Erratum: Gross–Zagier Formula On Shimura Curves

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Erratum

1. (p. 3, line 23) Change

   "It is a regular scheme over F, locally noetherian but not of finite type"

   to

   "If $\#\Sigma > 1$, it is a regular scheme over F, locally noetherian but not of finite type."

2. (p. 8, line 10) Change the equation

$$\Sigma(A,\chi) = \left\{ \text{places } v \text{ of } F: \ \epsilon(\frac{1}{2}, \pi_{A,v}, \chi_v) \neq \chi_v(-1)\eta_v(-1) \right\}$$

   to

$$\Sigma(A,\chi) = \left\{ \text{places } v \text{ of } F: \ \epsilon(\frac{1}{2}, \pi_{A,v}, \chi_v) \neq \chi_v(-1) \right\}.$$

   See also the correction for (p. 10, Thm 1.3) below.

3. (p. 8, line -8) Change

   "for all places v"

   to

   "for all non-archimedean places v."

4. (p. 8, line -15) Change

   "Fix Haar measures dt_v on $E_v^\times/F_v^\times$ such that the product measure over all v gives the Tamagawa measure on $E_{\mathbb{A}}^\times/\mathbb{A}^\times$"

   to

   "Fix Haar measures dt_v on $E_v^\times/F_v^\times$ such that the product measure over all v gives $\mathrm{vol}(E^\times\backslash E_{\mathbb{A}}^\times/\mathbb{A}^\times) = 2L(1,\eta)$."

   See also (p. 23, line -2).

5. (p. 10, Thm 1.3) Change the equation

$$\epsilon(\frac{1}{2}, \pi, \chi) = \chi(-1)\eta(-1)\epsilon(B)$$

   to

$$\epsilon(\frac{1}{2}, \pi, \chi) = \chi(-1)\epsilon(B).$$

In the literature, there are two conventional ways to define the root number $\epsilon(\frac{1}{2}, \pi, \chi)$, by viewing the L-function as either the base change L-function over E (twisted by χ) or the Rankin–Selberg L-function over F. The ratio of these two definitions are $\eta(-1)$. This paper takes the first convention, as written right before the theorem.

6. (p. 13, line 3) The formula

$$\theta(g, (h_1, h_2), \Phi) = \sum_{u \in F^\times} r(g, (h_1, h_2))\Phi(x, u)$$

should be

$$\theta(g, (h_1, h_2), \Phi) = \sum_{u \in F^\times} \sum_{x \in B} r(g, (h_1, h_2))\Phi(x, u).$$

7. (p. 13, line -9) In the summation, $\Phi(x)$ should be $\Phi(x_1, u)$.

8. (p. 15, line 4) In the formula, $\Phi(x, aq(x)^{-1})$ should be $\phi(x, aq(x)^{-1})$.

9. (p. 17, line -6) "$v$nonsplit" should be "v nonsplit". This type of typo is very common in the book due to a misoperation on the latex file. Namely, in many places there should be a space before the word "nonsplit". For example, in (p. 22, line 6), "ornonsplit" should be "or nonsplit"; in (p. 22, line 15), "isnonsplit" should be "is nonsplit".

10. (p. 23, line -2) Remove the sentence

    "All of them are Tamagawa measures."

    Note that the global measure on E^1 is not the Tamagawa measure due to the extra normalizing factor $L(1, \eta)$.

11. (p. 29, line -11) Change

$$\{(x_i, x_j)\}_{1 \leq i,j \leq \dim V}$$

to

$$\{\langle x_i, x_j \rangle\}_{1 \leq i,j \leq \dim V}.$$

12. (p. 31, line -2) Change

$$\mathcal{S}(V_v \times F_v)$$

to

$$\mathcal{S}(V_v \times F_v^\times).$$

13. (p. 33, line -9) Remove the sentence

    "The Tamagawa number of SO(V) in the above cases are respectively $2L(1, \eta), 2, 2$."

    See also (p. 23, line -2).

14. (p. 40, line -5) Change

$$\int_{Z(F_v)\backslash T(F_v)} \chi(t)dt$$

to

$$\int_{Z(F_v)\backslash T(F_v)} \chi_v(t)dt.$$

15. (p. 43, line -9) Change

    "Here the factor $4L(1, \eta)^2$ comes from the Tamagawa measure" to

"Here the factor $4L(1,\eta)^2$ comes from the measure."

See also (p. 23, line -2).

16. (p. 59, line 9) See (p. 3, line 23).
17. (p. 63, line 3) Change "$f(UxU) = x$" to "$f(UxU) = f(x)$".
18. (p. 64, line 7) Change the second "$H^{1,0}(X_{U,\tau})$" to "$H^{0,1}(X_{U,\tau})$".
19. (p. 68, line 9) Change

$$\mathrm{Pic}(X_U \times X_U) \longrightarrow \mathrm{Hom}^0(J_U, J_U^\vee)$$

to

$$\mathrm{Pic}(X_U \times X_U) \longrightarrow \mathrm{Hom}(J_U, J_U^\vee).$$

20. (p. 69, line -1) The definition should be

$$P_v(T, M) := \det_{M_\ell}(1 - \mathrm{Frob}(v)|V_\ell(A)^{I_v})$$

to

$$P_v(T, M) := \det_{M_\ell}(1 - \mathrm{Frob}(v)T|V_\ell(A)^{I_v}).$$

21. (p. 77, line -4) Change the equation

$$L_v(s, A, M) = L(s, \pi_{A,v})$$

to

$$L_v(s, A, M) = L\big(s - \frac{1}{2}, \pi_{A,v}\big).$$

22. (p. 78, line 18) The isomorphism should be

$$J_U \simeq \bigoplus_{\pi \in \mathcal{A}(\mathbb{B}^\times, \mathbb{Q})} \widetilde{\pi}^U \otimes_{\mathrm{End}(\pi)} A_\pi.$$

23. (p. 82, line -2) Change

$$P_\chi(f_1) = \int_{\mathrm{Gal}(\overline{E}/E)} f_1(P^\tau) \otimes_M \chi(\tau) d\tau$$

to

$$P_\chi(f_1) = \int_{\mathrm{Gal}(E^{\mathrm{ab}}/E)} f_1(P^\tau) \otimes_M \chi(\tau) d\tau.$$

24. (p. 84, line -6) Change the first "$H^{1,0}(X_{U,\tau})^\vee$" to "$H^{1,0}(X_{U,\tau})$".
25. (p. 88, line 11) Change

$$\mathrm{Hom}^0(J_U, J_U)$$

to

$$\mathrm{Hom}^0(J_U, J_U^\vee).$$

26. (p. 91, line 1) Change "$f_1 \in \pi_{A^\vee}^U$" to "$f_2 \in \pi_{A^\vee}^U$".
27. (p. 95, line -11) Change "The whole normalizing factor is always 2 if $F = \mathbb{Q}$" to "The whole normalizing factor is always 1/2 if $F = \mathbb{Q}$".

28. (p. 97, line 10) Change "tha" to "that".

29. (p. 98, line 6) Change

$$C_0^\infty(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})), \chi|_{\mathbb{A}^\times})$$

to

$$C_0^\infty(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}), \chi|_{\mathbb{A}^\times}).$$

30. (p. 105, line 15) Change

$$\Sigma = \{v : \epsilon(\frac{1}{2}, \pi_v, \chi_v) \neq \chi_v\eta_v(-1)\}$$

to

$$\Sigma = \{v : \epsilon(\frac{1}{2}, \pi_v, \chi_v) \neq \chi_v(-1)\}.$$

31. (p. 109, line 10) The goal of this item is not to state a mistake but to explain the expression

$$\theta(g,\phi)_K = \sum_{u\in\mu_K^2\backslash F^\times} r(g)\phi(0,u) + w_K \sum_{(x,u)\in\mu_K\backslash((V-\{0\})\times F^\times)} r(g)\phi(x,u).$$

Then it suffices to check

$$\sum_{u\in\mu_K^2\backslash F^\times}\sum_{x\in V'} r(g)\phi(x,u) = w_K \sum_{(x,u)\in\mu_K\backslash(V'\times F^\times)} r(g)\phi(x,u).$$

Here we denote $V' = V - \{0\}$.

Let A be a subset of V' such that the induced map $A \to \omega_K\backslash V'$ is bijective, and let B be a subset of $F^\times$ such that the induced map $B \to \mu_K^2\backslash F^\times$ is bijective. Here $\omega_K = \{\pm 1\}\cap K^\times$, and thus $w_K = |\omega_K|$. One checks that the natural map

$$A \times B \longrightarrow \mu_K\backslash(V'\times F^\times)$$

is bijective. As a consequence, we have

$$\sum_{u\in\mu_K^2\backslash F^\times}\sum_{x\in\omega_K\backslash V'} r(g)\phi(x,u) = \sum_{(x,u)\in\mu_K\backslash(V'\times F^\times)} r(g)\phi(x,u).$$

This gives the desired equality.

32. (p. 112, line 9) Change "different" to "the same".

33. (p. 114, line -2) Change

$$\mathbb{V}_f(b) = \{x \in \mathbb{V} : q(x) = b\}$$

to

$$\mathbb{V}_f(b) = \{x \in \mathbb{V}_f : q(x) = b\}.$$

34. (p. 115, line -9 and line -7; p. 116, line 4) Change

$$M'^{\circ}_{K'}$$

to

$$M'^{\circ}_{K^h}$$

35. (p. 116, line 16) Change

$$\ell(Z(g,\Phi)_U|_{M_K^\circ})$$

to

$$\ell(Z(g,\phi)_U|_{M_K^\circ}).$$

36. (p. 116, line 20) Change $\phi = \phi_f \times \phi_\infty$ to $\phi = \phi_f \otimes \phi_\infty$.

37. (p. 117, line 2) Change

$$M_K - M_K^\circ = (X_U \times \{\text{cusps}\}) \cup (X_U \times \{\text{cusps}\})$$

to

$$M_K - M_K^\circ = (X_U \times \{\text{cusps}\}) \cup (\{\text{cusps}\} \times X_U).$$

38. (p. 120, line -4) Change $\mathbb{B}_f^\times$ to $\mathbb{B}^\times$.

39. (p. 126, line -5) Change

$$\phi \in \overline{\mathcal{S}}(\mathbb{A} \times \mathbb{A}^\times)$$

to

$$\phi \in \overline{\mathcal{S}}(\mathbb{V} \times \mathbb{A}^\times).$$

40. (p. 137, line 8) Change

"The first identity is just the result for the Tamagawa number of"

to

"The first identity is just the result for the measure of"

See also (p. 23, line -2).

41. (p. 140, Proposition 4.11) In the main formula, there should be a negative sign in the second summation of the right-hand side. Hence, the correct formula is

$$\Delta^* Z(g,\phi)_U = \sum_{u \in \mathbb{Q}^\times} \theta(g,u,\phi^0) C(g,u,\phi_0)_U - \sum_{u \in \mathbb{Q}^\times} W_0(g,u,\phi)\, L_U + D(g,\phi)_U.$$

Accordingly, in the proof of the proposition, the expression of S should be negated. This proposition is used in the proof of Theorem 4.15, but the formula in the second paragraph has the corrected form.

42. (p. 184, line -10) Change "§6.2" to "§6.3".

43. (p. 185, line 2) $I(s,g,\phi)_U$ cannot be written that way until it is explained below. At present, it should be

$$I(s,g,\phi)_U = \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{\gamma \in P^1(F) \backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s \sum_{x_1 \in E} r(\gamma g)\phi(x_1,u).$$

44. (p. 186, line -3) Change "$\prod_v \gamma_{u,v} = 1$" to "$\prod_v \gamma_{u,v} = -1$".

45. (p. 187, line 4) Change

$$W_0^\circ(0,g,\phi) = r(g)\phi(0,u)$$

to

$$W_0^\circ(0,g,\phi_2) = r(g)\phi_2(0,u).$$

46. (p. 191, line -2) Change "$y = y_1 + y_2 \in V_v - V_{1v}$" to "$y = y_1 + y_2 \in V_v$".

47. (p. 196, line -2) Change "Proposition 6.6" to "Lemma 6.6".

48. (p. 198, line -11) Change

$$\mathcal{P}r(f)_\psi(g) = W^{(2)}(g_\infty)\, \mathcal{P}r(f)_\psi(g)$$

to

$$\mathcal{P}r(f)_\psi(g) = W^{(2)}(g_\infty)\, \mathcal{P}r(f)_{\psi,f}(g).$$

49. (p. 200, Proof of Lemma 6.13) The quotient $F^\times\backslash\mathbb{A}^\times/F_\tau^\times Q$ is not finite if $F \neq \mathbb{Q}$, but $F^\times\backslash\mathbb{A}^\times/F_\infty^\times Q$ is always finite. We should replace $F^\times\backslash\mathbb{A}^\times/F_\tau^\times Q$ by $F^\times\backslash\mathbb{A}^\times/F_\infty^\times Q$ in the proof.

50. (p. 219, line 8) Delete $\sum_v$ before $i_{\overline{v}}(t_1x, t_2)$.

51. (p. 225, line 14; p. 227, line 5) There is a negative sign missing on the right-hand side of the formula. The coefficient 2 before the averaged integral should be -2.

52. (p. 225, line 17) Change

$$\sum_{v\nmid\infty \text{ nonsplit}} \overline{\mathcal{K}}_\phi^{(v)}(g,(t_1,t_2))$$

to

$$\sum_{v\nmid\infty \text{ nonsplit}} \mathcal{K}_\phi^{(v)}(g,(t_1,t_2)).$$

53. (p. 228, line 10) Change

$$P_\chi(f_i) = \int_{T(F)\backslash T(\mathbb{A})/Z(\mathbb{A})} f_i(t)dt$$

to

$$P_\chi(f_i) = \int_{T(F)\backslash T(\mathbb{A})/Z(\mathbb{A})} f_i(t)\chi(t)dt.$$

54. (p. 228, line 14) Change

$$\mathrm{Hom}_{E\mathbb{A}^\times}(\pi(v)\otimes\chi,\mathbb{C})$$

to

$$\mathrm{Hom}_{E_{\mathbb{A}}^\times}(\pi(v)\otimes\chi,\mathbb{C}).$$

55. (p. 231, line 9; p. 232, line 7) The Green's function $\widetilde{\lim}_{s\to 0} g_s$ is not admissible in the sense of §7.1.15, since it does not satisfy the vanishing condition on (p. 212, line -6). Then there should be some extra terms in the computation of the archimedean local heights. However, the contributions of these extra terms are actually zero under Assumption 5.4 in page 174. Therefore, this mistake does not affect the main results. See clarification and correction of this issue in the preprint

   X. Yuan, Modular Heights of Quaternionic Shimura Curves, arXiv:2205.13995.

   See especially §4.1, Lemma 4.1(2), Remark 4.3, and Remark 4.5 of the paper.

56. (p. 233, line 15) The formula

$$\int_1^\infty \frac{1}{t(1-\lambda t)^{s+1}}dt = 2Q_s(1-2\lambda) + O(|\lambda|^{-s-2})$$

should be

$$\int_1^\infty \frac{1}{t(1-\lambda t)^{s+1}}dt = 2\frac{\Gamma(2s+2)}{\Gamma(s+1)\Gamma(s+2)}Q_s(1-2\lambda) + O(|\lambda|^{-s-2}).$$

This follows from the calculation in p. 304 of Gross–Zagier's paper and especially the formula in line 12 of the page.

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