor process to descend $\| \cdot \|_{\mathrm{Ar}}$ from $C(\overline{K})$ to

$$C^{\mathrm{an}} = C(\overline{K})/\mathrm{Gal}(\overline{K}/K).$$

A similar process is also needed for $\| \cdot \|_{\Delta,Ar}$. Our statement and proof of Theorem A.1 also work in the archimedean case.

If K is a trivially valued field, then we set the metrics $\|\cdot\|_a$ and $\|\cdot\|_{\Delta,a}$ to be the *canonical metrics* of the line bundles. Recall that for a projective variety X over K and a line bundle L on X, the line bundle L induces a metric of itself on X^{an} by the usual way, called the *canonical metric* or the *admissible metric*; see [YZ2, §3.4.1] for example. Note that in this case, any line bundle L has a *unique* admissible metric (instead of unique up to multiplicative constants).

Let K be either a number field or the function field of one variable over a base field. With the above metrics over the complete fields, we have an adelic line bundle

$$\overline{\omega}_{C/K,a} = (\omega_{C/K}, \{\|\cdot\|_{a,v}\}_v)$$

over C, and an adelic line bundle

$$\overline{\mathcal{O}}(\Delta)_a = (\mathcal{O}(\Delta), \{\|\cdot\|_{\Delta,a,v}\}_v)$$

over C^2.

To check the metrics are adelic, we need to check that in Theorem A.1, if C has good reduction over O_K, then the metrics $\|\cdot\|_a$ and $\|\cdot\|_{\Delta,a}$ are induced by the natural integral models $\omega_{\mathcal{C}/O_K}$ and $\mathcal{O}_{\mathcal{C}^2}(\Delta_{\mathcal{C}})$ of $\omega_{C/K}$ and $\mathcal{O}(\Delta)$. Here $\mathcal{C}$ is the smooth projective model of C over O_K, and $\Delta_{\mathcal{C}}$ denotes the diagonal morphism $\mathcal{C} \to \mathcal{C}^2 = \mathcal{C} \times_{O_K} \mathcal{C}$. By uniqueness, it suffices to check that the induced metrics satisfy the properties of the theorem. We omit it here.

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