EFFECTIVE BOUND OF LINEAR SERIES ON ARITHMETIC SURFACES

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Contents

1. Introduction

The results of this paper lie in the arithmetic intersection theory of Arakelov, Faltings and Gillet–Soulé.

We prove effective upper bounds on the number of effective sections of a hermitian line bundle over an arithmetic surface. The first two results are respectively for general arithmetic divisors and for nef arithmetic diviors. They can be viewed as effective versions of the arithmetic Hilbert–Samuel formula.

The third result improves the upper bound substantially for special nef line bundles, which particularly includes the Arakelov canonical bundle. As a consequence, we obtain effective lower bounds on the Faltings height and on the self-intersection of the canonical bundle in terms of the number of singular points on fibers of the arithmetic surface. It recovers a result of Bost.

Throughout this paper, K denotes a number field, and X denotes a regular and geometrically connected arithmetic surface of genus g over O_K . That is, X is a two-dimensional regular scheme, projective and flat over $\mathrm{Spec}(O_K)$, such that $X_{\overline{K}}$ is connected curve of genus g.

1.1. **Effective bound for arbitrary line bundles.** By a *hermitian line bundle* over X, we mean a pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$, where $\mathcal{L}$ is an invertible sheaf over X, and $\|\cdot\|$ is a *continuous metric* on the line bundle $\mathcal{L}(\mathbb{C})$ over $X(\mathbb{C})$, invariant under the complex conjugation. Denote by $\widehat{\mathrm{Pic}}(X)$ the group of isometry class of hermitian line bundles on X.

For any hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ over X, denote

$$\widehat{H}^0(\overline{\mathcal{L}}) = \{s \in H^0(X, \mathcal{L}) : \|s\|_{\sup} \leq 1\}.$$

It is the set of *effective sections*. Define

$$\widehat{h}^0(\overline{\mathcal{L}}) = \log \# \widehat{H}^0(\overline{\mathcal{L}}).$$

and

$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) = \limsup_{n\to\infty} \frac{2}{n^2} \widehat{h}^0(n\overline{\mathcal{L}}).$$

Here we always write tensor product of (hermitian) line bundles additively, so $n\overline{\mathcal{L}}$ means $\overline{\mathcal{L}}^{\otimes n}$.

By Chen [Ch], the "limsup" in the right-hand side is actually a limit. Thus we have the expansion

$$\widehat{h}^0(n\overline{\mathcal{L}}) = \frac{1}{2}\, \widehat{\mathrm{vol}}(\overline{\mathcal{L}})\; n^2 + o(n^2), \quad n \to \infty.$$

The first main theorem of this paper is the following *effective* version of the above expansion in one direction.

Theorem A. *Let X be a regular and geometrically connected arithmetic surface of genus g over O_K. Let $\overline{\mathcal{L}}$ be a hermitian line bundle on X. Denote $d^\circ = \deg(\mathcal{L}_K)$, and denote by r' the O_K-rank of the O_K-submodule of $H^0(\mathcal{L})$ generated by $\widehat{H}^0(\overline{\mathcal{L}})$. Assume $r' \geq 2$.*

(1) *If $g > 0$, then*

$$\widehat{h}^0(\overline{\mathcal{L}}) \leq \frac{1}{2}\, \widehat{\mathrm{vol}}(\overline{\mathcal{L}}) + 4d\log(3d).$$

Here $d = d^\circ[K:\mathbb{Q}]$.

(2) *If $g = 0$, then*

$$\widehat{h}^0(\overline{\mathcal{L}}) \leq \left(\frac{1}{2} + \frac{1}{2(r'-1)}\right) \widehat{\mathrm{vol}}(\overline{\mathcal{L}}) + 4r\log(3r).$$

Here $r = (d^\circ + 1)[K:\mathbb{Q}]$.

1.2. **Effective bound for nef line bundles.** Theorem A will be reduced to the case of nef hermitian line bundles.

Recall that a hermitian line bundle $\overline{\mathcal{L}}$ over X is *nef* if it satisfies the following conditions:

- $\widehat{\deg}(\overline{\mathcal{L}}|_Y) \geq 0$ for any integral subscheme Y of codimension one in X.
- The metric of $\overline{\mathcal{L}}$ is semipositive, i.e., the curvature current of $\overline{\mathcal{L}}$ on $X(\mathbb{C})$ is positive.

The conditions imply $\deg(\mathcal{L}_K) \geq 0$. They also imply that the self-intersection number $\overline{\mathcal{L}}^2 \geq 0$. It is a consequence of [Zh1, Theorem 6.3]. See also [Mo2, Proposition 2.3].

The arithmetic nefness is a direct analogue of the nefness in algebraic geometry. It generalizes the arithmetic ampleness of S. Zhang [Zh1], and serves as the limit notion of the arithmetic ampleness. In particular, a nef hermitian line bundle $\overline{\mathcal{L}}$ on X satisfies the following properties:

- The degree $\deg(\mathcal{L}_K) \geq 0$, which follows from the definition.
- The self-intersection number $\overline{\mathcal{L}}^2 \geq 0$. It is a consequence of [Zh1, Theorem 6.3]. See also [Mo2, Proposition 2.3].
- It satisfies the arithmetic Hilbert–Samuel formula

$$\widehat{h}^0(n\overline{\mathcal{L}}) = \frac{1}{2}n^2\overline{\mathcal{L}}^2 + o(n^2), \quad n \to \infty.$$

  Therefore, $\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) = \overline{\mathcal{L}}^2$. The formula is essentially due to Gillet–Soulé and S. Zhang. See [Yu1, Corollary 2.7] for more details.

The following result is an *effective* version of the Hilbert–Samuel formula in one direction.

Theorem B. *Let X be a regular and geometrically connected arithmetic surface of genus g over O_K. Let $\overline{\mathcal{L}}$ be a nef hermitian line bundle on X with $d^\circ = \deg(\mathcal{L}_K) > 0$.*

(1) *If $g > 0$ and $d^\circ > 1$, then*
$$\widehat{h}^0(\overline{\mathcal{L}}) \leq \frac{1}{2}\overline{\mathcal{L}}^2 + 4d\log(3d).$$
Here $d = d^\circ[K:\mathbb{Q}]$.

(2) *If $g = 0$ and $d^\circ > 0$, then*
$$\widehat{h}^0(\overline{\mathcal{L}}) \leq (\frac{1}{2} + \frac{1}{2d^\circ})\overline{\mathcal{L}}^2 + 4r\log(3r).$$
Here $r = (d^\circ + 1)[K:\mathbb{Q}]$.

The theorem is new even in the case that $\overline{\mathcal{L}}$ is ample. It is not a direct consequence of the arithmetic Riemann–Roch theorem of Gillet and Soulé, due to difficulties on effectively estimating the analytic torsion and the contribution of $H^1(\mathcal{L})$.

Theorem B is a special case of Theorem A under slightly weaker assumptions, but Theorem B actually implies Theorem A. To obtain Theorem A, we decompose
$$\overline{\mathcal{L}} = \overline{\mathcal{L}}_1 + \overline{\mathcal{E}}$$
with a nef hermitian line bundle $\overline{\mathcal{L}}_1$ and an effective hermitian line bundle $\overline{\mathcal{E}}$, which induces a bijection $\widehat{H}^0(\overline{\mathcal{L}}_1) \to \widehat{H}^0(\overline{\mathcal{L}})$. The effectivity of $\overline{\mathcal{E}}$ also gives $\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) \geq \widehat{\mathrm{vol}}(\overline{\mathcal{L}}_1) = \overline{\mathcal{L}}_1^2$. Then the result is obtained by applying Theorem B to $\overline{\mathcal{L}}_1$. See Theorem 3.1.

The following are some consequences and generalities related to the theorems:

- In the setting of Theorem B, for $\deg(\mathcal{L}_K) = 1$ and any genus $g \geq 0$, we can bound $\widehat{h}^0(\overline{\mathcal{L}})$ in terms of $\overline{\mathcal{L}}^2$ (with coefficient 1). See Proposition 4.8.
- In both theorems, the assumption that X is regular can be removed by the resolution of singularity proved by Lipman [Li].
- In §6, we generalize Theorem A, Theorem B and Theorem C below to arithmetic $\mathbb{R}$-divisors of C^0-type in the sense of Moriwaki [Mo6] and to adelic line bundles in the sense of S. Zhang [Zh3].

- The theorems easily induce upper bounds for the Euler characteristic
$$\chi_{\sup}(\overline{\mathcal{L}}) = \log \frac{\mathrm{vol}(B_{\sup}(\overline{\mathcal{L}}))}{\mathrm{vol}(H^0(X,\mathcal{L})_{\mathbb{R}}/H^0(X,\mathcal{L}))}.$$
Here $B_{\sup}(\overline{\mathcal{L}})$ is the unit ball in $H^0(X,\mathcal{L})_{\mathbb{R}}$ bounded by the supremum norm $\|\cdot\|_{\sup}$. In fact, Minkowski's theorem gives
$$\chi_{\sup}(\overline{\mathcal{L}}) \leq \widehat{h}^0(\overline{\mathcal{L}}) + d\log 2.$$
The bounds are "accurate" if $\overline{\mathcal{L}}$ is nef.

Our implication from Theorem B to Theorem A is inspired by the arithmetic Zariski decomposition of Moriwaki [Mo6], though we do not use it in this paper. To fit the setting of the Zariski decomposition, let $\overline{D}$ be an arithmetic divisor linearly equivalent to $\overline{\mathcal{L}}$. The Zariski decomposition (for $\overline{D}$ big) writes
$$\overline{D} = \overline{P} + \overline{N}$$
for a nef arithmetic $\mathbb{R}$-divisor $\overline{P}$ and an effective arithmetic $\mathbb{R}$-divisor $\overline{N}$. The decomposition induces
$$\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{P}) = \overline{P}^2, \quad \widehat{H}^0(n\overline{D}) = \widehat{H}^0(n\overline{P}), \quad n \geq 0.$$
If $\overline{P}$ is a $\mathbb{Z}$-divisor, apply Theorem B to $\overline{P}$. We obtain the bound of $\widehat{h}^0(\overline{D})$ in Theorem A. If $\overline{P}$ is not a $\mathbb{Z}$-divisor (which often happens even when $\overline{D}$ is a $\mathbb{Z}$-divisor), the argument can still go through by the results in §6.

1.3. **Effective bound for special line bundles.** Theorem B is very accurate when $\deg(\mathcal{L}_K)$ is large by the arithmetic Hilbert–Samuel formula. However, it may be too weak if $\deg(\mathcal{L}_K)$ is very small. Here we present a substantial improvement of Theorem B for special line bundles, and consider the application to the Arakelov canonical bundle.

Recall that a line bundle L on a projective and smooth curve over a field is *special* if both $h^0(L) > 0$ and $h^1(L) > 0$. In particular, the canonical bundle is special if the genus is positive. The following is the improvement of Theorem B in the special case. One can easily obtain the improvement of Theorem A along the line.

Theorem C. *Let X be a regular and geometrically connected arithmetic surface of genus $g > 1$ over O_K. Let $\overline{\mathcal{L}}$ be a nef hermitian line bundle on X with $d^\circ = \deg(\mathcal{L}_K) > 1$. Assume that $\mathcal{L}_K$ is a special line bundle on X_K. Then*
$$\widehat{h}^0(\overline{\mathcal{L}}) \leq \left(\frac{1}{4} + \frac{2+\varepsilon}{4d^\circ}\right)\overline{\mathcal{L}}^2 + 4d\log(3d).$$

Here $d = d^\circ[K:\mathbb{Q}]$. The number $\varepsilon = 1$ if X_K is hyperelliptic and d° is odd; otherwise, $\varepsilon = 0$.

The most interesting case of Theorem C happens when $\overline{\mathcal{L}}$ is the canonical bundle. Following [Ar], let $\overline{\omega}_X = (\omega_X, \|\cdot\|_{\mathrm{Ar}})$ be the Arakelov canonical bundle of X over O_K. That is, $\omega_X = \omega_{X/O_K}$ is the relative dualizing sheaf of X over O_K and $\|\cdot\|_{\mathrm{Ar}}$ is the Arakelov metric on ω_X. By Faltings [Fa], $\overline{\omega}_X$ is nef if X is semistable over O_K.

Theorem D. *Let X be a semistable regular arithmetic surface of genus $g > 1$ over O_K. Then*

$$\widehat{h}^0(\overline{\omega}_X) \leq \frac{g}{4(g-1)}\overline{\omega}_X^2 + 4d\log(3d).$$

Here $d = (2g-2)[K:\mathbb{Q}]$.

Next we state a consequence of the theorem. Recall from Faltings [Fa] that $\chi_{\mathrm{Fal}}(\omega_X)$ is defined as the arithmetic degree of the hermitian O_K-module $H^0(X, \omega_X)$ endowed with the natural metric

$$\|\alpha\|_{\mathrm{nat}}^2 = \frac{i}{2}\int_{X(\mathbb{C})} \alpha \wedge \overline{\alpha}, \quad \alpha \in H^0(X(\mathbb{C}), \Omega^1_{X(\mathbb{C})}).$$

It is usually called the Faltings height of X. The arithmetic Noether formula proved by Faltings (cf. [Fa, MB1]) gives

$$\chi_{\mathrm{Fal}}(\omega_X) = \frac{1}{12}(\overline{\omega}_X^2 + \delta_X) - \frac{1}{3}g[K:\mathbb{Q}]\log(2\pi).$$

Here the delta invariant of X is defined by

$$\delta_X = \sum_v \delta_v,$$

where the summation is over all places v of K. If v is non-archimedean, δ_v is just the product of $\log q_v$ with the number of singular points on the fiber of X above v. Here q_v denotes the cardinality of the residue field of v. If v is archimedean, δ_v is an invariant of the corresponding Riemann surface.

To state the consequence, we introduce another archimedean invariant. Let M be a compact Riemann surface of genus $g \geq 1$. There are two norms on $H^0(M, \Omega^1_M)$. One is the canonical inner product $\|\cdot\|_{\mathrm{nat}}$, and the other one is the supremum norm $\|\cdot\|_{\sup}$ of the Arakelov metric $\|\cdot\|_{\mathrm{Ar}}$. Denote by $B_{\mathrm{nat}}(\Omega^1_M)$ and $B_{\sup}(\Omega^1_M)$ the unit balls in $H^0(M, \Omega^1_M)$ corresponding to $\|\cdot\|_{\mathrm{nat}}$ and $\|\cdot\|_{\sup}$. Denote

$$\gamma_M = \frac{1}{2}\log\frac{\mathrm{vol}(B_{\mathrm{nat}}(\Omega^1_M))}{\mathrm{vol}(B_{\sup}(\Omega^1_M))}.$$

The volumes are defined by choosing a Haar measure on $H^0(M, \Omega^1_M)$, and the quotient does not depend on the choice of the Haar measure.

It is easy to see that both the invariants δ and γ define real-valued continuous functions on the moduli space $M_g(\mathbb{C})$ of compact Riemann surfaces of genus g.

Corollary E. *Let X be a semistable regular arithmetic surface of genus $g > 1$ over O_K. Denote*

$$\gamma_{X_\infty} = \sum_{\sigma: K \hookrightarrow \mathbb{C}} \gamma_{X_\sigma}.$$

Then

$$\begin{aligned} \left(2 + \frac{3}{g-1}\right) \overline{\omega}_X^2 &\geq \delta_X - 12\gamma_{X_\infty} - 3\ C(g, K), \\ \left(8 + \frac{4}{g}\right) \chi_{\mathrm{Fal}}(\omega_X) &\geq \delta_X - \frac{4(g-1)}{g}\gamma_{X_\infty} - C(g, K). \end{aligned}$$

Here

$$C(g, K) = 2g \log |D_K| + 18d \log d + 25d,$$

where D_K denotes the absolute discriminant of K.

The inequalities are equivalent up to error terms by Faltings's arithmetic Noether formula. We describe briefly how to deduce them from Theorem D. It is standard to use Minkowski's theorem to transfer the upper bound for $\widehat{h}^0(\overline{\omega}_X)$ to an upper bound for $\chi_{\sup}(\overline{\omega}_X)$. It further gives an upper bound of $\chi_{\mathrm{Fal}}(\omega_X)$ since the difference $\chi_{\mathrm{Fal}}(\omega_X) - \chi_{\sup}(\overline{\omega}_X)$ is essentially given by γ_{X_∞}. Now the inequalities are obtained by the arithmetic Noether formula.

The first inequality is in the opposite direction of the conjectural arithmetic Bogomolov–Miyaoka–Yau inequality proposed by Parshin [Pa] and Moret-Baily [MB2]. Recall that the conjecture asserts

$$\overline{\omega}_X^2 \leq A(\delta_X + (2g-2)\log|D_K|) + \sum_\sigma \xi_{X_\sigma}.$$

Here A is an absolute constant, and ξ is a continuous real-valued function on $M_g(\mathbb{C})$. Note that both δ and γ are such functions.

Many results similar to Corollary E are known in the literature.

Let us first compare the corollary with a result of Bost [Bo]. The second inequality of the corollary is an effective version of [Bo, Theorem IV], with explicit "error terms" γ_{X_∞} and $C(g, K)$. Note that our proofs are completely different. Bost obtained his result as a special case of his inequality between the slope of a hermitian vector bundle and the height of a semi-stable cycle, while our result is a consequence of the estimation of the corresponding linear series.

Many explicit bounds of the above type are previously in the literature. Moriwaki [Mo1] proved an explicit lower bound of $\overline{\omega}_X^2$ in terms of reducible fibers. Using Weierstrass points, Robin de Jong [Jo] obtained an explicit bound on the Faltings height. Their bounds are weaker than ours.

It will also be interesting to compare the first inequality with the main result of S. Zhang [Zh3], which proves a formula expressing ω_a^2 in terms of the Beilinson–Bloch height $\langle \Delta_\xi, \Delta_\xi \rangle$ of the Gross–Schoen cycle and some canonical local invariants of X_K. Note that the difference between $\overline{\omega}_X^2 - \omega_a^2$ is well understood by [Zh2]. Then our result gives a lower bound of $\langle \Delta_\xi, \Delta_\xi \rangle$ by some local invariants. It is also worth noting that if X_K is hyperelliptic, then $\langle \Delta_\xi, \Delta_\xi \rangle = 0$. Thus the comparison gives an inequality between two different sums of local invariants of X_K.

1.4. **Classical Noether inequalities.** The main results of this paper can be viewed as arithmetic versions of Noether type inequalities. The classical Noether inequality sits naturally in the geography theory of surfaces. We recall the theory briefly, and refer readers to [BHPV] for more details.

Theorem 1.1. *let X be a complex minimal surface of general type. Denote the Chern numbers $c_1^2 = c_1(X)^2 = \deg(\omega_X)$ and $c_2 = c_2(X) = \deg c_2(\Omega_X)$. The following are true:*

(a) *Noether formula*

$$\chi(\omega_X) = \frac{1}{12}(c_1^2 + c_2).$$

(b) *Noether inequality*

$$h^0(\omega_X) \leq \frac{1}{2}c_1^2 + 2.$$

(c) *Bogomolov–Miyaoka–Yau inequality*

$$c_1^2 \leq 3c_2.$$

The geography theory asks for what pair in $\mathbb{Z}^2$ can be equal to $(c_1(X)^2, c_2(X))$ for a minimal surface X. The following concise result is almost a complete answer of the question.

Theorem 1.2 (geography theorem)**.** *Let X be a complex minimal surface. Then $(c_1^2, c_2) = (c_1(X)^2, c_2(X)) \in \mathbb{Z}^2$ satisfies the following conditions:*

(1) $12 \mid (c_1^2 + c_2)$,
(2) $c_1^2 > 0,\ c_2 > 0$,
(3) $c_1^2 \leq 3c_2$,
(4) $5c_1^2 - c_2 + 36 \geq 0$ *when* $2|c_1^2$,
(5) $5c_1^2 - c_2 + 30 \geq 0$ *when* $2 \nmid c_1^2$.

Conversely, for any pair $(m,n) \in \mathbb{Z}^2$ *satisfying the above conditions for* (c_1^2, c_2)*, there is a complex minimal surface* X *of general type with* $(c_1(X)^2, c_2(X)) = (m,n)$*, except for some points on the following* 348 *lines:*

$$m - 3n + 4k = 0, \quad k = 0, 1, \cdots, 347.$$

The conditions (1)-(5) are easily derived from Theorem 1.1. For example, (4) is obtained by combining (a) and (b) with the naive bound

$$\chi(\omega_X) = h^0(\omega_X) - h^1(\omega_X) + h^2(\omega_X) \leq h^0(\omega_X) + 1.$$

And (5) is obtained with an extra simple divisibility argument.

Consider arithmetic surfaces in the the setting of Arakelov geometry. The arithmetic Noether formula was proved by Faltings [Fa]. The arithmetic Noether inequality is proved in this paper. We have actually proved a more delicate bound in Theorem D in terms of the genus of the generic fiber. Then Corollary E is the arithmetic version of (4) and (5) in the geography theorem.

The arithmetic Bogomolov–Miyaoka–Yau inequality, as proposed by Parshin [Pa] and Moret-Baily [MB2], is equivalent to the abc conjecture. Recently, Shinichi Mochizuki announced a proof of the conjecture.

Go back to the classical setting, delicate inequalities for $\deg(\pi_*\omega_{X/B})$ of Noether type were obtained by Xiao [Xi] and Cornalba–Harris [CH] for fibered algebraic surfaces $\pi : X \to B$ via stability consideration. The treatment of Bost [Bo] can be viewed as an arithmetic analogue of [CH]. Theorem D, which treats $\widehat{h}^0$ instead of $\widehat{\deg}$, has the same leading coefficients as their results. In a forthcoming paper, we will address a classical version of Theorem D, i.e., a delicate upper bound of $h^0(\omega_{X/B})$ for the fibration $\pi : X \to B$.

In the end, we mention a result of Shin [Sh]. He proves that, on a complex algebraic surface X with non-negative Kodaira dimension,

$$h^0(L) \leq \frac{1}{2}L^2 + 2$$

for any nef and big line bundle L on X such that the rational map $X \dashrightarrow \mathbb{P}(H^0(X, L))$ is generically finite (cf. [Sh, Theorem 2]).

Theorem B of this paper is an arithmetic analogue of Shin's result, but the proof in [Sh] is not available here due to the essential use of the adjunction formula.

1.5. **Idea of proof.** Our proofs of Theorem B and Theorem C are very similar. Theorem C is sharper than Theorem B by the application of Clifford's theorem, which gives a very good bound on linear series of special line bundles on curves.

Now we describe the main idea to prove Theorem B. Let $\overline{\mathcal{L}}$ be a nef line bundle. Denote

$$\Delta(\overline{\mathcal{L}}) = \widehat{h}^0(\overline{\mathcal{L}}) - \frac{1}{2}\overline{\mathcal{L}}^2.$$

We first find the largest constant $c \geq 0$ such that

$$\overline{\mathcal{L}}(-c) = (\mathcal{L}, e^c \|\cdot\|)$$

is still nef on X. It is easy to control $\Delta(\overline{\mathcal{L}})$ by $\Delta(\overline{\mathcal{L}}(-c))$. Then the problem is reduced to $\overline{\mathcal{L}}(-c)$.

Denote by $\mathcal{E}_1$ the line bundle associated to the codimension one part of the base locus of the strictly effective sections of $\overline{\mathcal{L}}(-c)$. We obtain a decomposition

$$\overline{\mathcal{L}}(-c) = \overline{\mathcal{L}}_1 + \overline{\mathcal{E}}_1.$$

In Theorem 3.2, we construct hermitian metrics such that $\overline{\mathcal{L}}_1$ is nef and $\overline{\mathcal{E}}_1$ is effective, and such that strictly effective sections of $\overline{\mathcal{L}}_1$ can be transfered to those on $\overline{\mathcal{L}}(-c)$. Then it is easy to control $\Delta(\overline{\mathcal{L}}(-c))$ by $\Delta(\overline{\mathcal{L}}_1)$. Then the problem is reduced to $\overline{\mathcal{L}}_1$.

The key property for $\overline{\mathcal{L}}(-c)$ is that, it usually has a large base locus, due to the lack of effective sections. In particular, $\deg(\mathcal{E}_{1,K}) > 0$. It gives a strict inequality $\deg(\mathcal{L}_{1,K}) < \deg(\mathcal{L}_K)$.

Keep the reduction process. We obtain $\overline{\mathcal{L}}_2, \overline{\mathcal{L}}_3, \cdots$. The process terminates due to the strict decreasing of the degree. We eventually end up with $\overline{\mathcal{L}}_n$ such that $\overline{\mathcal{L}}_n(-c_n)$ has no strictly effective sections. It leads to the proof of the theorem.

The successive minima of Gillet and Soulé is used to control the error terms in the reduction process.

The structure of the paper is as follows. In §2, we state some results bounding lattice points on normed modules. They will be used in the proof of the main theorems. In §3, we explore our major construction of the decomposition $\overline{\mathcal{L}}(-c) = \overline{\mathcal{L}}_1 + \overline{\mathcal{E}}_1$, and reduce Theorem A to Theorem B. In §4, we prove Theorem B. In §5, we prove Theorem C and Corollary E.

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2. Some results on normed modules

By a normed $\mathbb{Z}$-module, we mean a pair $\overline{M} = (M, \|\cdot\|)$ consisting of a $\mathbb{Z}$-module M and an $\mathbb{R}$-norm $\|\cdot\|$ on $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. We say that $\overline{M}$ is a normed free $\mathbb{Z}$-module of finite rank, if M is a free $\mathbb{Z}$-module of finite rank. It is the case which we will restrict to.

Let $\overline{M} = (M, \|\cdot\|)$ be a normed free $\mathbb{Z}$-module of finite rank. Define

$$\widehat{H}^0(\overline{M}) = \{m \in M : \|m\| \leq 1\}, \quad \widehat{H}^0_{\mathrm{sef}}(\overline{M}) = \{m \in M : \|m\| < 1\},$$

and

$$\widehat{h}^0(\overline{M}) = \log \# \widehat{H}^0(M), \quad \widehat{h}^0_{\mathrm{sef}}(\overline{M}) = \log \# \widehat{H}^0_{\mathrm{sef}}(M).$$

The Euler characteristic of $\overline{M}$ is defined by

$$\chi(\overline{M}) = \log \frac{\mathrm{vol}(B(M))}{\mathrm{vol}(M_{\mathbb{R}}/M)},$$

where $B(M) = \{x \in M_{\mathbb{R}} : \|x\| \leq 1\}$ is a convex body in $M_{\mathbb{R}}$.

2.1. **Change of norms.**

Let $\overline{M} = (M, \|\cdot\|)$ be a normed free $\mathbb{Z}$-module of finite rank. For any $\alpha \in \mathbb{R}$, define

$$\overline{M}(\alpha) = (M, e^{-\alpha}\|\cdot\|).$$

Since $\widehat{h}^0_{\mathrm{sef}}(\overline{M})$ is finite, it is easy to have

$$\widehat{h}^0_{\mathrm{sef}}(\overline{M}) = \lim_{\alpha \to 0^-} \widehat{h}^0(\overline{M}(\alpha)).$$

Then many results on $\widehat{h}^0$ can be transfered to $\widehat{h}^0_{\mathrm{sef}}$. We first present a simple result on the change of effective sections.

Proposition 2.1. *Let $\overline{M} = (M, \|\cdot\|)$ be a normed free module of rank r. The following are true:*

(1) *For any $\alpha \geq 0$, one has*

$$\begin{array}{rcccl} \widehat{h}^0(\overline{M}(-\alpha)) & \leq & \widehat{h}^0(\overline{M}) & \leq & \widehat{h}^0(\overline{M}(-\alpha)) + r\alpha + r\log 3, \\ \widehat{h}^0_{\mathrm{sef}}(\overline{M}(-\alpha)) & \leq & \widehat{h}^0_{\mathrm{sef}}(\overline{M}) & \leq & \widehat{h}^0_{\mathrm{sef}}(\overline{M}(-\alpha)) + r\alpha + r\log 3. \end{array}$$

(2) *One has*

$$\widehat{h}^0_{\mathrm{sef}}(\overline{M}) \leq \widehat{h}^0(\overline{M}) \leq \widehat{h}^0_{\mathrm{sef}}(\overline{M}) + r\log 3.$$

Proof. The first inequality of (1) implies the other two inequalities. In fact, (2) is obtained by setting $\alpha \to 0$ in the first inequality of (1). It is also easy to deduce the second inequality of (1) by the first inequality of (1). In fact, replace $\overline{M}$ by $\overline{M}(-\beta)$ with $\beta > 0$ in the first inequality. Set $\beta \to 0$. The limit gives the second inequality.

Now we prove the first inequality. For any $\beta > 0$, denote

$$B(\beta) = \{x \in M_{\mathbb{R}} : \ \|x\| \leq \beta\}.$$

It is a symmetric convex body in $M_{\mathbb{R}}$. Then $B(1)$ and $B(e^{-\alpha})$ are exactly the unit balls of the metrics of $\overline{M}$ and $\overline{M}(-\alpha)$. Consider the set

$$S = \{x + B(2^{-1}e^{-\alpha}) : \ x \in \widehat{H}^0(\overline{M})\}.$$

All convex bodies in S are contained in the convex body $B(1+2^{-1}e^{-\alpha})$. Comparing the volumes, we conclude that there is a point $y \in B(1 + 2^{-1}e^{-\alpha})$ covered by at least N convex bodies in S, where

$$\begin{aligned} N &\geq \frac{\#S \cdot \mathrm{vol}(B(2^{-1}e^{-\alpha}))}{\mathrm{vol}(B(1+2^{-1}e^{-\alpha}))} \\ &= \#S \cdot \frac{(2^{-1}e^{-\alpha})^r}{(1+2^{-1}e^{-\alpha})^r} = \#S \cdot \frac{1}{(1+2e^{\alpha})^r}. \end{aligned}$$

Then

$$\log N \geq \log \#S - r\log(1+2e^{\alpha}) \geq \widehat{h}^0(\overline{M}) - r(\alpha + \log 3).$$

Let $x_1, \cdots, x_N$ be the centers of these N convex bodies. Then $x_i - y \in B(2^{-1}e^{-\alpha})$, and thus $x_i - x_j \in B(e^{-\alpha})$. In particular, we have

$$\{x_i - x_1 : i = 1, \cdots, N\} \subset \widehat{H}^0(\overline{M}(-\alpha)).$$

Therefore,

$$\widehat{h}^0(\overline{M}(-\alpha)) \geq \log N \geq \widehat{h}^0(\overline{M}) - r(\alpha + \log 3).$$

It proves the result. $\square$

Remark 2.2. There are many bounds for $\widehat{h}^0(\overline{M}) - \widehat{h}^0(\overline{M}(-\alpha))$ in the literature. See [GS1, Mo3, Yu2, Mo5] for example.

The following filtration version of the proposition will be used in the proof of our main theorem.

Proposition 2.3. *Let $\overline{M} = (M, \|\cdot\|)$ be a normed free $\mathbb{Z}$-module of finite rank. Let $0 = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$ be an increasing sequence.*

For $0 \leq i \leq n$, denote by r_i the rank of the submodule of M generated by $\widehat{H}^0(\overline{M}(-\alpha_i))$. Then

$$\widehat{h}^0(\overline{M}) \leq \widehat{h}^0(\overline{M}(-\alpha_n)) + \sum_{i=1}^{n} r_{i-1}(\alpha_i - \alpha_{i-1}) + 4r_0 \log r_0 + 2r_0 \log 3,$$

$$\widehat{h}^0(\overline{M}) \geq \sum_{i=1}^{n} r_i(\alpha_i - \alpha_{i-1}) - 2r_0 \log r_0 - r_0 \log 3.$$

The same results hold for the pair $(\widehat{h}^0_{\mathrm{sef}}(\overline{M}), \widehat{h}^0_{\mathrm{sef}}(\overline{M}(-\alpha_n)))$.

Remark 2.4. We will only use the first inequality in this paper. In a forthcoming paper, Yuan will use both inequalities to improve [Yu2, Theorem A].

The proposition is a consequence of the successive minima of Gillet and Soulé. One may try to use Proposition 2.1 to prove the first inequality. Namely, for each $i = 1, \cdots, n$, one has

$$h^0(\overline{M}(-\alpha_{i-1})) \leq \widehat{h}^0(\overline{M}(-\alpha_i)) + r_{i-1}(\alpha_i - \alpha_{i-1}) + r_{i-1} \log 3.$$

Summing over i, we obtain

$$\widehat{h}^0(\overline{M}) \leq \widehat{h}^0(\overline{M}(-\alpha_n)) + \sum_{i=1}^{n} r_{i-1}(\alpha_i - \alpha_{i-1}) + (r_0 + \cdots + r_{n-1}) \log 3.$$

The error term may be bigger than that in Proposition 2.3, if the sequence $\{r_i\}_i$ has too many terms and decays too slowly. It would actually be the case in our application.

2.2. **Successive minima.** Here we prove Proposition 2.3. We first recall the successive minima of Gillet and Soulé.

Let $\overline{M} = (M, \|\cdot\|)$ be a normed free $\mathbb{Z}$-module of finite rank r. For $i = 1, \cdots, r$, *the i-th logarithmic minimum of $\overline{M}$* is defined to be

$$\mu_i(\overline{M}) = \sup\{\mu \in \mathbb{R} : \mathrm{rank}\langle \widehat{H}^0(\overline{M}(-\mu))\rangle_{\mathbb{Z}} \geq i\}.$$

Here $\langle \widehat{H}^0(\overline{M}(-\mu))\rangle_{\mathbb{Z}}$ denotes the $\mathbb{Z}$-submodule of M generated by $\widehat{H}^0(\overline{M}(-\mu))$.

The following classical result gives a way to estimate $\widehat{h}^0(\overline{M})$ and $\chi(\overline{M})$ in terms of the minima of $\overline{M}$.

Theorem 2.5 (successive minima)**.** *Let $\overline{M} = (M, \|\cdot\|)$ be a normed free $\mathbb{Z}$-module of finite rank r. Then*

$$r \log 2 - \log(r!) \leq \chi(\overline{M}) - \sum_{i=1}^{r} \mu_i(\overline{M}) \leq r \log 2,$$

and

$$\left|\widehat{h}^0(\overline{M}) - \sum_{i=1}^{r} \max\{\mu_i(\overline{M}), 0\}\right| \leq r\log 3 + 2r\log r.$$

The second result still holds if replacing $\widehat{h}^0(\overline{M})$ by $\widehat{h}^0_{\mathrm{sef}}(\overline{M})$.

Proof. The first result is a restatement of Minkowski's second theorem on successive minima.

The second result for $\widehat{h}^0(\overline{M})$ is essentially due to Gillet–Soulé [GS1], where the error term is not explicit. It implies the same result for $\widehat{h}^0_{\mathrm{sef}}(\overline{M})$. In fact, apply it to $\overline{M}(-\alpha)$ for $\alpha > 0$, we have

$$\left|\widehat{h}^0(\overline{M}(-\alpha)) - \sum_{i=1}^{r} \max\{\mu_i(\overline{M}) - \alpha, 0\}\right| \leq r\log 3 + 2r\log r.$$

Set $\alpha \to 0$. Note that $\widehat{h}^0(\overline{M}(-\alpha))$ converges to $\widehat{h}^0_{\mathrm{sef}}(\overline{M})$. It gives the bound for $\widehat{h}^0_{\mathrm{sef}}(\overline{M})$.

Now we check the explicit error terms. We will use some effective error terms collected by Moriwaki [Mo3]. We will use similar notations.

Without loss of generality, assume $M = \mathbb{Z}^r$. Define by M_0 the submodule of M generated by $\widehat{H}^0(\overline{M})$, and denote $r_0 = \mathrm{rank}(M_0)$. Denote

$$B = \{x \in M_{\mathbb{R}} : \|x\| \leq 1\},$$

which is a convex centrally symmetric bounded absorbing set in $\mathbb{R}^r$. Let $B_0 = B \cap (M_0 \otimes_{\mathbb{Z}} \mathbb{R})$ and let B_0^* be the polar body of B_0. That is,

$$B_0^* = \{x \in M_0 \otimes_{\mathbb{Z}} \mathbb{R} : |\langle x, y\rangle| \leq 1 \,\mathrm{for\,all}\, y \in B_0\}.$$

Since M_0 is generated by $M_0 \cap B_0$, we have $\#(M_0 \cap B_0^*) = 1$.

As in [Mo3], we have

$$6^{-r_0} \leq \frac{\#\widehat{H}^0(\overline{M})}{\mathrm{vol}(B_0)} \leq \frac{6^{r_0}(r_0!)^2}{4^{r_0}}.$$

Apply Minkowski's second theorem on successive minima to B_0, we obtain

$$\frac{2^{r_0}}{r_0!}\prod_{i=1}^{r_0}\frac{1}{\lambda_i(B_0)} \leq \mathrm{vol}(B_0) \leq 2^{r_0}\prod_{i=1}^{r_0}\frac{1}{\lambda_i(B_0)},$$

where $\lambda_i(B_0)$ is the i-th successive minimum of B_0. Note that we used a different normalization of the minima here, but the relation is simply $\log \lambda_i(B_0) = \mu_i(\overline{M})$ for $i = 1, \cdots r_0$. Thus we can get

$$\frac{1}{3^{r_0}r_0!}\prod_{i=1}^{r_0}\frac{1}{\lambda_i(B_0)} \leq \widehat{H}^0(\overline{M}) \leq 3^{r_0}(r_0!)^2\prod_{i=1}^{r_0}\frac{1}{\lambda_i(B_0)}$$

Therefore we finally get

$$\left|\sum_{i=1}^{n} \max\{\mu_i(\overline{M}), 0\} - \widehat{h}^0(\overline{M})\right| \leq r_0 \log 3 + 2r_0 \log r_0.$$

It proves the second result.

□

Proof of Proposition 2.3. By the same limit trick as above, the results for h^0 implies that for h^0_{sef}.

We first prove the first inequality. By definition,

$$r_0 \geq r_1 \geq \cdots \geq r_n.$$

If $r_{i-1} = r_i$ for some i, the inequality does not depend on $\overline{M}_i$. We can remove $\overline{M}_i$ from the data. Thus we can assume that

$$r_0 > r_1 > \cdots > r_n.$$

For $j = 1, \cdots, r_0$, denote by μ_j the j-th logarithmic successive minimum of $\overline{M}$. By the definition, it is easy to have

$$\alpha_{i-1} \leq \mu_{r_{i-1}} \leq \mu_{1+r_i} < \alpha_i, \quad i = 1, \cdots, n.$$

Then we can bound the sequence $\{\mu_j\}_j$ by the sequence $\{\alpha_i\}_i$.

By Theorem 2.5,

$$\widehat{h}^0(\overline{M}) \leq \sum_{j=1}^{r_0} \max\{\mu_j, 0\} + r_0 \log 3 + 2r_0 \log r_0.$$

Replace μ_j by α_i for any $r_i + 1 \leq j \leq r_{i-1}$ in the bound. It gives

$$\widehat{h}^0(\overline{M}) \leq \sum_{i=1}^{n} (r_{i-1} - r_i)\alpha_i + \sum_{j=1}^{r_n} \max\{\mu_j, 0\} + r_0 \log 3 + 2r_0 \log r_0.$$

Applying Theorem 2.5 to $\overline{M}(-\alpha_n)$, we obtain

$$\begin{aligned}
\widehat{h}^0(\overline{M}(-\alpha_n)) &\geq \sum_{j=1}^{r_n} \max\{\mu_j - \alpha_n, 0\} - r_n \log 3 - 2r_n \log r_n \\
&\geq \sum_{j=1}^{r_n} \max\{\mu_j, 0\} - r_n\alpha_n - r_0 \log 3 - 2r_0 \log r_0.
\end{aligned}$$

It follows that

$$\begin{aligned}\widehat{h}^0(\overline{M}) &\leq \sum_{i=1}^{n}(r_{i-1}-r_i)\alpha_i + \widehat{h}^0(\overline{M}(-\alpha_n)) + r_n\alpha_n \\ &\quad +2r_0\log 3 + 4r_0\log r_0 \\ &= \widehat{h}^0(\overline{M}(-\alpha_n)) + \sum_{i=1}^{n} r_{i-1}(\alpha_i-\alpha_{i-1}) \\ &\quad +2r_0\log 3 + 4r_0\log r_0.\end{aligned}$$

It proves the first inequality.

Now we prove the second inequality. Still apply Theorem 2.5. We have

$$\widehat{h}^0(\overline{M}) \geq \sum_{j=1}^{r_0}\max\{\mu_j,0\} - r_0\log 3 - 2r_0\log r_0.$$

It follows that

$$\widehat{h}^0(\overline{M}) \geq \sum_{j=1}^{r_1}\max\{\mu_j,0\} - r_0\log 3 - 2r_0\log r_0.$$

Replace μ_j by α_{i-1} for any $r_i+1\leq j\leq r_{i-1}$. It gives

$$\begin{aligned}\widehat{h}^0(\overline{M}) &\geq r_n\alpha_n + \sum_{i=2}^{n}(r_{i-1}-r_i)\alpha_{i-1} - r_0\log 3 - 2r_0\log r_0 \\ &= \sum_{i=1}^{n} r_i(\alpha_i-\alpha_{i-1}) - r_0\log 3 - 2r_0\log r_0.\end{aligned}$$

It finishes the proof.

□

3. The key decompositions

The key idea of the proof the main theorems is to reduce the sections of $\overline{\mathcal{L}}$ to sections of nef line bundles of smaller degree. The goal here is to introduce this process.

3.1. **Notations and preliminary results.** Let X be an arithmetic surface, and $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ be a hermitian line bundle over X. We introduce the following notations.

Effective sections. Recall that the set of *effective sections* is

$$\widehat{H}^0(X,\overline{\mathcal{L}}) = \{s \in H^0(X,\mathcal{L}) : \|s\|_{\sup} \le 1\}.$$

Define the set of *strictly effective sections* to be

$$\widehat{H}^0_{\rm sef}(X,\overline{\mathcal{L}}) = \{s \in H^0(X,\mathcal{L}) : \|s\|_{\sup} < 1\}.$$

Denote

$$\widehat{h}^0(X,\overline{\mathcal{L}}) = \log \#\widehat{H}^0(X,\overline{\mathcal{L}}), \quad \widehat{h}^0_{\rm sef}(X,\overline{\mathcal{L}}) = \log \#\widehat{H}^0_{\rm sef}(X,\overline{\mathcal{L}}).$$

We say that $\overline{\mathcal{L}}$ is *effective* (resp. *strictly effective*) if $\widehat{h}^0(X,\overline{\mathcal{L}}) \neq 0$ (resp. $\widehat{h}^0_{\rm sef}(X,\overline{\mathcal{L}}) \neq 0$).

We usually omit X in the above notations. For example, $\widehat{H}^0(X,\overline{\mathcal{L}})$ is written as $\widehat{H}^0(\overline{\mathcal{L}})$.

Note that $\overline{M} = (H^0(X,\mathcal{L}), \|\cdot\|_{\sup})$ is a normed $\mathbb{Z}$-module. The definitions are compatible in that

$$\widehat{H}^0(\overline{\mathcal{L}}), \quad \widehat{H}^0_{\rm sef}(\overline{\mathcal{L}}), \quad \widehat{h}^0(\overline{\mathcal{L}}), \quad \widehat{h}^0_{\rm sef}(\overline{\mathcal{L}})$$

are identical to

$$\widehat{H}^0(\overline{M}), \quad \widehat{H}^0_{\rm sef}(\overline{M}), \quad \widehat{h}^0(\overline{M}), \quad \widehat{h}^0_{\rm sef}(\overline{M}).$$

Hence, the results in last section can be applied here.

For example, Proposition 2.1 gives

$$\widehat{h}^0_{\rm sef}(\overline{\mathcal{L}}) \le \widehat{h}^0(\overline{\mathcal{L}}) \le \widehat{h}^0_{\rm sef}(\overline{\mathcal{L}}) + h^0(\mathcal{L}_{\mathbb{Q}}) \log 3.$$

Note that if X is also defined over $\mathrm{Spec}(O_K)$ for some number field K. Then we obtain two projective curves $X_{\mathbb{Q}} = X \times_{\mathbb{Z}} \mathbb{Q}$ and $X_K = X \times_{O_K} K$, and two line bundles $\mathcal{L}_{\mathbb{Q}}$ and $\mathcal{L}_K$. It is easy to have $h^0(\mathcal{L}_{\mathbb{Q}}) = [K:\mathbb{Q}] h^0(\mathcal{L}_K)$.

Change of metrics. For any continuous function $f : X(\mathbb{C}) \to \mathbb{R}$, denote

$$\overline{\mathcal{L}}(f) = (\mathcal{L}, e^{-f}\|\cdot\|).$$

In particular, $\overline{\mathcal{O}}(f) = (\mathcal{O}, e^{-f})$ is the trivial line bundle with the metric sending the section 1 to e^{-f}. The case $\overline{\mathcal{O}}_X = \overline{\mathcal{O}}(0)$ is exactly the trivial hermitian line bundle on X.

If $c > 0$ is a constant, one has

$$\widehat{h}^0(\overline{\mathcal{L}}(-c)) \le \widehat{h}^0(\overline{\mathcal{L}}) \le \widehat{h}^0(\overline{\mathcal{L}}(-c)) + h^0(\mathcal{L}_{\mathbb{Q}})(c + \log 3)$$

$$\widehat{h}^0_{\rm sef}(\overline{\mathcal{L}}(-c)) \le \widehat{h}^0(\overline{\mathcal{L}}) \le \widehat{h}^0_{\rm sef}(\overline{\mathcal{L}}(-c)) + h^0(\mathcal{L}_{\mathbb{Q}})(c + \log 3).$$

These also follow from Proposition 2.1.

Base loci. Let H denote $\widehat{H}^0(\overline{\mathcal{L}})$ or $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}})$ in the following. Consider the natural map

$$H \times \mathcal{L}^\vee \longrightarrow \mathcal{L} \times \mathcal{L}^\vee \longrightarrow \mathcal{O}_X.$$

The image of the composition generates an ideal sheaf of $\mathcal{O}_X$. The zero locus of this ideal sheaf, defined as a closed subscheme of X, is called *the base locus of* H in X. The union of the irreducible components of codimension one of the base locus is called *the fixed part of* H in X.

Absolute minima. For any irreducible horizontal divisor D of X, define the normalized height function

$$h_{\overline{\mathcal{L}}}(D) = \frac{\widehat{\deg}(\overline{\mathcal{L}}|_D)}{\deg D_{\mathbb{Q}}}.$$

Define the *absolute minimum* $e_{\overline{\mathcal{L}}}$ of $\overline{\mathcal{L}}$ to be

$$e_{\overline{\mathcal{L}}} = \inf_D h_{\overline{\mathcal{L}}}(D).$$

It is easy to verify that

$$e_{\overline{\mathcal{L}}(\alpha)} = e_{\overline{\mathcal{L}}} + \alpha, \quad \alpha \in \mathbb{R}.$$

By definition, the absolute minimum $e_{\overline{\mathcal{L}}} \geq 0$ if $\overline{\mathcal{L}}$ is nef. Then $\overline{\mathcal{L}}(-e_{\overline{\mathcal{L}}})$ is a nef line bundle whose absolute minimum is zero. It is a very important fact in our treatment in the following.

We refer to [Zh1] for more results on the minima of $\overline{\mathcal{L}}$ for nef hermitian line bundles.

3.2. **The key decompositions.** The goal of this section is to prove two basic decompositions of hermitian line bundles. They are respectively decompositions keeping $\widehat{H}^0(\overline{\mathcal{L}})$ and $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}})$. The proofs are the same, but we state them in separate theorems since they will be used for different purposes.

Theorem 3.1. *Let X be a regular arithmetic surface, and $\overline{\mathcal{L}}$ be a hermitian line bundle with $\widehat{h}^0(\overline{\mathcal{L}}) \neq 0$. Then there is a decomposition*

$$\overline{\mathcal{L}} = \overline{\mathcal{E}} + \overline{\mathcal{L}}_1$$

where $\overline{\mathcal{E}}$ is an effective hermitian line bundle on X, and $\overline{\mathcal{L}}_1$ is a nef hermitian line bundle on X satisfying the following conditions:

- *There is an effective section $e \in \widehat{H}^0(\overline{\mathcal{E}})$ such that* $\mathrm{div}(e)$ *is the fixed part of $\widehat{H}^0(\overline{\mathcal{L}})$ in X.*

- *The map $\mathcal{L}_1 \to \mathcal{L}$ defined by tensoring with e induces a bijection*
$$\widehat{H}^0(\overline{\mathcal{L}}_1) \xrightarrow{\otimes e} \widehat{H}^0(\overline{\mathcal{L}}).$$
Furthermore, the bijection keeps the supremum norms, i.e.,
$$\|s\|_{\sup} = \|e \otimes s\|_{\sup}, \quad \forall\ s \in \widehat{H}^0(\overline{\mathcal{L}}_1).$$

Theorem 3.2. *Let X be a regular arithmetic surface, and $\overline{\mathcal{L}}$ be a hermitian line bundle with $\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) \neq 0$. Then there is a decomposition*
$$\overline{\mathcal{L}} = \overline{\mathcal{E}} + \overline{\mathcal{L}}_1$$
where $\overline{\mathcal{E}}$ is an effective hermitian line bundle on X, and $\overline{\mathcal{L}}_1$ is a nef hermitian line bundle on X satisfying the following conditions:

- *There is an effective section $e \in \widehat{H}^0(\overline{\mathcal{E}})$ such that $\mathrm{div}(e)$ is the fixed part of $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}})$ in X.*
- *The map $\mathcal{L}_1 \to \mathcal{L}$ defined by tensoring with e induces a bijection*
$$\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_1) \xrightarrow{\otimes e} \widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}).$$
Furthermore, the bijection keeps the supremum norms, i.e.,
$$\|s\|_{\sup} = \|e \otimes s\|_{\sup}, \quad \forall\ s \in \widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_1).$$

Before proving the theorems, we deduce Theorem A from Theorem B using Theorem 3.1.

Let $\overline{\mathcal{L}}$ be as in Theorem A. The theorem is trivial if $\widehat{h}^0(\overline{\mathcal{L}}) = 0$. Assume that $\widehat{h}^0(\overline{\mathcal{L}}) \neq 0$. As in Theorem 3.1, decompose
$$\overline{\mathcal{L}} = \overline{\mathcal{L}}_1 + \overline{\mathcal{E}}.$$
It particularly gives $\widehat{h}^0(\overline{\mathcal{L}}) = \widehat{h}^0(\overline{\mathcal{L}}_1)$. For any $n \geq 1$, we have an injection
$$\widehat{H}^0(n\overline{\mathcal{L}}_1) \xrightarrow{e^{\otimes n}} \widehat{H}^0(n\overline{\mathcal{L}}).$$
It follows that $\widehat{h}^0(n\overline{\mathcal{L}}) \geq \widehat{h}^0(n\overline{\mathcal{L}}_1)$, and thus
$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) \geq \widehat{\mathrm{vol}}(\overline{\mathcal{L}}_1) = \overline{\mathcal{L}}_1^2.$$

By $\widehat{H}^0(\overline{\mathcal{L}}) = \widehat{H}^0(\overline{\mathcal{L}}_1)$, we have $h^0(\mathcal{L}_{1,K}) \geq r' \geq 2$. It yields that $\deg(\mathcal{L}_{1,K}) \geq r' \geq 2$ if $g > 0$, and $\deg(\mathcal{L}_{1,K}) \geq r' - 1 \geq 1$ if $g = 0$. Then we are exactly in the situation to apply Theorem B to $\overline{\mathcal{L}}_1$. It gives exactly Theorem A since $\deg(\mathcal{L}_{1,K}) \leq \deg(\mathcal{L}_K) = d^\circ$.

3.3. **Construction of the decompositions.** Now we prove Theorem 3.2. We will see that Theorem 3.1 can be proved in the same way. The proof is very similar to the arithmetic Fujita approximation in [Yu2]. We prove Theorem 3.2 by the following steps.

Step 1. Denote by Z the fixed part of $\widehat{H}^0_{\mathrm{sef}}(X,\overline{\mathcal{L}})$. Set $\mathcal{E}$ to be the line bundle on X associated to Z, and let $e\in H^0(\mathcal{E})$ be the section defining Z. Define a line bundle $\mathcal{L}_1$ on X by the decomposition

$$\mathcal{L}=\mathcal{E}+\mathcal{L}_1.$$

We need to construct suitable metrics on $\mathcal{E}$ and $\mathcal{L}_1$.

For convenience, in the following we write

$$\widehat{H}^0_{\mathrm{sef}}(X,\overline{\mathcal{L}})=\{0,s_1,s_2,\cdots,s_k\}.$$

Denote $t_i=e^{-1}s_i$ for any $i=1,\cdots,k$, viewed as a global section of $\mathcal{L}_1$.

Step 2. Define a metric $\|\cdot\|_{\mathcal{E}}$ on $\mathcal{E}$ by assigning any $x\in X(\mathbb{C})$ to

$$\|e(x)\|_{\mathcal{E}}=\max\{\|s_i(x)\|/\|s_i\|_{\sup}: i=1,\cdots,k\}.$$

It is easy to see that $\|e\|_{\mathcal{E},\sup}=1$. Define a metric $\|\cdot\|_{\mathcal{L}_1}$ on $\mathcal{L}_1$ by the decomposition

$$\overline{\mathcal{L}}=(\mathcal{E},\|\cdot\|_{\mathcal{E}})+(\mathcal{L}_1,\|\cdot\|_{\mathcal{L}_1}).$$

Set $\overline{\mathcal{E}}=(\mathcal{E},\|\cdot\|_{\mathcal{E}})$ and $\overline{\mathcal{L}}_1=(\mathcal{L}_1,\|\cdot\|_{\mathcal{L}_1})$. We will prove that the decomposition $\overline{\mathcal{L}}=\overline{\mathcal{E}}+\overline{\mathcal{L}}_1$ satisfies the theorem. We first verify that $\|\cdot\|_{\mathcal{E}}$ and $\|\cdot\|_{\mathcal{L}_1}$ are continuous metrics.

It suffices to prove that $\|\cdot\|_{\mathcal{L}_1}$ is continuous. By definition, for any local section t of $\mathcal{L}_1(\mathbb{C})$ at a point $x\in X(\mathbb{C})$,

$$\begin{aligned}\|t(x)\|_{\mathcal{L}_1}=\frac{\|(et)(x)\|}{\|e(x)\|_{\mathcal{E}}} &= \min_i\left(\|s_i\|_{\sup}\frac{\|(et)(x)\|}{\|s_i(x)\|}\right)\\ &= \min_i\left(\|s_i\|_{\sup}\cdot|(t/t_i)(x)|\right).\end{aligned}$$

It is a continous metric since the set $\{t_1,\cdots,t_k\}$ is base-point-free on $X(\mathbb{C})$ by definition.

Step 3. We claim that the map $\mathcal{L}_1\to\mathcal{L}$ defined by tensoring with e induces a bijection

$$\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_1)\longrightarrow\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}})$$

which keeps the supremum norms. In other words, we have

$$\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_1)=\{0,t_1,t_2,\cdots,t_k\},$$

and $\|t_i\|_{\mathcal{L}_1,\sup}=\|s_i\|_{\sup}$ for $i=1,\cdots,k$.

In fact, it suffices to verify $\|t_i\|_{\mathcal{L}_1,\sup}=\|s_i\|_{\sup}$ for each i. By definition of the metrics, $\|e(x)\|_{\mathcal{E}}\leq 1$ and thus

$$\|s_i(x)\|\leq\|t_i(x)\|_{\mathcal{L}_1}\leq\|s_i\|_{\sup},\quad x\in X(\mathbb{C}).$$

Taking supremum, we have

$$\|s_i\|_{\sup}\leq\|t_i\|_{\mathcal{L}_1,\sup}\leq\|s_i\|_{\sup}.$$

The equality is obtained.

Step 4. We show that the continuous metric $\|\cdot\|_{\mathcal{L}_1}$ in Step 2 is semipositive. For any point $x \in X(\mathbb{C})$, take a trivialization of $\mathcal{L}(\mathbb{C})$ in neighborhood of x. Recall that for any local section t of $\mathcal{L}_1(\mathbb{C})$ at a point $x \in X(\mathbb{C})$, the metric

$$\|t(x)\|_{\mathcal{L}_1} = \min_i \left(\|s_i\|_{\sup} \cdot |(t/t_i)(x)|\right) = |t(x)|e^{-\phi(x)},$$

where

$$\phi(x) = \max_i \left(\log|t_i(x)| - \log\|s_i\|_{\sup}\right).$$

Note that each function $\log|t_i(x)| - \log\|s_i\|_{\sup}$ is pluri-subharmonic. Then ϕ is pluri-subharmonic since being pluri-subharmonic is stable under taking maximum. It follows that the metric is semipositive.

Step 5. Finally, we prove that the hermitian line bundle $\overline{\mathcal{L}}_1$ is nef on X. We only need to show $\widehat{\deg}(\overline{\mathcal{L}}_1|_Y) \geq 0$ for any integral subscheme Y. By definition, the set $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_1)$ has no fixed part. For any integral subscheme Y of X, we can find a section $s \in \widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_1)$ nonvanishing on Y. Then $\widehat{\deg}(\overline{\mathcal{L}}_1|_Y) \geq 0$ by this section.

4. Proof of Theorem B

In this section, we use the construction above to prove Theorem B. We first prove a trivial bound, and then prove the theorem.

4.1. **A trivial Bound.**

The following is an easy bound on $\widehat{h}^0(\overline{\mathcal{L}})$, which serves as the last step of our reduction.

Proposition 4.1. *Let $\overline{\mathcal{L}}$ be a nef hermitian line bundle on X with $\deg(\mathcal{L}_K) > 0$. Denote by r^{-} the $\mathbb{Z}$-rank of the $\mathbb{Z}$-submodule of $H^0(\mathcal{L})$ generated by $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}})$. Then we have*

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) \leq \frac{r^{-}}{\deg(\mathcal{L}_{\mathbb{Q}})}\overline{\mathcal{L}}^2 + r^{-}\log 3.$$

The same result holds for $\widehat{h}^0(\overline{\mathcal{L}})$ by replacing r^{-} by the $\mathbb{Z}$-rank of the $\mathbb{Z}$-submodule of $H^0(\mathcal{L})$ generated by $\widehat{H}^0(\overline{\mathcal{L}})$.

Proof. We can assume that the metric of $\overline{\mathcal{L}}$ is positive and smooth, since we can always approximate semipositive and continuous metrics by positive and smooth metrics uniformly. Denote

$$\alpha = \frac{1}{r^{-}}\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - \log 3 - \epsilon, \quad \epsilon > 0.$$

By Proposition 2.1,

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}(-\alpha)) \geq \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - (\alpha + \log 3)r^- = \epsilon r^- > 0.$$

It follows that there is a section $s \in H^0(\mathcal{L})$ with

$$-\log \|s\|_{\sup} > \alpha.$$

Then we have

$$\overline{\mathcal{L}}^2 = \overline{\mathcal{L}} \cdot \mathrm{div}(s) - \int_{X(\mathbb{C})} \log \|s\| c_1(\overline{\mathcal{L}}) > \alpha \ \deg(\mathcal{L}_{\mathbb{Q}}).$$

Therefore, we have

$$\frac{1}{r^-} \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - \log 3 - \epsilon < \frac{\overline{\mathcal{L}}^2}{\deg(\mathcal{L}_{\mathbb{Q}})}.$$

Take $\epsilon \to 0$. The inequality follows.

□

Remark 4.2. The result can be extended to arithmetic varieties of any dimensions without extra work.

4.2. **The reduction process.** Let $\overline{\mathcal{L}}$ be a nef line bundle. We are going to apply Theorem 3.2 to reduce $\overline{\mathcal{L}}$ to "smaller" nef line bundles. The problem is that the fixed part of $\overline{\mathcal{L}}$ may be empty, and then Theorem 3.2 is a trivial decomposition. The idea is to enlarge the metric of $\overline{\mathcal{L}}$ by constant multiples to create base points. To keep the nefness, the largest constant multiple we can use gives the case that the absolute minimum is 0. The following proposition says that the situation exactly meets our requirement.

Proposition 4.3. *Let X be a regular arithmetic surface, and $\overline{\mathcal{L}}$ be a nef hermitian line bundle on X satisfying*

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) > 0, \quad e_{\overline{\mathcal{L}}} = 0.$$

Then the base locus of $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}})$ contains some horizontal divisor of X.

Proof. Denote by S the set of horizontal irreducible divisors D of X such that $h_{\overline{\mathcal{L}}}(D) = 0$. The result follows from the properties that S is non-empty and contained in the base locus of $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}})$.

First, S is non-empty. Note that the absolute minimum of $\overline{\mathcal{L}}$ is 0, so it suffices to prove that 0 is not an accumulation point of the range of $h_{\overline{\mathcal{L}}}$. Choose any nonzero section $s \in \widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}})$. For any horizontal irreducible divisor D not contained in the support of $\mathrm{div}(s)$, one has

$$h_{\overline{\mathcal{L}}}(D) = \frac{1}{\deg(D_{\mathbb{Q}})} (\mathrm{div}(s) \cdot D - \log \|s\|(D(\mathbb{C}))) \geq -\log \|s\|_{\sup} > 0.$$

It follows that 0 is not an accumulation point, and there must be an irreducible component of $\mathrm{div}(s)$ lying in X.

Second, every element of S is contained in the base locus of $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}})$. Take any $D \in S$ and $s \in \widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}})$. If s does not vanish on D, then the above estimate gives

$$h_{\overline{\mathcal{L}}}(D) \geq -\log \|s\|_{\sup} > 0.$$

It is a contradiction.

$\square$

Now let us try to prove Theorem B by Theorem 3.2. Denote by $c = e_{\overline{\mathcal{L}}}$ the absolute minimum. By definition, $\overline{\mathcal{L}}(-c)$ is still nef, and its absolute minimum is 0. If $\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}(-c)) \neq 0$, applying Theorem 3.2 to $\overline{\mathcal{L}}(-c)$, we obtain a decomposition

$$\overline{\mathcal{L}}(-c) = \overline{\mathcal{E}} + \overline{\mathcal{L}}_1$$

with $\overline{\mathcal{E}}$ effective and $\overline{\mathcal{L}}_1$ nef, which gives

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}(-c)) = \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_1).$$

By Proposition 2.1,

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) \leq \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}(-c)) + (c + \log 3) h^0(\mathcal{L}_{\mathbb{Q}}).$$

Note that

$$\overline{\mathcal{L}}(-c)^2 - \overline{\mathcal{L}}_1^2 = \overline{\mathcal{E}} \cdot (\overline{\mathcal{L}}(-c) + \overline{\mathcal{L}}_1) \geq 0.$$

Thus

$$\overline{\mathcal{L}}^2 = \overline{\mathcal{L}}(-c)^2 + 2cd \geq \overline{\mathcal{L}}_1^2 + 2cd.$$

Therefore,

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - \frac{1}{2}\overline{\mathcal{L}}^2 \leq \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_1) - \frac{1}{2}\overline{\mathcal{L}}_1^2 + \deg(\mathcal{L}_{\mathbb{Q}}) \log 3.$$

By Proposition 4.3, the degree decreases:

$$\deg(\mathcal{L}_{\mathbb{Q}}) > \deg(\mathcal{L}_{1,\mathbb{Q}}).$$

Then we can reduce the theorem for $\overline{\mathcal{L}}$ to that for $\overline{\mathcal{L}}_1$. One problem is that, when we keep the reduction process to obtain $\overline{\mathcal{L}}_2, \cdots$, the accumulated error term

$$\deg(\mathcal{L}_{\mathbb{Q}}) \log 3 + \deg(\mathcal{L}_{1,\mathbb{Q}}) \log 3 + \deg(\mathcal{L}_{2,\mathbb{Q}}) \log 3 + \cdots$$

may grow as

$$(d + (d-1) + \cdots + 1) \log 3 = \frac{1}{2} d(d+1) \log 3.$$

It is too big for our consideration.

The key of our solution of the problem is Proposition 2.3. We put all the sections $\widehat{H}^0(\overline{\mathcal{L}}_i)$ in one space, the error term will be decreased to a multiple of $d\log d$ in Proposition 4.5. See also the remark after Proposition 2.3.

For convenience of application, we describe the total construction as a theorem. The proof of Theorem B will be given in next section.

Theorem 4.4. *Let X be a regular arithmetic surface, and let $\overline{\mathcal{L}}$ be a nef hermitian line bundle on X. There is an integer $n \geq 0$, and a sequence of triples*

$$\{(\overline{\mathcal{L}}_i, \overline{\mathcal{E}}_i, c_i): \ i = 0, 1, \cdots, n\}$$

satisfying the following properties:

- $(\overline{\mathcal{L}}_0, \overline{\mathcal{E}}_0, c_0) = (\overline{\mathcal{L}}, \overline{\mathcal{O}}_X, e_{\overline{\mathcal{L}}})$.
- *For any $i = 0, \cdots, n$, the constant $c_i = e_{\overline{\mathcal{L}}_i} \geq 0$ is the absolute minimum of $\overline{\mathcal{L}}_i$.*
- $\widehat{h}^0_{\mathrm{sef}}(X, \overline{\mathcal{L}}_i(-c_i)) > 0$ *for any $i = 0, \cdots, n-1$.*
- *For any $i = 0, \cdots, n-1$,*
$$\overline{\mathcal{L}}_i(-c_i) = \overline{\mathcal{L}}_{i+1} + \overline{\mathcal{E}}_{i+1}$$
is a decomposition of $\overline{\mathcal{L}}_i(-c_i)$ as in Theorem 3.2.
- $\widehat{h}^0_{\mathrm{sef}}(X, \overline{\mathcal{L}}_n(-c_n)) = 0$.

The following are some properties by the construction:

- *For any $i = 0, \cdots, n$, $\overline{\mathcal{L}}_i$ is nef and every $\overline{\mathcal{E}}_i$ is effective.*
- $\deg(\mathcal{L}_{0,\mathbb{Q}}) > \deg(\mathcal{L}_{1,\mathbb{Q}}) > \cdots > \deg(\mathcal{L}_{n,\mathbb{Q}})$.
- *For any $i = 0, \cdots, n-1$, there is a section $e_{i+1} \in \widehat{H}^0(\overline{\mathcal{E}}_{i+1})$ inducing a bijection*
$$\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_{i+1}) \longrightarrow \widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_i(-c_i))$$
which keeps the supremum norms.

Proof. The triple $(\overline{\mathcal{L}}_{i+1}, \overline{\mathcal{E}}_{i+1}, c_{i+1})$ is obtained by decomposing $\overline{\mathcal{L}}_i(-c_i)$. The extra part is that Proposition 4.3 ensures the degrees on the generic fiber decreases strictly. The process terminates if $\widehat{h}^0_{\mathrm{sef}}(X, \overline{\mathcal{L}}_i(-c_i)) = 0$. It always terminates since $\deg(\mathcal{L}_{i,\mathbb{Q}})$ decreases. $\square$

4.3. **Case of positive genus.** Here we prove Theorem B in the case $g > 0$. Assume the notations of Theorem 4.4. We first bound the changes of $\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_j)$ and $\overline{\mathcal{L}}_j^2$.

Recall that Theorem 4.4 starts with a nef line bundle $\overline{\mathcal{L}}_0 = \overline{\mathcal{L}}$ and constructs the sequence

$$(\overline{\mathcal{L}}_i, \overline{\mathcal{E}}_i, c_i), \quad i = 0, \cdots, n.$$

Here $\overline{\mathcal{L}}_i$ is nef and $\overline{\mathcal{E}}_i$ is effective, and $c_i = e_{\overline{\mathcal{L}}_i} \geq 0$. In particular, $\overline{\mathcal{L}}_i(-c_i)$ is still nef. For any $i = 0, \cdots, n-1$, the decomposition

$$\overline{\mathcal{L}}_i(-c_i) = \overline{\mathcal{L}}_{i+1} + \overline{\mathcal{E}}_{i+1}$$

yields a bijection

$$\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_{i+1}) \longrightarrow \widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_i(-c_i)).$$

It is given by tensoring some distinguished element $e_i \in \widehat{H}^0(\overline{\mathcal{E}}_i)$. It is very important that the bijection keeps the supremum norms. In the following, we denote

$$\overline{\mathcal{L}}'_i = \overline{\mathcal{L}}_i(-c_i), \quad i = 0, \cdots, n.$$

Proposition 4.5. *For any $j = 0, \cdots, n$, one has*

$$\begin{aligned} \overline{\mathcal{L}}^2 &\geq \overline{\mathcal{L}}'^2_j + 2d_0c_0 + \sum_{i=1}^{j}(d_{i-1} + d_i)c_i \geq \overline{\mathcal{L}}'^2_j + 2\sum_{i=0}^{j} d_i c_i, \\ \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) &\leq \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}'_j) + \sum_{i=0}^{j} r_i c_i + 4r_0\log r_0 + 2r_0 \log 3. \end{aligned}$$

Here we denote $d_i = \deg(\mathcal{L}_{i,\mathbb{Q}})$ and $r_i = h^0(\mathcal{L}_{i,\mathbb{Q}})$.

Proof. Denote $\alpha_0 = 0$ and

$$\alpha_i = c_0 + \cdots + c_{i-1}, \quad i = 1, \cdots, n.$$

The key is the bijection

$$\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_i) \longrightarrow \widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}(-\alpha_i)).$$

It is given by tensoring the section $e_1 \otimes \cdots \otimes e_i$. Denote by r_i^- the rank of the $\mathbb{Z}$-submodule of $H^0(\mathcal{L})$ generated by $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}(-\alpha_i))$. Apply Proposition 2.3 to $\overline{M} = (H^0(\mathcal{L}), \|\cdot\|_{\sup})$. We obtain

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) \leq \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_j(-c_j)) + \sum_{i=0}^{j} r_i^- c_i + 4r \log r_0^- + 2r_0^- \log 3.$$

The result follows since $r_i^- \leq r_i$.

It is also easy to bound the intersection numbers. By definition, we have

$$\overline{\mathcal{L}}'_i = \overline{\mathcal{L}}_{i+1} + \overline{\mathcal{E}}_{i+1} = \overline{\mathcal{L}}'_{i+1} + \overline{\mathcal{E}}_{i+1} + \overline{\mathcal{O}}(c_{i+1}).$$

Here $\overline{\mathcal{L}}'_i$ and $\overline{\mathcal{L}}'_{i+1}$ are nef, and $\overline{\mathcal{E}}_{i+1}$ is effective. It follows that

$$\begin{aligned} \overline{\mathcal{L}}'^2_i - \overline{\mathcal{L}}'^2_{i+1} &= (\overline{\mathcal{L}}'_i + \overline{\mathcal{L}}'_{i+1}) \cdot (\overline{\mathcal{E}}_{i+1} + \overline{\mathcal{O}}(c_{i+1})) \\ &\geq (\overline{\mathcal{L}}'_i + \overline{\mathcal{L}}'_{i+1}) \cdot \overline{\mathcal{O}}(c_{i+1}) \\ &= (d_i + d_{i+1})c_{i+1}. \end{aligned}$$

Summing over $i = 0, \cdots, j-1$, we can get

$$\overline{\mathcal{L}}_0'^2 \geq \overline{\mathcal{L}}_j'^2 + \sum_{i=1}^{j} (d_{i-1} + d_i) c_i.$$

Then the conclusion follows from the fact that

$$\overline{\mathcal{L}}_0^2 = \overline{\mathcal{L}}_0'^2 - 2d_0 c_0.$$

□

Now we prove Theorem B. By Proposition 2.1, it suffices to prove

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) \leq \frac{1}{2}\overline{\mathcal{L}}^2 + 4d\log d + 3d\log 3.$$

It is classical that $g > 0$ implies

$$r_i \leq d_i, \quad i = 0, \cdots, n-1.$$

It also holds for $i = n$ if $\deg(\mathcal{L}_{n,\mathbb{Q}}) \neq 0$. Then the proposition gives, for $j = 0, \cdots, n-1$,

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - \frac{1}{2}\overline{\mathcal{L}}^2 \leq \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_j') - \frac{1}{2}\overline{\mathcal{L}}_j'^2 + 4d\log d + 2d\log 3.$$

If $\deg(\mathcal{L}_{n,\mathbb{Q}}) > 0$, the inequality also holds for $j = n$. Then the theorem is proved since $\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_n') = 0$ and $\overline{\mathcal{L}}_n'^2 \geq 0$.

It remains to treat the case $\deg(\mathcal{L}_{n,\mathbb{Q}}) = 0$. Note that $\mathcal{L}_{n,\mathbb{Q}}$ is trivial since $\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_n)$ is base-point-free on the generic fiber by construction. The inequality is not true for $j = n$. We use the case $j = n-1$ instead.

To bound $\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_{n-1}')$, we apply Proposition 4.1. It gives

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_{n-1}') \leq \frac{r_n^-}{\deg(\mathcal{L}_{n-1,\mathbb{Q}})}\overline{\mathcal{L}}_{n-1}'^2 + r_n^- \log 3.$$

Here r_n^- is the $\mathbb{Z}$-rank of $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_{n-1}') = \widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_n)$. It is easy to have $r_n^- \leq [K:\mathbb{Q}]$ since $\mathcal{L}_{n,\mathbb{Q}}$ is trivial.

We claim that $\deg(\mathcal{L}_{n-1,\mathbb{Q}}) \geq 2[K:\mathbb{Q}]$, or equivalently $\deg \mathcal{L}_{n-1,K} \geq 2$. If $n = 1$, it is true by the assumption on $\overline{\mathcal{L}}$. Otherwise, by the construction from $\overline{\mathcal{L}}_{n-2}$, the base locus of $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_{n-1})$ is empty or has dimension zero on X. It follows that $\mathcal{L}_{n-1,K}$ is base-point-free on X_K. Its degree is at least two since $g > 0$.

By the claim, we have

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_{n-1}') \leq \frac{1}{2}\overline{\mathcal{L}}_{n-1}'^2 + [K:\mathbb{Q}]\log 3.$$

It finishes proving Theorem B for $g > 0$.

4.4. **Case of genus zero.** Here we prove Theorem B in the case $g = 0$. Still apply Proposition 4.5. We use the more delicate bounds

$$\begin{aligned} \overline{\mathcal{L}}^2 &\geq \overline{\mathcal{L}}_n'^2 + 2d_0c_0 + \sum_{i=1}^{n}(d_{i-1}+d_i)c_i, \\ \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) &\leq \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_n') + \sum_{i=0}^{n} r_ic_i + 4r_0\log r_0 + 2r_0\log 3. \end{aligned}$$

We still need to compare them.

Denote $\kappa = [K:\mathbb{Q}]$. Then

$$d_i = \deg(\mathcal{L}_{i,\mathbb{Q}}) = \kappa\deg(\mathcal{L}_{i,K}), \quad r_i = h^0(\mathcal{L}_{i,\mathbb{Q}}) = \kappa h^0(\mathcal{L}_{i,K}).$$

Note that we do not have $r_i \leq d_i$ any more. But we have

$$h^0(\mathcal{L}_{i,K}) = \deg(\mathcal{L}_{i,K}) + 1,$$

and thus

$$r_i = d_i + \kappa, \quad i = 0, \cdots, n.$$

Hence, the inequalities yield

$$\begin{aligned} &\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - \frac{1}{2}\overline{\mathcal{L}}^2 \\ \leq\ & \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_n') - \frac{1}{2}\overline{\mathcal{L}}_n'^2 + \kappa\sum_{i=0}^{n} c_i - \frac{1}{2}\sum_{i=1}^{n}(d_{i-1}-d_i)c_i + 4r_0\log r_0 + 2r_0\log 3 \\ \leq\ & \kappa\sum_{i=0}^{n} c_i - \frac{1}{2}\sum_{i=1}^{n}(d_{i-1}-d_i)c_i + 4r_0\log r_0 + 2r_0\log 3. \end{aligned}$$

Note that

$$d_{i-1} - d_i = \kappa(\deg(\mathcal{L}_{i-1,K}) - \deg(\mathcal{L}_{i,K})) \geq \kappa.$$

We have

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - \frac{1}{2}\overline{\mathcal{L}}^2 \leq \kappa c_0 + \frac{1}{2}\kappa\sum_{i=1}^{n} c_i + 4r_0\log r_0 + 2r_0\log 3.$$

The proof is completed by the following result.

Lemma 4.6. *In the setting of Theorem 4.4, for any genus $g \geq 0$,*

$$c_0 + \sum_{i=0}^{n} c_i \leq \frac{1}{\deg(\mathcal{L}_{\mathbb{Q}})}\overline{\mathcal{L}}^2.$$

Proof. Denote $\beta = c_1 + \cdots + c_n$ and $\overline{\mathcal{F}} = \overline{\mathcal{E}}_1 + \cdots + \overline{\mathcal{E}}_n$. We have the decomposition

$$\overline{\mathcal{L}}_0'(-\beta) = \overline{\mathcal{L}}_n' + \overline{\mathcal{F}}.$$

Note that $\overline{\mathcal{L}}'_0(-\beta)$ is not nef any more. But we can still have a weaker bound as follows:

$$\begin{aligned}\overline{\mathcal{L}}'^2_0 &= \overline{\mathcal{L}}'_0 \cdot (\overline{\mathcal{L}}'_n + \overline{\mathcal{F}} + \overline{\mathcal{O}}(\beta)) \\ &\geq \overline{\mathcal{L}}'_0 \cdot \overline{\mathcal{L}}'_n + d_0\beta \\ &= (\overline{\mathcal{L}}'_n + \overline{\mathcal{F}} + \overline{\mathcal{O}}(\beta)) \cdot \overline{\mathcal{L}}'_n + d_0\beta \\ &\geq \overline{\mathcal{L}}'^2_n + d_n\beta + d_0\beta.\end{aligned}$$

Here $d_i = \deg(\mathcal{L}_{i,\mathbb{Q}})$ as usual.

Combine with

$$\overline{\mathcal{L}}^2 = \overline{\mathcal{L}}(-c_0)^2 + 2d_0c_0.$$

We have

$$\overline{\mathcal{L}}^2 \geq \overline{\mathcal{L}}'^2_n + d_n\beta + d_0\beta + 2d_0c_0 \geq d_0(2c_0 + \beta).$$

The result follows.

□

Remark 4.7. The result is in the spirit of the successive minima of S. Zhang [Zh1].

4.5. **Extra case of degree one.**

Proposition 4.8. *Let X be a regular and geometrically connected arithmetic surface of genus g over O_K. Let $\overline{\mathcal{L}}$ be a nef hermitian line bundle on X with $\deg(\mathcal{L}_K) = 1$.*

(1) *If $g > 0$, then*

$$\widehat{h}^0(\overline{\mathcal{L}}) \leq \overline{\mathcal{L}}^2 + [K:\mathbb{Q}]\log 3.$$

(2) *If $g = 0$, then*

$$\widehat{h}^0(\overline{\mathcal{L}}) \leq \overline{\mathcal{L}}^2 + 5[K:\mathbb{Q}]\log 3.$$

Proof. If $g > 0$, the result follows from Proposition 4.1. If $g = 0$, we use the method of §4.3 to get a good bound. Denote $\kappa = [K:\mathbb{Q}]$ as usual. We still have

$$\overline{\mathcal{L}}^2 = \overline{\mathcal{L}}'^2 + 2\kappa c_0, \quad \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) \leq \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}') + 2\kappa c_0 + 2\kappa\log 3.$$

It follows that

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - \overline{\mathcal{L}}^2 \leq \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}') - \overline{\mathcal{L}}'^2 + 2\kappa\log 3.$$

Because the O_K-rank $\widehat{H}^0(\overline{\mathcal{L}}')$ is at most one, Proposition 4.1 gives

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}') \leq \overline{\mathcal{L}}'^2 + \kappa\log 3.$$

It follows that

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) \leq \overline{\mathcal{L}}^2 + 3\kappa\log 3.$$

The result follows from Proposition 2.1.

□

5. Proof of Theorem C

Our tool to get the strong bound in Theorem C is Clifford's theorem in the classical setting. For convenience, we recall it here.

Let C be a projective, smooth and geometrically connected curve over a field k. Recall that a line bundle L on C is special if

$$h^0(L) > 0, \quad h^1(L) > 0.$$

The following is Clifford's theorem (cf.[Ha, Theorem IV.5.4]).

Theorem 5.1 (Clifford)**.** *If L is a special line bundle on C, then*

$$h^0(L) \leq \frac{1}{2}\deg(L) + 1.$$

Furthermore, if C is not hyperelliptic, then the equality is obtained if and only if $L \simeq \mathcal{O}_C$ or $L \simeq \omega_{C/k}$.

We also need the following basic fact, whose proof we include for convenience of readers.

Lemma 5.2. *Let L be a special line bundle on a hyperelliptic curve C. If L is base-point-free, then* $\deg(L)$ *is even.*

Proof. Let ι be the hyperelliptic involution of C. Then any divisor in $|\omega_{C/k}|$ is of the form $D_0 + \iota^* D_0$ for some divisor D_0 on C. In fact, let $\pi : C \to \mathbb{P}^1$ be the quotient map of degree two. Then $\omega_{C/k}$ is isomorphic to the pull-back of some line bundle M on $\mathbb{P}^1$, and any global section of $\omega_{C/k}$ is the pull-back of some global section of M by counting dimensions.

Prove the lemma by contradiction. Assume that $\deg(L)$ is odd. Fix a divisor $D \in |\omega_{C/k} - L|$. Then $\deg(D)$ is also odd. There is a closed point P_0 on C such that one of the following holds:

- $P_0 \neq \iota(P_0)$, and the support of D contains P_0 but does not contain $\iota(P_0)$.
- $P_0 = \iota(P_0)$, and the multiplicity of P_0 in D is odd.

In both cases, any divisor $E \in |L|$ contains $\iota(P_0)$ since $D + E \in |\omega_{C/k}|$ is of the form $D_0 + \iota^* D_0$. In another word, $\iota(P_0)$ is a base point of L. It is a contradiction. □

5.1. **Hyperelliptic case.** We first prove Theorem C in the hyperelliptic case. The proof is very similar to that in §4.4.

By Proposition 4.5,

$$\begin{aligned} \overline{\mathcal{L}}^2 &\geq \overline{\mathcal{L}}_n'^2 + 2d_0c_0 + \sum_{i=1}^{n}(d_{i-1}+d_i)c_i, \\ \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) &\leq \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_n') + \sum_{i=0}^{n} r_ic_i + 4r_0\log r_0 + 2r_0\log 3. \end{aligned}$$

Here $d_i = \deg(\mathcal{L}_{i,\mathbb{Q}})$ and $r_i = h^0(\mathcal{L}_{i,\mathbb{Q}})$.

By construction, each $\mathcal{L}_{i,K}$ is special. Clifford's theorem gives

$$h^0(\mathcal{L}_{i,K}) \leq \frac{1}{2}\deg(\mathcal{L}_{i,K}) + 1,$$

and thus

$$r_i \leq \frac{1}{2}d_i + \kappa, \quad \kappa = [K:\mathbb{Q}].$$

Hence, the inequalities yield

$$\begin{aligned} &\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - \frac{1}{4}\overline{\mathcal{L}}^2 \\ \leq\ & \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_n') - \frac{1}{4}\overline{\mathcal{L}}_n'^2 + \kappa\sum_{i=0}^{n} c_i - \frac{1}{4}\sum_{i=1}^{n}(d_{i-1}-d_i)c_i + 4r_0\log r_0 + 2r_0\log 3 \\ \leq\ & \kappa\sum_{i=0}^{n} c_i - \frac{1}{4}\sum_{i=1}^{n}(d_{i-1}-d_i)c_i + 4r_0\log r_0 + 2r_0\log 3. \end{aligned}$$

By definition,

$$d_{i-1} - d_i = \kappa(\deg(\mathcal{L}_{i-1,K}) - \deg(\mathcal{L}_{i,K})) \geq \kappa$$

for $i = 1, \cdots, n$. It follows that

$$\begin{aligned} \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) &\leq \frac{1}{4}\overline{\mathcal{L}}^2 + \kappa c_0 + \frac{3}{4}\kappa\sum_{i=1}^{n} c_i + 4r_0\log r_0 + 2r_0\log 3 \\ &\leq (\frac{1}{4} + \frac{3}{4d^\circ})\overline{\mathcal{L}}^2 + 4r_0\log r_0 + 2r_0\log 3. \end{aligned}$$

Here we have used Lemma 4.6, which asserts

$$c_0 + \sum_{i=0}^{n} c_i \leq \frac{1}{d_0}\overline{\mathcal{L}}^2.$$

If $\deg(\mathcal{L}_K)$ is even, we can have a stronger estimate. In fact, for $i = 1, \cdots, n$, the line bundle $\mathcal{L}_i$ is base-point-free by construction, so

$\deg(\mathcal{L}_{i,K})$ is even by Lemma 5.2. Thus we have

$$d_{i-1} - d_i = \kappa(\deg(\mathcal{L}_{i-1,K}) - \deg(\mathcal{L}_{i,K})) \geq 2\kappa$$

for $i = 1, 2, \cdots, n$. It especially holds for $i = 1$ by the assumption that $\deg(\mathcal{L}_K)$ is even. In that case, we have

$$\begin{aligned} \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) &\leq \frac{1}{4}\overline{\mathcal{L}}^2 + \kappa c_0 + \frac{1}{2}\kappa \sum_{i=1}^{n} c_i + 4r_0 \log r_0 + 2r_0 \log 3 \\ &\leq (\frac{1}{4} + \frac{1}{2d^\circ})\overline{\mathcal{L}}^2 + 4r_0 \log r_0 + 2r_0 \log 3. \end{aligned}$$

The result is proved.

5.2. **Non-hyperelliptic case.** For $i = 1, \cdots, n-1$, Clifford's theorem gives a stronger bound

$$h^0(\mathcal{L}_{i,K}) \leq \frac{1}{2}\deg(\mathcal{L}_{i,K}) + \frac{1}{2},$$

and thus

$$r_i \leq \frac{1}{2}d_i + \frac{1}{2}\kappa.$$

It is also true for $i = 0$ or $i = n$ as long as $\mathcal{L}_{i,K}$ is neither the canonical bundle nor the trivial bundle. For $i = 0$ or $i = n$, it is always safe to use the bound

$$r_i \leq \frac{1}{2}d_i + \kappa.$$

The proof of Theorem C is similar, but more subtle due to the possible failure of the strong bound for $i = 0$ and $i = n$.

We first assume that $\mathcal{L}_{n,K}$ is non-trivial. By the strong bounds, Proposition 4.5 gives

$$\begin{aligned} &\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - \frac{1}{4}\overline{\mathcal{L}}^2 \\ \leq\ &\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}'_n) - \frac{1}{4}\overline{\mathcal{L}}'^2_n + \frac{1}{2}\kappa(c_0 + \sum_{i=0}^{n} c_i) + 4r_0 \log r_0 + 2r_0 \log 3. \end{aligned}$$

By Lemma 4.6,

$$c_0 + \sum_{i=0}^{n} c_i \leq \frac{1}{d_0}\overline{\mathcal{L}}^2.$$

It follows that

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) \leq \frac{1}{4}\overline{\mathcal{L}}^2 + \frac{1}{2d^\circ}\overline{\mathcal{L}}^2 + 4r_0 \log r_0 + 2r_0 \log 3.$$

It gives

$$\widehat{h}^0(\overline{\mathcal{L}}) \leq (\frac{1}{4} + \frac{1}{2d^\circ})\overline{\mathcal{L}}^2 + 4r_0 \log r_0 + 3r_0 \log 3.$$

It remains to treat the case that $\mathcal{L}_{n,K}$ is trivial. As in the proof of Theorem B in §4.3, we go back to $n-1$. Proposition 4.5 gives

$$\begin{aligned}
&\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - \frac{1}{4}\overline{\mathcal{L}}^2 \\
\leq\ & \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}'_{n-1}) - \frac{1}{4}\overline{\mathcal{L}}'^2_{n-1} + \frac{1}{2}\kappa(c_0 + \sum_{i=0}^{n-1} c_i) + 4r_0\log r_0 + 2r_0 \log 3.
\end{aligned}$$

As in §4.3, $\deg(\mathcal{L}_{n-1,K}) > 1$ since $\mathcal{L}_{n-1,K}$ is base-point-free by construction, and $\widehat{H}^0_{\mathrm{sef}}(\overline{\mathcal{L}}'_{n-1})$ has $\mathbb{Z}$-rank at most κ. Apply Proposition 4.1. We have

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}'_{n-1}) \leq \frac{1}{2}\overline{\mathcal{L}}'^2_{n-1} + \kappa\log 3.$$

This is actually a special case of Theorem B, but the error term here is better. Thus

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}'_{n-1}) - \frac{1}{4}\overline{\mathcal{L}}'^2_{n-1} \leq \frac{1}{2}\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}'_{n-1}) + \frac{1}{2}\kappa\log 3.$$

By Proposition 2.1,

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}'_{n-1}) = \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}_n) \leq \widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}'_n) + \kappa c_n + \kappa\log 3 = \kappa c_n + \kappa \log 3.$$

It follows that

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}'_{n-1}) - \frac{1}{4}\overline{\mathcal{L}}'^2_{n-1} \leq \frac{1}{2}\kappa c_n + \kappa\log 3.$$

Therefore, the bound on $\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}})$ becomes

$$\begin{aligned}
&\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) - \frac{1}{4}\overline{\mathcal{L}}^2 \\
\leq\ & \frac{1}{2}\kappa(c_0 + \sum_{i=0}^{n} c_i) + 4r_0\log r_0 + 2r_0 \log 3 + \kappa\log 3.
\end{aligned}$$

By Lemma 4.6,

$$c_0 + \sum_{i=0}^{n} c_i \leq \frac{1}{d_0}\overline{\mathcal{L}}^2.$$

It follows that

$$\widehat{h}^0_{\mathrm{sef}}(\overline{\mathcal{L}}) \leq \frac{1}{4}\overline{\mathcal{L}}^2 + \frac{1}{2d^\circ}\overline{\mathcal{L}}^2 + 4r_0\log r_0 + 2r_0\log 3 + \kappa\log 3.$$

Thus

$$\widehat{h}^0(\overline{\mathcal{L}}) \leq (\frac{1}{4} + \frac{1}{2d^\circ})\overline{\mathcal{L}}^2 + 4r_0\log r_0 + 4r_0\log 3.$$

It finishes the proof.

5.3. **Application to the canonical bundle.** Theorem D is a special case of Theorem C by Faltings's result that $\overline{\omega}_X$ is nef on X. Corollary E is an easy consequence of Theorem D and Faltings's arithmetic Noether formula. Here we briefly track the "error terms" in Corollary E.

Recall that $\overline{\omega}_X = (\omega_X, \|\cdot\|_{\mathrm{Ar}})$ is endowed with the Arakelov metric $\|\cdot\|_{\mathrm{Ar}}$. It induces on $H^0(X,\omega_X)_{\mathbb{C}}$ the supremum norm

$$\|\alpha\|_{\sup} = \sup_{z\in M} \|\alpha(z)\|_{\mathrm{Ar}}, \quad \alpha \in H^0(X(\mathbb{C}), \Omega^1_{X(\mathbb{C})}).$$

Consider

$$\chi_{\sup}(\overline{\omega}_X) = \log \frac{\mathrm{vol}(B_{\sup}(\omega_X))}{\mathrm{vol}(H^0(X,\omega_X)_{\mathbb{R}}/H^0(X,\omega_X))}.$$

Here $B_{\sup}(\omega_X)$ is the unit ball in $H^0(X,\omega_X)_{\mathbb{R}}$ associated to $\|\cdot\|_{\sup}$.

By Minkowski' theorem,

$$\widehat{h}^0(\overline{\omega}_X) \geq \chi_{\sup}(\overline{\omega}_X) - r\log 2.$$

Here

$$r = g[K:\mathbb{Q}] \leq (2g-2)[K:\mathbb{Q}] = d.$$

Thus Theorem D implies

$$\chi_{\sup}(\overline{\omega}_X) \leq \frac{g}{4(g-1)}\overline{\omega}_X^2 + 4d\log(3d) + r\log 2.$$

Now we compare $\chi_{\sup}(\overline{\omega}_X)$ with the Faltings height $\chi_{\mathrm{Fal}}(\omega_X)$. The latter is the arithmetic degree of the hermitian O_K-module $H^0(X,\omega_X)$ endowed with the natural metric

$$\|\alpha\|^2_{\mathrm{nat}} = \frac{i}{2}\int_{X(\mathbb{C})} \alpha\wedge\overline{\alpha}, \quad \alpha \in H^0(X(\mathbb{C}), \Omega^1_{X(\mathbb{C})}).$$

By definition, it is easy to obtain

$$\chi_{\mathrm{Fal}}(\omega_X) = \log \frac{\mathrm{vol}(B_{\mathrm{nat}}(\omega_X))}{\mathrm{vol}(H^0(X,\omega_X)_{\mathbb{R}}/H^0(X,\omega_X))} - \chi(O_K^g).$$

Here $B_{\mathrm{nat}}(\omega_X)$ is the unit ball in $H^0(X,\omega_X)_{\mathbb{R}}$ associated to $\|\cdot\|_{\mathrm{nat}}$, and

$$\chi(O_K^g) = r_1\log V(g) + r_2\log V(2g) - \frac{1}{2}g\log|D_K|.$$

Here D_K is the absolute discriminant of K, r_1 (resp. $2r_2$) is the number of real (resp. complex) embeddings of K in $\mathbb{C}$, and $V(n) = \pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}+1)$ is the volume of the standard unit ball in the Euclidean space $\mathbb{R}^n$.

It follows that

$$\chi_{\mathrm{Fal}}(\omega_X) - \chi_{\sup}(\overline{\omega}_X) = \log\frac{\mathrm{vol}(B_{\mathrm{nat}}(\omega_X))}{\mathrm{vol}(B_{\sup}(\omega_X))} - \chi(O_K^g) = \gamma_{X_\infty} - \chi(O_K^g).$$

The second equality follows from the definition of

$$\gamma_{X_\infty} = \sum_{\sigma: K \hookrightarrow \mathbb{C}} \gamma_{X_\sigma}.$$

Therefore,

$$\chi_{\mathrm{Fal}}(\omega_X) \leq \frac{g}{4(g-1)}\overline{\omega}_X^2 + \gamma_{X_\infty} - \chi(O_K^g) + 4d\log(3d) + r\log 2.$$

Stirling's approximation gives

$$\chi(O_K^g) > \frac{1}{2}r\log(2\pi) - \frac{1}{2}r\log r - \frac{1}{2}g\log|D_K|.$$

Thus the inequality implies

$$\chi_{\mathrm{Fal}}(\omega_X) \leq \frac{g}{4(g-1)}\overline{\omega}_X^2 + \gamma_{X_\infty} + \frac{1}{2}g\log|D_K| + \frac{9}{2}d\log d + 4d\log 3.$$

The coefficient before $\overline{\omega}_X^2$ is exactly the same as the inequality of Bost [Bo] concerning the height and the slope in this case. In the following, denote

$$C' = \frac{1}{2}g\log|D_K| + \frac{9}{2}d\log d + 4d\log 3.$$

Combine with Faltings's arithmetic Noether formula

$$\chi_{\mathrm{Fal}}(\omega_X) = \frac{1}{12}(\overline{\omega}_X^2 + \delta_X) - \frac{1}{3}r\log(2\pi).$$

We have

$$\begin{aligned} & \left(2 + \frac{3}{g-1}\right)\overline{\omega}_X^2 \\ \geq\ & \delta_X - 12\gamma_{X_\infty} - 12C' - 4r\log(2\pi) \\ \geq\ & \delta_X - 12\gamma_{X_\infty} - 6g\log|D_K| - 54d\log d - 61d. \end{aligned}$$

Similarly,

$$\begin{aligned} & \left(8 + \frac{4}{g}\right)\chi_{\mathrm{Fal}}(\overline{\omega}_X) \\ \geq\ & \delta_X - \frac{4(g-1)}{g}\gamma_{X_\infty} - \frac{4(g-1)}{g}C' - 4r\log(2\pi) \\ \geq\ & \delta_X - \frac{4(g-1)}{g}\gamma_{X_\infty} - 4C' - 4r\log(2\pi) \\ \geq\ & \delta_X - \frac{4(g-1)}{g}\gamma_{X_\infty} - 2g\log|D_K| - 18d\log d - 25d. \end{aligned}$$

It completes the inequalities.

6. Extension to $\mathbb{R}$-divisors and adelic line bundles

In this section, we extend the main results to arithmetic $\mathbb{R}$-divisors and adelic line bundles. The proofs are similar to the integral case. We will focus on the terminology, and only sketch the proofs.

6.1. Arithmetic $\mathbb{R}$-divisors.

We first recall the definition of arithmetic $\mathbb{R}$-divisors following Moriwaki [Mo4, Mo6]. To be compatible with the setting of this article, we only work on regular arithmetic varieties.

Let K be a number field, and X be a regular arithmetic surface over O_K. By an *arithmetic* $\mathbb{Z}$-*divisor* (resp. *arithmetic* $\mathbb{R}$-*divisor*) (of C^0-type) on X, we mean a pair $\overline{D} = (D, g)$ consisting of a finite formal sum

$$D = \sum_i a_i C_i$$

of integral subschemes C_i of codimension one on X with coefficients $a_i \in \mathbb{Z}$ (resp. $a_i \in \mathbb{R}$), and a *continuous function*

$$g_D : X(\mathbb{C}) - \mathrm{supp}(D(\mathbb{C})) \to \mathbb{R}$$

with logarithmic singularity along the divisor $D(\mathbb{C})$ and invariant under the complex conjugation on $X(\mathbb{C}) = \coprod_{\sigma: K \hookrightarrow \mathbb{C}} X_\sigma(\mathbb{C})$. Here $\mathrm{supp}(D(\mathbb{C}))$ denotes the support of the divisor $D(\mathbb{C})$ on $X(\mathbb{C})$.

Denote by $\widehat{\mathrm{Div}}(X)$ the additive group of arithmetic $\mathbb{Z}$-divisors on X, and by $\widehat{\mathrm{Div}}(X)_\mathbb{R}$ the $\mathbb{R}$-vector space of arithmetic $\mathbb{R}$-divisors on X. Note that $\widehat{\mathrm{Div}}(X)_\mathbb{R} \neq \widehat{\mathrm{Div}}(X) \otimes_\mathbb{Z} \mathbb{R}$, but

$$\widehat{\mathrm{Div}}(X)_\mathbb{R} = \frac{\widehat{\mathrm{Div}}(X) \otimes_\mathbb{Z} \mathbb{R}}{\{\sum_{i=1}^k (0, \phi_i) \otimes a_i : \phi_i \in C^0(X),\ a_i \in \mathbb{R},\ \sum_i a_i \phi_i = 0\}}.$$

Here $C^0(X)$ denotes the space of real-valued continuous functions on $X(\mathbb{C})$, invariant under the complex conjugation.

The arithmetic $\mathbb{R}$-divisor $\overline{D} = (D, g_D)$ is called *effective* (resp. *strictly effective*) if all coefficients $a_i \geq 0$ and the Green function $g_D \geq 0$ (resp. $g_D > 0$) on $X(\mathbb{C}) - |D(\mathbb{C})|$. Effectivity defines a *partial order* "$\geq$" on $\widehat{\mathrm{Div}}(X)_\mathbb{R}$. Namely, the notation $D_1 \geq D_2$ for $D_1, D_2 \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$ means that $D_1 - D_2$ is effective. Similarly, strict effectivity defines a *partial order* "$\geq_{\mathrm{sef}}$" on $\widehat{\mathrm{Div}}(X)_\mathbb{R}$.

For any $\overline{D} \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$, denote

$$\widehat{H}^0(\overline{D}) := \{\phi \in K(X)^\times : \widehat{\mathrm{div}}(\phi) + \overline{D} \geq 0\} \cup \{0\}.$$

Here $K(X)$ denotes the function field of X, and the *principal divisor*

$$\widehat{\mathrm{div}}(\phi) := (\mathrm{div}(\phi), -\log|\phi|^2).$$

We will see that $\widehat{H}^0(\overline{D})$ is a finite subset of $K(X)$. Define

$$\widehat{h}^0(\overline{D}) := \log \#\widehat{H}^0(\overline{D})$$

and

$$\widehat{\mathrm{vol}}(\overline{D}) := \limsup_{n\to\infty} \frac{2}{n^2}\widehat{h}^0(n\overline{D}).$$

We can still associate a normed $\mathbb{Z}$-module $\overline{M} = (M, \|\cdot\|_{D,\sup})$ to $\overline{D}$. In fact, we set

$$M = H^0(D) := \{\phi \in K(X)^\times : \mathrm{div}(\phi) + D \geq 0\} \cup \{0\}.$$

It is a finite $\mathbb{Z}$-module since we simply have

$$H^0(D) = H^0(\lfloor D \rfloor),$$

where the integral part

$$\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor C_i.$$

The norm on $M_{\mathbb{C}} = H^0(D_{\mathbb{C}}) = H^0(\lfloor D_{\mathbb{C}} \rfloor)$ is defined as

$$\|f\|_{\overline{D},\sup} = \sup_{z\in X(\mathbb{C})} |f(z)|e^{-\frac{1}{2}g_D(z)}.$$

Then we simply have $\widehat{H}^0(\overline{D}) = \widehat{H}^0(\overline{M})$. In particular, $\widehat{H}^0(\overline{D})$ is finite.

An arithmetic $\mathbb{Z}$-divisor is also called an *arithmetic divisor*. For an arithmetic divisor $\overline{D} = (D, g_D)$, one introduces a hermitian line bundle $\mathcal{O}(\overline{D}) = (\mathcal{O}(D), \|\cdot\|_{\overline{D}})$ consisting of the line bundle $\mathcal{O}(D)$ on X and the metric

$$\|f\|_{\overline{D}} = |f|e^{-\frac{1}{2}g_D},$$

where f is any local section of $\mathcal{O}(D)$. The correspondence $\overline{D} \mapsto \mathcal{O}(\overline{D})$ keeps $\widehat{H}^0, \widehat{h}^0, \widehat{\mathrm{vol}}$.

By Chen [Ch], the "limsup" in the definition of $\widehat{\mathrm{vol}}(\overline{D})$ is a limit if D is an arithmetic $\mathbb{Z}$-divisor. Moriwaki [Mo3, Mo4] extends Chen's result to all arithmetic $\mathbb{R}$-divisors, and proves that $\widehat{\mathrm{vol}} : \widehat{\mathrm{Div}}(X)_{\mathbb{R}} \to \mathbb{R}$ defines a continuous function on $\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, homogeneous of degree 2.

Now Theorem A is extended to the following result.

Theorem A1. *Let X be a regular and geometrically connected arithmetic surface of genus g over O_K. Let $\overline{D}$ be an arithmetic $\mathbb{R}$-divisor on X. Denote $d^\circ = \deg(D_K)$, and denote by r' the O_K-rank of the O_K-submodule of $H^0(D)$ generated by $\widehat{H}^0(\overline{D})$. Assume that $r' \geq 2$.*

(1) *If* $g > 0$, *then*

$$\widehat{h}^0(\overline{D}) \leq \frac{1}{2}\ \widehat{\mathrm{vol}}(\overline{D}) + 4d\log(3d).$$

Here $d = d^\circ[K:\mathbb{Q}]$.

(2) *If* $g = 0$, *then*

$$\widehat{h}^0(\overline{D}) \leq (\frac{1}{2} + \frac{1}{2(r'-1)})\ \widehat{\mathrm{vol}}(\overline{D}) + 4r\log(3r).$$

Here $r = (d^\circ + 1)[K:\mathbb{Q}]$.

An arithmetic $\mathbb{R}$-divisor $\overline{D} = (D, g_D)$ on X is called *nef* if it satisfies the following conditions:

- $\widehat{\deg}(\overline{D}|_C) \geq 0$ for any integral subscheme C of dimension one in X.
- The Green function g_D is pluri-subharmonic, i.e., the curvature current
$$\omega_{\overline{D}} := -\frac{\partial\overline{\partial}}{2\pi i} g_D + \delta_{D(\mathbb{C})}$$
on $X(\mathbb{C})$ is positive.

The basic properties we listed for nef hermitian line bundles also hold for nef arithmetic $\mathbb{R}$-divisors. In particular, we still have the arithmetic Hilbert–Samuel formula

$$\widehat{h}^0(n\overline{D}) = \frac{1}{2} n^2 \overline{D}^2 + o(n^2), \quad n \to \infty.$$

Theorem B is extended to the following result.

Theorem B1. *Let* X *be a regular and geometrically connected arithmetic surface of genus* g *over* O_K. *Let* $\overline{D}$ *be a nef arithmetic* $\mathbb{R}$*-divisor on* X *with* $d^\circ = \deg(D_K)$.

(1) *If* $g > 0$ *and* $d^\circ \geq 2$, *then*

$$\widehat{h}^0(\overline{D}) \leq \frac{1}{2}\overline{D}^2 + 4d\log(3d).$$

Here $d = d^\circ[K:\mathbb{Q}]$.

(2) *If* $g = 0$ *and* $d^\circ \geq 1$, *then*

$$\widehat{h}^0(\overline{D}) \leq (\frac{1}{2} + \frac{1+\epsilon}{2d^\circ})\overline{D}^2 + 4r\log(3r).$$

Here $r = (d^\circ + 1)[K:\mathbb{Q}]$. *The number* $\epsilon = 0$ *if* $d^\circ \in \mathbb{Z}$; *otherwise,* $\epsilon = 1$.

We say an $\mathbb{R}$-divisor E on a projective and smooth curve Y over a field is *special* if its integer part $\lfloor E \rfloor$ is special.

Theorem C1. *Let X be a regular and geometrically connected arithmetic surface of genus $g > 1$ over O_K. Let $\overline{D}$ be a nef arithmetic $\mathbb{R}$-divisor on X with $d^\circ = \deg(D_K) \geq 2$. Assume that D_K is a special divisor on the generic fiber X_K. Then*

$$\widehat{h}^0(\overline{D}) \leq (\frac{1}{4} + \frac{2+\varepsilon}{4d^\circ})\overline{D}^2 + 4d\log(3d).$$

Here $d = d^\circ[K:\mathbb{Q}]$. If X_K is non-hyperelliptic, the number $\varepsilon = 0$. If X_K is hyperelliptic, $\varepsilon = 0$ if $d^\circ \in 2\mathbb{Z}$, $\varepsilon = 1$ if $d^\circ \in 2\mathbb{Z}+1$, and $\varepsilon = 2$ if $d^\circ \notin \mathbb{Z}$.

6.2. **Adelic line bundles.** Here we generalize the theorems to adelic line bundles introduced by S. Zhang [Zh3]. We first recall the definition of adelic line bundles briefly.

Let Y be a projective variety over a number field K, and let L be a line bundle on Y. A K_v*-metrics* $\|\cdot\|_v$ on L is a collection of $\overline{K}_v$-metrics on $L(x)$ over $x \in Y(\overline{K}_v)$, which is continuous and invariant under the action of $\mathrm{Gal}(\overline{K}_v/K_v)$ when varying x. An *adelic metric* on L is a *coherent* collection $\{\|\cdot\|_v\}_v$ of bounded K_v*-metrics* $\|\cdot\|_v$ on L over all places v of K. That the collection $\{\|\cdot\|_v\}_v$ is *coherent* means that, there exist a finite set S of non-archimedean places of K and a (projective and flat) integral model $(\mathcal{Y}, \mathcal{L})$ of (Y, L) over O_K, such that the K_v-norm $\|\cdot\|_v$ is induced by $(\mathcal{Y}_{O_{K_v}}, \mathcal{L}_{O_{K_v}})$ for all non-archimedean places $v \notin S$.

In the above situation, we write $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ and call it an *adelic line bundle on* X. We further call L *the generic fiber of* $\overline{L}$. An adelic line bundle is called *nef* (or *semipositive* by S. Zhang) if its adelic metric is a uniform limit of metrics induced by integer models $(\mathcal{Y}_m, \overline{\mathcal{L}}_m)$ with $\overline{\mathcal{L}}_m \in \widehat{\mathrm{Pic}}(\mathcal{Y}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$ nef on $\mathcal{Y}_m$.

Let $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ be an adelic line bundle on Y. Define

$$\widehat{H}^0(\overline{L}) := \{s \in H^0(L) : \|s\|_{v,\sup} \leq 1,\ \forall v\},$$

where the supremum norm $\|s\|_{v,\sup} = \sup_{z \in X(\overline{K}_v)} \|s(z)\|_v$. Define

$$\widehat{h}^0(\overline{L}) := \log \#\widehat{H}^0(\overline{L}),$$

and

$$\widehat{\mathrm{vol}}(\overline{L}) := \limsup_{n\to\infty} \frac{2}{n^2}\widehat{h}^0(n\overline{L}).$$

By approximation, Chen's result on $\widehat{\mathrm{vol}}$ holds for adelic line bundles, and the arithmetic Hilbert–Samuel formula holds for nef adelic line bundles. Now the theorems can be generalized as follows.

Theorem A2. *Let Y be a smooth and geometrically connected curve of genus g over a number field K. Let $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ be an adelic line bundle on Y. Denote $d^\circ = \deg(L)$, and denote by r' the dimension of the K-subspace of $H^0(L)$ generated by $\widehat{H}^0(\overline{L})$. Assume that $r' \geq 2$.*

(1) *If $g > 0$, then*

$$\widehat{h}^0(\overline{L}) \leq \frac{1}{2}\ \widehat{\mathrm{vol}}(\overline{L}) + 4d\log(3d).$$

Here $d = d^\circ[K:\mathbb{Q}]$.

(2) *If $g = 0$, then*

$$\widehat{h}^0(\overline{L}) \leq (\frac{1}{2} + \frac{1}{2(r'-1)})\ \widehat{\mathrm{vol}}(\overline{L}) + 4r\log(3r).$$

Here $r = (d^\circ + 1)[K:\mathbb{Q}]$.

Theorem B2. *Let Y be a smooth and geometrically connected curve of genus g over a number field K. Let $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ be a nef adelic line bundle on Y with $d^\circ = \deg(L)$.*

(1) *If $g > 0$ and $d^\circ \geq 2$, then*

$$\widehat{h}^0(\overline{L}) \leq \frac{1}{2}\overline{L}^2 + 4d\log(3d).$$

Here $d = d^\circ[K:\mathbb{Q}]$.

(2) *If $g = 0$ and $d^\circ \geq 1$, then*

$$\widehat{h}^0(\overline{L}) \leq (\frac{1}{2} + \frac{1}{2d^\circ})\overline{L}^2 + 4r\log(3r).$$

Here $r = (d^\circ + 1)[K:\mathbb{Q}]$.

Theorem C2. *Let Y be a smooth and geometrically connected curve of genus $g > 1$ over a number field K. Let $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ be a nef adelic line bundle on Y with $d^\circ = \deg(L) \geq 2$. Assume that L is a special divisor on Y. Then*

$$\widehat{h}^0(\overline{L}) \leq (\frac{1}{4} + \frac{2+\varepsilon}{4d^\circ})\overline{L}^2 + 4d\log(3d).$$

Here $d = d^\circ[K:\mathbb{Q}]$. The number $\varepsilon = 1$ if Y is hyperelliptic and d° is odd; otherwise, $\varepsilon = 0$.

As a counterpart of Theorem D, we can apply Theorem C2 to the admissible canonical bundle ω_a on Y introduced by S. Zhang [Zh2].

It is easy to deduce Theorems A2, B2, C2 from Theorems A1, B1, C1. In the following, we take Theorem A2 as an example.

Let $(Y, \overline{L})$ be as in the Theorem. The adelic metric $\{\|\cdot\|_v\}_v$ of $\overline{L}$ is a uniform limit of a sequence of adelic metrics $\{\|\cdot\|_{m,v}\}_v$ on L, where for each $m \geq 1$, the adelic metric $\{\|\cdot\|_{m,v}\}_v$ is induced by an integral model $(\mathcal{Y}_m, \overline{\mathcal{L}}_m)$ of (Y, L) over O_K. Here we allow $\overline{\mathcal{L}}_m \in \widehat{\mathrm{Pic}}(\mathcal{Y}) \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e., $\overline{\mathcal{L}}_m$ a hermitian $\mathbb{Q}$-line bundle on $\mathcal{Y}_m$ with $\mathcal{L}_{m,K} = L$. We can assume $\mathcal{Y}_m$ to be regular.

Both $\widehat{H}^0(\overline{\mathcal{L}}_m)$ and $\widehat{H}^0(\overline{L})$ are subsets of $H^0(Y, L)$. Note that $\widehat{H}^0(\overline{L})$ is a finite set. We have $\widehat{H}^0(\overline{\mathcal{L}}_m) = \widehat{H}^0(\overline{L})$ for sufficiently large m. By continuity, it is not hard to see that $\widehat{\mathrm{vol}}(\overline{\mathcal{L}}_m)$ converges to $\widehat{\mathrm{vol}}(\overline{L})$. Theorem A1 holds for $\overline{\mathcal{L}}_m$ by the correspondence between hermitian $\mathbb{Q}$-line bundles and arithmetic $\mathbb{Q}$-divisors. Taking limit as $m \to \infty$, we get the result for $\overline{L}$.

6.3. **Proofs for $\mathbb{R}$-divisors.** Now we sketch the proofs for Theorem A1, B1, C1. We only focus on the parts that are different from the integral case.

Let $\overline{D} = (D, g_D)$ be an arithmetic $\mathbb{R}$-divisor on X. Recall that

$$\widehat{h}^0(\overline{D}) = \log \# \widehat{H}^0(\overline{D}),$$

where

$$\widehat{H}^0(\overline{D}) = \{\phi \in K(X)^{\times} : \widehat{\mathrm{div}}(\phi) + \overline{D} \geq 0\} \cup \{0\}.$$

We also introduce

$$\widehat{h}^0_{\mathrm{sef}}(\overline{D}) := \log \# \widehat{H}^0_{\mathrm{sef}}(\overline{D}),$$

where

$$\widehat{H}^0_{\mathrm{sef}}(\overline{D}) := \{\phi \in K(X)^{\times} : \widehat{\mathrm{div}}(\phi) + \overline{D} \geq_{\mathrm{sef}} 0\} \cup \{0\}.$$

Denote

$$|\overline{D}| = \{\widehat{\mathrm{div}}(\phi) + \overline{D} : \ \phi \in \widehat{H}^0(\overline{D}), \ \phi \neq 0\},$$

$$|\overline{D}|_{\mathrm{sef}} = \{\widehat{\mathrm{div}}(\phi) + \overline{D} : \ \phi \in \widehat{H}^0_{\mathrm{sef}}(\overline{D}), \ \phi \neq 0\}.$$

For $S = |\overline{D}|$ or $|\overline{D}|_{\mathrm{sef}}$, the *fixed part* E_S *of* S is an effective $\mathbb{R}$-divisor on X defined by

$$E_S = \sum_{C \subset X} (\min_{\overline{D}' \in S} \mathrm{ord}_C \overline{D}') \cdot C.$$

Here the summation is over all prime divisors of X. Note that $D - E_S$ is a $\mathbb{Z}$-divisor on X. It follows that $E_S \neq 0$ if $\overline{D}$ is not an arithmetic $\mathbb{Z}$-divisor.

To translate the proofs of the theorem, we need the following $\mathbb{R}$-version of Theorem 3.2. One can also formulate the $\mathbb{R}$-version of Theorem 3.1 in the same manner.

Theorem 6.1. *Let X be a regular arithmetic surface, and $\overline{D}$ be an arithmetic $\mathbb{R}$-divisor on X with $\widehat{h}^0_{\mathrm{sef}}(\overline{D}) \neq 0$. Then there is a decomposition*

$$\overline{D} = \overline{E} + \overline{D}_1$$

where $\overline{E}$ is an effective arithmetic $\mathbb{R}$-divisor on X, and $\overline{D}_1$ is a nef arithmetic $\mathbb{Z}$-divisor on X satisfying the following conditions:

- *The finite part E of $\overline{E}$ is the fixed part of $|\overline{D}|_{\mathrm{sef}}$ in X.*
- *The natural inclusion*

$$\widehat{H}^0_{\mathrm{sef}}(\overline{D}_1) \longrightarrow \widehat{H}^0_{\mathrm{sef}}(\overline{D})$$

as subsets of $K(X)$ is bijective. Furthermore, the bijection keeps the supremum norms, i.e.,

$$\|f\|_{\overline{D}_1,\sup} = \|f\|_{\overline{D},\sup}, \quad \forall\ f \in \widehat{H}^0_{\mathrm{sef}}(\overline{D}_1).$$

Proof. The proof is similar to that of Theorem 3.2. The statement already defines E to be the fixed part of $|\overline{D}|_{\mathrm{sef}}$. Set $\overline{D} = (E, g_E)$ with

$$g_E = \min_{\overline{D}' \in |\overline{D}|_{\mathrm{sef}}} (g_{D'} - \inf_{X(\mathbb{C})} g_{D'}).$$

One checks that g_E is a Green function for E. Set $\overline{D}_1 = \overline{D} - \overline{E}$. Then we can prove the theorem as in the proof of Theorem 3.2. □

It is straightforward to define the heights and the absolute minimum associated to an arithmetic $\mathbb{R}$-divisor. Thus one can also formulate Proposition 4.3 and Theorem 4.4.

Recall that we have associated the normed $\mathbb{Z}$-module $\overline{M} = (H^0(D), \|\cdot\|_{\overline{D},\sup})$ to $\overline{D}$. Under the relation, we simply have

$$\widehat{H}^0(\overline{D}) = \widehat{H}^0(\overline{M}), \quad \widehat{H}^0_{\mathrm{sef}}(\overline{D}) = \widehat{H}^0_{\mathrm{sef}}(\overline{M}).$$

It follows that there is no problem to apply the results of normed $\mathbb{Z}$-modules to the current setting. Then we obtain the proofs of Theorems A1, B1, C1. In the following, we mention a few places different from the integral case.

It is worth nothing that, even if $\overline{D}$ is an arithmetic $\mathbb{R}$-divisor, the resulting $\overline{D}_1$ in Theorem 6.1 is an arithmetic $\mathbb{Z}$-divisor. In particular, in the proof of Theorem A1, we only need Theorem B, the integral version of Theorem B1.

Similarly, in the counterpart of Theorem 4.4, start with an arithmetic $\mathbb{R}$-divisor $\overline{D}_0 = \overline{D}$, we get a series of arithmetic $\mathbb{Z}$-divisors $\overline{D}_1, \cdots, \overline{D}_n$.

In the proof of Theorem A1(2), one needs to use the bound

$$\deg(D_{i-1,K}) - \deg(D_{i,K}) \geq 1, \quad i = 2, \cdots, n.$$

If $\deg(D_K) \in \mathbb{Z}$, the bound also holds for $i = 1$, and we get the same result as in Theorem A(2). If $\deg(D_K) \notin \mathbb{Z}$, we can only use the weaker bound $\deg(D_{0,K}) - \deg(D_{1,K}) \geq 0$. It gives the case $\epsilon = 1$.

As for Theorem C1, Clifford's theorem extends to $\mathbb{R}$-divisors on curves by reducing to their integral parts. In the hyperelliptic case, the occurrence of $\epsilon = 0, 1, 2$ still comes from the degrees. Since $\deg(D_{i,K})$ is even for $i = 1, \cdots, n$ as in Lemma 5.2, we have

$$\deg(D_{i-1,K}) - \deg(D_{i,K}) \geq 2, \quad i = 2, \cdots, n.$$

If $\deg(D_K) \in 2\mathbb{Z}$, the bound also holds for $i = 1$, and we get $\epsilon = 0$. If $\deg(D_K) \in 2\mathbb{Z} + 1$, we use $\deg(D_{0,K}) - \deg(D_{1,K}) \geq 1$, which gives $\epsilon = 1$. If $\deg(D_K) \notin \mathbb{Z}$, we can only use $\deg(D_{0,K}) - \deg(D_{1,K}) \geq 0$, which gives $\epsilon = 2$.

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