Relative Noether inequality on fibered surfaces

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Contents

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1 Introduction

This paper is the algebraic version of the previous works [YZ, Zh] of the authors about linear series on arithmetic surfaces.

We work over an algebraically closed field k of any characteristic. Let $f : X \to Y$ be a *surface fibration of genus* g over k. That is:

(1) X is a smooth projective surface over k;

(2) Y is a smooth projective curve over k;

(3) f is flat with connected fibers;

(4) The general fiber F of f is a geometrically integral curve of arithmetic genus $g := p_a(F)$.

Note that here we do *not* assume the general fiber F to be smooth. If $\operatorname{char} k = 0$, then F is a smooth curve of geometric genus g. However, it is not always true if $\operatorname{char} k > 0$.

A reduced and irreducible curve C over k is called *hyperelliptic* if its arithmetic genus $p_a(C) \geq 2$ and if there exists a flat morphism of degree 2 from C onto $\mathbb{P}^1_k$. It follows that C is automatically Gorenstein [Li2]. If F is hyperelliptic, then f is called a *hyperelliptic fibration*. Otherwise, f is called a *non-hyperelliptic fibration*.

1.1 Relative Noether inequality

The following is the main theorem of this paper.

Theorem 1.1. *Let $f : X \to Y$ be a surface fibration of genus $g \geq 2$ over k, and L be a nef line bundle on X. Denote $d = \deg(L|_F)$, where F is a general fiber of f. If $2 \leq d \leq 2g - 2$, then*

$$h^0(L) \leq (\frac{1}{4} + \frac{2+\varepsilon}{4d})L^2 + \frac{d+2+\varepsilon}{2}.$$

Here, $\varepsilon = 1$ if F is hyperelliptic and d is odd. Otherwise, $\varepsilon = 0$.

The most interesting case of Theorem 1.1 occurs when L is the *relative dualizing sheaf* $\omega_f = \omega_{X/Y} = \omega_{X/k} \otimes f^*\omega_{Y/k}^\vee$. Let $f : X \to Y$ be a surface fibration of $g \geq 2$. We say that f is *relatively minimal* if X contains no (-1)-curves in fibers. In this situation, it is known that ω_f is nef. We have the following theorem.

Theorem 1.2 (Relative Noether inequality). *Let $f : X \to Y$ be a relatively minimal fibration of genus $g \geq 2$ over k. Then*

$$h^0(\omega_f) \leq \frac{g}{4g-4}\omega_f^2 + g.$$

If the equality holds, then f is either a trivial fibration or hyperelliptic.

The theorem can be viewed as a relative version of the classical Noether inequality on algebraic surfaces. Recall that if X is a minimal surface of general type over k, and ω_X is the canonical bundle of X, then the Noether inequality asserts that

$$h^0(\omega_X) \leq \frac{1}{2}\omega_X^2 + 2.$$

See [BHPV] for $k = \mathbb{C}$ and [Li1, Li2] for $\operatorname{char} k > 0$. If the equality holds, then X has a hyperelliptic pencil. See [Ho].

As consequences of Theorem 1.2, we will obtain in the following the slope inequality, the Arakelov inequality, and Severi inequality for surfaces of maximal Albanese dimension over any field.

1.2 Slope inequality and Arakelov inequality

A surface fibration $f : X \to Y$ is called *semistable*, if all the fibers are semistable in the sense of Deligne and Mumford. The semistability implies that there are no (-1)-curves contained in fibers. In characteristic 0, the general fiber is automatically smooth. As shown in [Ta], it still holds in positive characteristic.

Theorem 1.3 (Slope inequality). *Let $f : X \to Y$ be a semistable fibration of genus $g \geq 2$ over k. Then*

$$\omega_f^2 \geq \frac{4g-4}{g}\deg(f_*\omega_f).$$

This theorem recovers the slope inequality of Cornalba-Harris [CH] and Xiao [Xi1] in characteristic 0, and that of Moriwaki [Mo] in arbitrary characteristics. But our proof is totally different from theirs. Their treatments respectively make essential uses of geometric invariant theory, the stability of vector bundles, or the Chow stability result of Bost [Bo]. But we derive the result from Theorem 1.2 by some arguments using coverings and reduction mod $\wp$.

Note that Theorem 1.2 can not be implied by the slope inequality. It turns out that Theorem 1.2 is *quite strong*, though the proof only involves the basic theory of linear series on fibered surfaces. In fact, since the slope inequality is sharp in characteristic 0, our bound is also the best in positive characteristic. For instance, if $\operatorname{char} k \neq 2$, one can construct a hyperelliptic fibration by taking the double cover of a ruled surface branched along a smooth curve. The slope inequality for this fibration is in fact an equality. See [Xi2] for examples.

If $g(Y) = 0$ or 1, we even do not need f to be semistable. The slope inequality for general fibrations in these cases is directly implied by Theorem 1.2. See Theorem 3.2.

We also establish a similar result for arbitrary line bundle of small degree when $\operatorname{char} k > 0$. See Theorem 3.4.

Using Theorem 1.3, we can prove the following Arakelov inequality in positive characteristic, which generalizes the original result in [Ar] over $\mathbb{C}$.

Theorem 1.4 (Arakelov inequality)**.** *Let f be a non-isotrivial semistable fibration of genus $g \geq 2$ over the field k of positive characteristic. Let $S \subset Y$ be the singular locus over which f degenerates. Then*

$$\deg(f_*\omega_f) < g^2 \deg \Omega_Y(S).$$

In [LSZ], Lu, Sheng and Zuo have used a modified version of the stability method to obtain a bound. Our bound is better than theirs. We refer to the recent survey article [Vi, Zu] for more information on Arakelov inequality. We also refer to the works by Viehweg and Zuo [VZ1, VZ2] for the generalized version of Arakelov inequality.

1.3 Severi conjecture

Let X be a smooth surface of general type over k. We say X is *of maximal Albanese dimension*, if the Albanese map $\mathrm{Alb}_X : X \to \mathrm{Alb}(X)$ is generically finite.

The Albanese map can be defined for X over an arbitrary field k. Note that since our field k is arbitrary, it is not necessary that $\dim \mathrm{Alb}(X) = h^{1,0}(X)$ in general. If $k = \mathbb{C}$, then it is true. However, for arbitrary fields, we only have $\dim \mathrm{Alb}(X) \leq h^{1,0}(X)$ [Ig]. See also [Li1] for a survey in positive characteristic.

We have the following theorem.

Theorem 1.5 (Severi inequality)**.** *Let X be a smooth minimal surface of general type over k and of maximal Albanese dimension. Then*

$$\omega_{X/k}^2 \geq 4\chi(\mathcal{O}_X).$$

Let us briefly introduce the history of this inequality. It was Severi who stated it as a theorem in [Se], whose proof was not correct unfortunately. It became again as a conjecture in [Ca, Re].

When $k = \mathbb{C}$, in the fundamental paper [Xi1], Xiao proved this conjecture assuming that X has a fibration over a curve of positive genus. Konno [Ko] proved this conjecture for surfaces with even canonical divisor, namely the surfaces whose canonical divisor $K \equiv 2L$ for certain divisor L. Also Manetti [Ma] proved this conjecture for surfaces with ample canonical bundle. The complete proof of this conjecture in the complex case was given by Pardini [Pa], which is a very clever application of the slope inequality for certain fibrations she has constructed.

It is worth mentioning that this theorem is new in positive characteristics. In positive characteristic, if the Albanese map is generically finite onto its image, it may even become purely inseparable (c.f. [Li3] for examples). Such effects make estimates on positivity and differentials especially subtle. The proofs of Konno and Manetti might be available as partial evidences for this conjecture in positive characteristic, but the direct generalization of Pardini's proof should be based on the slope inequality for surface fibrations in positive characteristic whose fibers are singular, which was still unknown as we mentioned before. However, our approach again comes from a completely different point of view. Without using the slope inequality, we just apply Theorem 1.2 to the fibrations we construct. One interesting fact is that, since Theorem 1.2 holds for arbitrary fields, the proof we offer here is also independent on the base field k.

1.4 Idea of the proofs

The idea of the proof of Theorem 1.1 comes from [YZ, Zh]. Let L be a line bundle on X as stated in Theorem 1.1. Denote

$$\Delta(L) = h^0(L) - \frac{1}{4}L^2.$$

We can find the largest integer c such that $L - (c-1)F$ is nef. Denote $L' = L - cF$. We manage to reduce our problem to $\Delta(L')$.

The key observation is that: the linear system $|L'|$ has a horizontal base locus. See Lemma 2.1. Let $|L_1|$ be the movable part of $|L'|$. Then the problem is reduced to $\Delta(L_1)$. Note that now

$$L_1F < L'F = LF.$$

Keep this reduction and we can get $L_2, L_3, \cdots$. The whole process will terminate since L_iF decreases strictly. Finally, we reach the proof of Theorem 1.1.

The proof of the slope inequality is quite novel. Using Theorem 1.2, we can *almost* get the slope inequality for arbitrary fields up to a constant related to the genera of the fiber and the base curve. See Lemma 3.1.

We first prove Theorem 1.3 in positive characteristic. By iterating the Frobenius base change, we successfully eliminate the constant term. Based on a result of Szpiro, we can prove Theorem 1.4. Then we reduce the slope inequality in characteristic 0 to positive characteristic, which leads to the final proof of Theorem 1.3.

The proof of the Severi inequality is also from the iteration of the base change. The difference is that the base change here is actually the multiplicative map of an abelian variety. This base change was used in [Pa]. By doing this, we can get a family of fibrations. Applying Theorem 1.2 to those fibrations and taking the limit, we eventually prove the Severi inequality once for all fields.

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2 Proof of the relative Noether inequality

In this section, we will prove Theorem 1.1. The idea comes from [YZ, Zh], where a very similar argument was used to prove an effective Hilbert-Samuel formula for arithmetic surfaces.

2.1 The reduction process

We first resume our notations. Let $f : X \to Y$ be a surface fibration, and F be a general fiber. For any nef divisor L on X, we can find an integer c such that

- $L - cF$ is not nef;
- $L - c'F$ is nef for any integer $c' < c$.

We denote this number c for L by e_L. In particular, since L itself is nef, we have

$$e_L > 0.$$

We have the following lemma.

Lemma 2.1. *Let $f : X \to Y$ be a surface fibration, and F be a general fiber. Let L be a nef line bundle on X such that $LF > 0$ and $h^0(L - e_L F) > 0$. Then the linear system $|L - e_L F|$ has a fixed part $Z > 0$. Moreover,*

$$ZF > 0.$$

Proof. By our assumption, we can write

$$|L - e_L F| = |L_1| + Z,$$

where $|L_1|$ is the movable part, and Z is the fixed part. By the definition of e_L, $L - e_L F$ is not nef. Then there exists an irreducible and reduced curve C on X such that

$$(L - e_L F)C < 0,$$

which implies $Z \geq C > 0$. Again by the definition of e_L, $L - (e_L - 1)F$ is nef. For the above C, we have

$$(L - (e_L - 1)F)C = (L - e_L F)C + FC \geq 0,$$

i.e., $FC > 0$. Therefore,

$$ZF \geq FC > 0.$$

□

We have the following general theorem.

Theorem 2.2. *Let $f : X \to Y$ be a surface fibration, and F be a general fiber. Let L be a nef divisor on X such that $LF > 0$ and $h^0(L) > 0$. We can get the following sequence of triples*

$$\{(L_i, Z_i, a_i) : i = 0, 1, \cdots, n\}$$

such that

- $(L_0, Z_0, a_0) = (L, 0, e_L)$ *and* $a_i = e_{L_i}$ *for* $i \geq 0$*;*
- *We have*
$$|L_{i-1} - e_{L_i}F| = |L_i| + Z_i,$$
where L_i *(resp.* Z_i*) is the movable part (resp. fixed part) of the linear system* $|L_{i-1} - e_{L_i}F|$ *for* $0 < i < n$*;*
- $h^0(L'_n) = 0$*;*
- $LF = L_0F > L_1F > \cdots > L_nF \geq 0.$

Proof. The triple $(L_{i+1}, Z_{i+1}, a_{i+1})$ can be obtained by applying Lemma 2.1 to the triple (L_i, Z_i, a_i). The whole process will terminate when $h^0(L_i - a_iF) = 0$. It always terminates because by Lemma 2.1, L_iF decreases strictly. □

Now, we denote $r_i = h^0(L_i|_F)$, $d_i = L_iF$ and $L'_i = L_i - a_iF$.

Proposition 2.3. *For any $j = 0, 1, \cdots n$, we have*

$$\begin{aligned} h^0(L_0) &\leq h^0(L'_j) + \sum_{i=0}^{j} a_i r_i; \\ {L_0}^2 &\geq {L'_j}^2 + 2a_0d_0 + \sum_{i=1}^{j} a_i(d_{i-1} + d_i) - 2(d_0 - d_j). \end{aligned}$$

Proof. We have the following exact sequence:

$$0 \longrightarrow H^0(L_{i+1} - F) \longrightarrow H^0(L_{i+1}) \longrightarrow H^0(L_{i+1}|_F).$$

Then it follows that

$$h^0(L_{i+1} - F) \leq h^0(L_{i+1}) - h^0(L_{i+1}|_F) = h^0(L_{i+1}) - r_{i+1}.$$

By induction, we have

$$h^0(L'_{i+1}) = h^0(L_{i+1} - a_{i+1}F) \leq h^0(L_{i+1}) - a_{i+1}r_{i+1} = h^0(L'_i) - a_{i+1}r_{i+1}.$$

Note that both $L'_i + F$ and $L'_{i+1} + F$ are nef, and Z_i is effective. We get

$$\begin{aligned} {L'_i}^2 - L'^2_{i+1} &= (L'_i + L'_{i+1})(L'_i - L'_{i+1}) \\ &= (L'_i + L'_{i+1})(a_{i+1}F + Z_{i+1}) \\ &= a_{i+1}(L'_i + L'_{i+1})F + [(L'_i + F) + (L'_{i+1} + F) - 2F]Z_{i+1} \\ &\geq a_{i+1}(d_i + d_{i+1}) - 2Z_{i+1}F. \end{aligned}$$

For any $j = 0, 1, \cdots, n-1$, summing over $i = 0, 1, \cdots, j$, we have

$$\begin{aligned} h^0(L'_0) &\leq h^0(L'_j) + \sum_{i=1}^{j} a_i r_i; \\ L'^2_0 &\geq L'^2_j + \sum_{i=1}^{j} a_i(d_{i-1} + d_i) - 2\sum_{i=1}^{j} Z_i F. \end{aligned}$$

Moreover, we have

$$\sum_{i=1}^{j} Z_i = L'_0 - L'_n - \sum_{i=1}^{j} a_i F.$$

It follows that

$$\sum_{i=1}^{j} Z_i F = (L'_0 - L'_j)F = d_0 - d_j.$$

Since

$$L_0^2 - {L'_0}^2 = 2a_0 d_0$$

and

$$h^0(L_0) \leq h^0(L'_0) + a_0 r_0,$$

the result follows. □

We also have the following lemma.

Lemma 2.4. *In the above setting, we have*

$$2a_0 + \sum_{i=1}^{n} a_i - 2 \leq \frac{L_0^2}{d_0}.$$

Proof. Denote $b = a_1 + \cdots + a_n$ and $Z = Z_1 + \cdots + Z_n$. We have the following linear equivalence

$$L_0' = L_n' + bF + Z.$$

Since $L_0' + F$ and $L_n' + F$ are both nef, it follows that

$$\begin{aligned} (L_0' + F)^2 &= (L_0' + F)(L_n' + F + bF + Z) \\ &\geq (L_n' + F + bF + Z)(L_n' + F) + bd_0 \\ &\geq (L_n' + F)^2 + b(d_0 + d_n). \end{aligned}$$

Combine with

$$L_0^2 - (L_0' + F)^2 = 2(a_0 - 1)d_0.$$

We get

$$L_0^2 \geq (L_n' + F)^2 + 2(a_0 - 1)d_0 + b(d_0 + d_n) \geq d_0(2a_0 + b - 2).$$

□

2.2 Linear systems on curves

We need several results on linear systems on algebraic curves.

First, recall the following Clifford's theorem for special line bundles on algebraic curves. Let C be a reduced, irreducible Gorenstein curve over k. We say a line bundle L is *special* if

$$h^0(L) > 0, \qquad h^1(L) > 0.$$

We have the following theorem.

Theorem 2.5. *Let C be a reduced Gorenstein curve over k, $p_a(C) \geq 2$. Let L be a line bundle on C such that $h^0(L) > 0$ and $\deg(L) \leq 2p_a(C) - 2$.*

1. [Clifford's Theorem] *If L is special, then*

$$h^0(L) \leq \frac{1}{2}\deg(L) + 1.$$

Moreover, if C is not hyperelliptic, then the equality holds if and only if $L = \mathcal{O}_C$ or $L = \omega_{C/k}$.

2. If $h^1(L) = 0$, then

$$h^0(L) \leq \frac{1}{2}\deg(L).$$

Proof. If L is special, then the theorem is just the generalized version of Clifford's theorem in [Li2]. If $h^1(L) = 0$, by the Riemann-Roch theorem,

$$h^0(L) = \deg(L) - p_a(C) + 1 \leq \frac{1}{2}\deg(L).$$

□

We also need the following lemma.

Lemma 2.6. *Let L be a special line bundle on a hyperelliptic curve C over k such that $|L|$ is base-point-free, then $\deg(L)$ is even.*

Proof. Denote $d_L = \deg(L)$. Since L is base-point-free, we can choose $D_1, D_2 \in |L|$ and define a morphism

$$\phi = (\phi_1, \phi_2) : C \to \mathbb{P}^1 \times \mathbb{P}^1.$$

Here ϕ_1 is the degree two map from C to $\mathbb{P}^1$ by the definition of hyperelliptic curves, and ϕ_2 is the degree d_L map C to $\mathbb{P}^1$ induced by D_1 and D_2. It is easy to see that either ϕ is birational from C to $\phi(C)$ or $\deg\phi = 2$.

If $\deg\phi = 2$, then we are done. Furthermore, we claim that ϕ can not be birational to its image.

If ϕ is birational, denote $C' = \phi(C)$. By the definition of ϕ, we can write $C' \in |d_L F_1 + 2F_2|$, where F_1 and F_2 are rules on $\mathbb{P}^1 \times \mathbb{P}^1$. And $L = \phi^*\mathcal{O}_{C'}(F_2)$. We have the following short exact sequence on C':

$$0 \longrightarrow \mathcal{O}_{C'}(F_2) \longrightarrow \phi_* L \longrightarrow \mathcal{E} \longrightarrow 0,$$

where $\mathcal{E}$ is a skyscraper sheaf. Hence we have a surjection

$$H^1(\mathcal{O}_{C'}(F_2)) \twoheadrightarrow H^1(\phi_* L).$$

Since L is special and ϕ is finite, we get $h^1(\mathcal{O}_{C'}(F_2)) \geq h^1(\phi_* L) = h^1(L) > 0$.

On the other hand, we have another exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-C' + F_2) \longrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(F_2) \longrightarrow O_{C'}(F_2) \longrightarrow 0,$$

which gives

$$H^1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(F_2)) \longrightarrow H^1(\mathcal{O}_{C'}(F_2)) \longrightarrow H^2(O_{\mathbb{P}^1\times\mathbb{P}^1}(-C'+F_2)).$$

Now by Serre duality,

$$h^2(O_{\mathbb{P}^1\times\mathbb{P}^1}(-C'+F_2)) = h^0(O_{\mathbb{P}^1\times\mathbb{P}^1}((d_L-2)F_1 - F_2) = 0.$$

Moreover, using the Riemann-Roch formula on $\mathbb{P}^1\times\mathbb{P}^1$, we get

$$h^1(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(F_2)) = 0,$$

which forces $h^1(\mathcal{O}_{C'}(F_2)) = 0$. □

We will divide the proof of Theorem 1.1 into two parts.

2.3 Hyperelliptic case

We first prove Theorem 1.1 when F is hyperelliptic.

We know that by our reduction process, $h^0(L_n') = 0$. By our construction, $L_n' + F$ is nef. So

$$L_n'^{\,2} = ((L_n'+F)-F)^2 \geq -2L_n'F = -2d_n.$$

By Proposition 2.3, we get

$$\begin{aligned} h^0(L_0) &\leq h^0(L_n') + \sum_{i=0}^{n} a_i r_i = \sum_{i=0}^{n} a_i r_i, \\ {L_0}^2 &\geq L_n'^{\,2} + 2a_0 d_0 + \sum_{i=1}^{n} a_i(d_{i-1}+d_i) - 2\sum_{i=1}^{n} Z_i F \\ &\geq 2a_0 d_0 + \sum_{i=1}^{n} a_i(d_{i-1}+d_i) - 2d_0. \end{aligned}$$

We need to deal with two different cases:

- Each $L_i|_F$ is special;
- There exists a $k>0$ such that $L_0|_F, \cdots, L_{k-1}|_F$ are not special and $L_k|_F, \cdots, L_n|_F$ are special.

First, we assume that each $L_i|_F$ is special. Applying Clifford's theorem, we have

$$r_i \leq \frac{1}{2}d_i + 1.$$

It yields

$$h^0(L_0) - \frac{1}{4}L_0^2 \leq \sum_{i=0}^{n} a_i - \frac{1}{4}\sum_{i=1}^{n}(d_{i-1} - d_i)a_i + \frac{1}{2}d_0.$$

On the other hand, since F is sufficiently general, and the linear system $|L_i|$ has no fixed part for $i = 1, \cdots, n$, the line bundle $L_i|_F$ is base-point-free by construction. By Lemma 2.6, d_i is even. Thus

$$d_{i-1} - d_i \geq 2$$

for $i = 2, \cdots, n$. By Lemma 2.4,

$$a_0 + \sum_{i=0}^{n} a_i - 2 \leq \frac{L_0^2}{d_0}.$$

Therefore, we have

$$\begin{aligned} h^0(L_0) &\leq \frac{1}{4}L_0^2 + a_0 + \frac{3}{4}\sum_{i=1}^{n} a_i + \frac{1}{2}d_0 \\ &\leq (\frac{1}{4} + \frac{3}{4d_0})L_0^2 + \frac{1}{2}d_0 + \frac{3}{2}. \end{aligned}$$

If d_0 is even, we also have

$$d_0 - d_1 \geq 2.$$

Hence, we have the strong bound

$$\begin{aligned} h^0(L_0) &\leq \frac{1}{4}L_0^2 + a_0 + \frac{1}{2}\sum_{i=1}^{n} a_i + \frac{1}{2}d_0 \\ &\leq (\frac{1}{4} + \frac{1}{2d_0})L_0^2 + \frac{1}{2}d_0 + 1. \end{aligned}$$

Now let us deal with the other case. From Theorem 2.5, we know

$$r_i \leq \frac{1}{2}d_i, \qquad i = 0, \cdots, k-1,$$

and

$$r_i \leq \frac{1}{2}d_i + 1, \qquad i = k+1, \cdots, n.$$

Similarly, we have

$$d_{i-1} - d_i \geq 1, \qquad i = 0, \cdots, k,$$

and

$$d_{i-1} - d_i \geq 2, \qquad i = k+1, \cdots, n.$$

Thus we have

$$\begin{aligned} h^0(L_0) - \frac{1}{4}L_0^2 &\leq \sum_{i=k+1}^{n} a_i - \frac{1}{4}\sum_{i=k+1}^{n}(d_{i-1} - d_i)a_i + \frac{1}{2}d_0 \\ &\quad + \left(r_k - \frac{1}{2}d_k - \frac{1}{4}(d_{k-1} - d_k)\right) a_k \\ &\leq \frac{1}{2}\sum_{i=k+1}^{n} a_i + \frac{1}{2}d_0 + \left(r_k - \frac{1}{2}d_k - \frac{1}{4}(d_{k-1} - d_k)\right) a_k. \end{aligned}$$

If $d_{k-1} - d_k \geq 2$, we have

$$h^0(L_0) - \frac{1}{4}L_0^2 \leq \frac{1}{2}\sum_{i=k}^{n} a_i + \frac{1}{2}d_0.$$

If $d_{k-1} - d_k = 1$, we know that $r_k \leq r_{k-1} - 1$. So

$$r_k \leq r_{k-1} - 1 \leq \frac{1}{2}d_{k-1} - 1 = \frac{1}{2}d_k + \frac{1}{2}.$$

Hence we still have

$$h^0(L_0) - \frac{1}{4}L_0^2 < \frac{1}{2}\sum_{i=k}^{n} a_i + \frac{1}{2}d_0.$$

By Lemma 2.4 again,

$$h^0(L_0) < (\frac{1}{4} + \frac{1}{2d_0})L_0^2 + \frac{1}{2}d_0 + 1.$$

Remark 2.7. We see from the proof that when f is hyperelliptic and d_0 is even, if the above equality holds, then

$$d_i - d_{i+1} = 2$$

for $0 < i < n$ and

$$r_i = \frac{1}{2}d_i + 1$$

for $0 \le i \le n$.

2.4 Non-hyperelliptic case

In this case, for $i = 1, \cdots, n-1$, we have the following stronger Clifford's theorem:

$$r_i \le \frac{1}{2}d_i + \frac{1}{2}.$$

For $i = 0$ or $i = n$, it also holds if $L_i|_F$ is neither trivial nor $\omega_{F/k}$.

First, we assume $L_n|_F$ is not trivial. Using the strong bound, we get

$$\begin{aligned} h^0(L_0) - \frac{1}{4}L_0^2 &\le \sum_{i=0}^n a_i r_i - \frac{1}{2}a_0 d_0 - \frac{1}{4}\sum_{i=1}^n a_i(d_{i-1} + d_i) + \frac{1}{2}d_0 \\ &\le a_0 + \frac{1}{2}\sum_{i=1}^n a_i - \frac{1}{4}\sum_{i=1}^n a_i(d_{i-1} - d_i) + \frac{1}{2}d_0 \\ &< a_0 + \frac{1}{2}\sum_{i=1}^n a_i + \frac{1}{2}d_0. \end{aligned}$$

By Lemma 2.4,

$$a_0 + \frac{1}{2}\sum_{i=1}^n a_i \le \frac{L_0^2}{2d_0} + 1.$$

We have

$$h^0(L_0) < (\frac{1}{4} + \frac{1}{2d_0})L_0^2 + \frac{1}{2}d_0 + 1.$$

If $L_n|_F$ is trivial, it gives

$$\begin{aligned} h^0(L_0) - \frac{1}{4}L_0^2 &\leq \sum_{i=0}^{n} a_i r_i - \frac{1}{2}\sum_{i=0}^{n} a_i d_i - \frac{1}{4}\sum_{i=1}^{n} a_i(d_{i-1} - d_i) + \frac{1}{2}d_0 \\ &\leq a_0 + \frac{1}{2}\sum_{i=1}^{n-1} a_i + a_n - a_n(d_{n-1} - d_n) + \frac{1}{2}d_0 \\ &\leq a_0 + \frac{1}{2}\sum_{i=1}^{n} a_i + \frac{1}{2}d_0. \end{aligned}$$

Here the equality will not hold unless $n = 1$ and $d_0 = 1$. By Lemma 2.4 again, it yields

$$h^0(L_0) \leq (\frac{1}{4} + \frac{1}{2d_0})L_0^2 + \frac{1}{2}d_0 + 1.$$

It ends the proof of Theorem 1.1.

2.5 Proof of Theorem 1.2

Now, Theorem 1.2 becomes straightforward. Let $f : X \to Y$ be a relatively minimal fibration. It is well-known that in this case, the relative dualizing sheaf ω_f is nef. Moveover, $\omega_f F = 2g - 2$. Applying Theorem 1.1, we can directly get Theorem 1.2.

Now we only need to characterize the case when the equality in Theorem 1.2 holds. We can assume that f is not a trivial fibration. Suppose the equality holds in 1.2 and f is not hyperelliptic. The only possibility is when $d_0 = 1$, which can not happen here because $d_0 = \omega_f F = 2g - 2$ has to be even.

3 Slope inequality

In this section, we prove Theorem 1.3. By the Riemann-Roch theorem, Theorem 1.2 implies a slope inequality with an "error term." To get rid of the "error term," we consider base change to coverings of Y, and take a limit. The argument is straight-forward if $g(Y) \leq 1$. In general, the covering trick only works in positive characteristics. To get the result in characteristic 0, we use a reduction argument.

3.1 Slope inequality for $g(Y) \leq 1$

In this section, we apply Theorem 1.2 to give a new proof of the slope inequality for fibered surfaces over $\mathbb{P}^1$ and elliptic curves. We first give the following lemma.

Lemma 3.1. *Let $f : X \to Y$ be a relatively minimal surface fibration of genus g. Then*

$$\deg(f_*\omega_f) \leq \frac{g}{4g-4}\omega_f^2 + gb.$$

Here $b = g(Y)$.

Proof. By Theorem 1.2, we have

$$h^0(\omega_f) \leq \frac{g}{4g-4}\omega_f^2 + g.$$

On the other hand, using the Riemann-Roch theorem on Y,

$$h^0(\omega_f) = h^0(f_*\omega_f) \geq \deg(f_*\omega_f) + g(1-b).$$

It follows that

$$\deg(f_*\omega_f) \leq \frac{g}{4g-4}\omega_f^2 + gb.$$

□

Theorem 3.2. *Let $f : X \to Y$ be a relatively minimal surface fibration of genus g. Assume that $g(Y) \leq 1$. Then*

$$\omega_f^2 \geq \frac{4g-4}{g}\deg(f_*\omega_f).$$

Proof. If $g(Y) = 0$, then the result is just Lemma 3.1.

Now, suppose that Y is an elliptic curve over k and that $\mu : Y \to Y$ is the multiplication by n such that n and $\operatorname{char} k$ are coprime with each other. Denote $X' = X \times_\pi Y$. We get a new fibration $f' : X' \to Y$ which is just the pull-back of f by μ. Applying Lemma 3.1 to f', it follows that

$$\deg(f'_*\omega_{f'}) \leq \frac{g}{4g-4}\omega_{f'}^2 + g.$$

On the other hand, since the base change is étale, we have the following facts:

$$\deg(f'_*\omega_{f'}) = n^2\deg(f_*\omega_f), \quad \omega_{f'}^2 = n^2\omega_f^2,$$

which gives us

$$n^2 \deg(f_*\omega_f) \leq \frac{n^2 g}{4g-4}\omega_f^2 + g,$$

i.e.,

$$\deg(f_*\omega_f) \leq \frac{g}{4g-4}\omega_f^2 + \frac{g}{n^2}.$$

We can prove our result by letting $n \to \infty$. $\square$

Remark 3.3. In the case $b = g(Y) \geq 2$, if we directly use an étale base change $\pi : Y' \to Y$, then $g(Y')$ also increases. Therefore, we can not prove the general slope inequality using the above argument. However, we still want to use the base change trick and control $g(Y')$ at the same time, which motivates us to consider the reduction mod $\wp$ method.

3.2 Slope inequality in positive characteristic

We first prove the slope inequality when $\operatorname{char} k = p > 0$. This is also crucial for us to prove the slope inequality for $\operatorname{char} k = 0$. Actually, the proof is quite similar to the proof of Theorem 3.2.

Let $f : X \to Y$ be a semistable fibration of genus g. By Lemma 3.1 to f, it follows that

$$\deg(f_*\omega_f) \leq \frac{g}{4g-4}\omega_f^2 + gb,$$

where $b = g(Y)$.

Now let $F_Y : Y \to Y$ be the absolute Frobenius morphism of Y. Via this base change, we get a new fibration

$$f' : X' \to Y,$$

where X' is the minimal desingularization of the normal surface $X \times_{F_Y} Y$. Thus f' is still semistable. Applying Lemma 3.1 again to f', it follows that

$$\deg(f'_*\omega_{f'}) \leq \frac{g}{4g-4}\omega_{f'}^2 + gb.$$

Moreover, we have the following facts:

$$\deg(f'_*\omega_{f'}) = p\deg(f_*\omega_f), \quad \omega_{f'}^2 = p\,\omega_f^2,$$

We obtain

$$p\deg(f_*\omega_f) \leq \frac{pg}{4g-4}\omega_f^2 + gb,$$

i.e.,

$$\deg(f_*\omega_f) \leq \frac{g}{4g-4}\omega_f^2 + \frac{gb}{p}.$$

We can prove our result by iterating this Frobenius base change.

Actually, using the same idea, we can get a more general result for line bundles of small degree in positive characteristics, which is similar to Theorem 1.3.

Theorem 3.4. *Let $f : X \to Y$ be a surface fibration of genus $g > 0$ over a field k of positive characteristic, and L be a nef line bundle on X. Assume that $2 \leq d = \deg(L|_F) \leq 2g-2$, where F is a general fiber of f. Then*

$$L^2 \geq \frac{4d}{d+2+\varepsilon}\deg(f_*L).$$

Here, $\varepsilon = 1$ if F is hyperelliptic and d is odd. Otherwise, $\varepsilon = 0$.

Proof. The proof is the same as the proof of the slope inequality in positive characteristic. We just sketch here.

First, by Theorem 1.1, we have

$$h^0(L) \leq (\frac{1}{4} + \frac{2+\varepsilon}{4d})L^2 + \frac{d+2+\varepsilon}{2}.$$

The Riemann-Roch theorem on Y gives us

$$h^0(L) = h^0(f_*L) \geq \deg(f_*L) + r(1-b),$$

where $b = g(Y)$ and $r = h^0(L|_F)$. Combine them together and we get

$$\deg(f_*L) \leq (\frac{1}{4} + \frac{2+\varepsilon}{4d})L^2 + \frac{d+2+\varepsilon}{2} + r(b-1).$$

Now we apply the Frobenius base change iteration as above. Finally, we eliminate the constant term. $\square$

3.3 Slope inequality in characteristic 0

Now we can prove the slope inequality for char $k = 0$. We have the following lemma.

Lemma 3.5. *Let X, Y, Z be integral schemes, and $f : X \to Y$ and $g : Y \to Z$ be proper and flat morphisms of relative dimension one. Assume that f is a local complete intersection and g is smooth. Then the numbers*

$$\deg f_{z*}(\omega_{X_z/Y_z}) \quad \textit{and} \quad \omega^2_{X_z/Y_z}$$

are independent of $z \in Z$. Here $f_z : X_z \to Y_z$ denotes the fiber of $f : X \to Y$ over z, and ω_{X_z/Y_z} denotes the relative dualizing sheaf of X_z over Y_z.

Proof. The invariance of $\deg f_{z*}(\omega_{X_z/Y_z})$ is an interpretation of the determinant line bundle. Recall that for any line bundle L on X, the determinant line bundle $\lambda_f(L)$ is a line bundle on Y such that, for any $y \in Y$, there is a canonical isomorphism

$$\lambda_f(L) \simeq \det H^*(X_y, L_y) = \det H^0(X_y, L_y) \otimes \det H^1(X_y, L_y)^\vee.$$

The construction is functorial.

In the setting of the lemma, consider the determinant line bundle $M = \lambda_f(\omega_{X/Y})$. Restricted to Y_z, we have

$$M|_{Y_z} = (\det f_{z*}\omega_{X_z/Y_z}) \otimes (\det R^1 f_{z*}\omega_{X_z/Y_z})^\vee = \det f_{z*}\omega_{X_z/Y_z}.$$

Here we used the canonical isomorphism $R^1 f_{z*}\omega_{X_z/Y_z} = \mathcal{O}_{Y_z}$ following the duality theorem. Therefore, we simply have

$$\deg f_{z*}\omega_{X_z/Y_z} = \deg(M|_{Y_z}).$$

It is independent of z.

The invariance of $\omega^2_{X_z/Y_z}$ follows from the definition of the Deligne pairing introduced in [De]. In fact, the Deligne pairing $N = \langle \omega_{X/Y}, \omega_{X/Y} \rangle$ is a line bundle on Y such that

$$\omega^2_{X_z/Y_z} = \deg(N|_{Y_z}), \quad \forall z \in Z$$

It is also independent of z. □

The proof does not work for arbitrary line bundles of small degree since we used the duality theorem above.

Go back to our setting. Suppose that $f : X \to Y$ is a semistable fibration over of characteristic 0. By the Lefschetz principle, one can assume that k is finitely generated over $\mathbb{Q}$. Let $\mathcal{Z}$ be an integral scheme of finite type over $\mathbb{Z}$

with the function field k. Shrink $\mathcal{Z}$ by an open subset and replace it by a finite cover if necessary. We are able to extend the composition $X \to Y \to \mathrm{Spec}(k)$ to $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ satisfying the conditions of the lemma.

Now choose a nonzero prime $\wp \in \mathcal{Z}$ such that $f_\wp : \mathcal{X}_\wp \to \mathcal{Y}_\wp$ is a semistable fibration over the field $k/\wp$. By the slope inequality in positive characteristic, we have

$$\omega_{f_\wp}^2 \geq \frac{4g-4}{g} \deg(f_{\wp *}\omega_{f_\wp}).$$

By Lemma 3.5, the slope inequality holds for f over k.

3.4 Arakelov inequality

Now Theorem 1.4 is straightforward. Suppose that $\mathrm{char}\, k > 0$. Let f be a non-isotrivial semistable fibration of genus $g \geq 2$ over k. Let $S \subset Y$ be the singular locus over which f degenerates. The proof of Theorem 1.4 is just the combination of the slope inequality in positive characteristic and the following Szpiro's inequality (cf. [Sz, Prop. 4.2]):

$$\omega_f^2 < (4g-4)g \deg \Omega_Y(S).$$

Therefore,

$$\deg(f_*\omega_f) \leq \frac{g}{4g-4}\omega_f^2 < g^2 \deg \Omega_Y(S).$$

4 Proof of Severi inequality

In this section, we give the proof of Theorem 1.5 using Theorem 1.2. We start this section by first recalling some basic facts about surface fibrations.

4.1 Basic facts

Let $f : X \to Y$ be a surface fibration of g over k. Denote $b = g(Y)$. Then

$$\omega_f = \omega_{X/k} \otimes f^*\omega_{Y/k}^\vee,$$

where $\omega_{X/k}$ (resp. $\omega_{Y/k}$) is the canonical bundle of X (resp. Y). We have the following formulae between the absolute invariants of X and the relative invariants of f:

$$\begin{aligned} \omega_f^2 &= \omega_{X/k}^2 - 8(g-1)(b-1), \\ \deg(f_*\omega_f) &= \chi(\mathcal{O}_X) - (g-1)(b-1). \end{aligned}$$

Here, by Serre duality,

$$\chi(\mathcal{O}_X) = h^0(\omega_{X/k}) - h^1(\mathcal{O}_X) + 1.$$

4.2 Construction of fibrations

In order to apply Theorem 1.2, we need to construct certain fibrations. The construction is essentially due to [Pa]. Here we use a slightly simpler one. However, for the completeness of the paper, we still give the details.

Let X be a minimal surface of general type with maximal Albanese dimension. Denote $\mathrm{Alb}_X : X \to A$ to be the Albanese map, where $A = \mathrm{Alb}(X)$ is an abelian variety of dimension m over k. We only have $m \leq h^{1,0}(X)$, but since X is of maximal Albanese dimension, it is always safe for us to use the bound

$$m \geq 2.$$

Let H be a very ample line bundle on A, and L be the pull-back of H on X. Set

$$\alpha = L^2, \quad \beta = \omega_{X/k} L.$$

Since X is of general type, α and β are both strictly positive.

Let $\mu : A \to A$ be the multiplication by n, where $n > 1$ is an integer. If $\mathrm{char}\, k = p > 0$, we assume n and p to be coprime. We have the following base change:

$$\begin{tikzcd} X' \arrow[r, "\tau"] \arrow[d, "\nu"'] & X \arrow[d, "\mathrm{Alb}_X"] \\ A \arrow[r, "\mu"] & A \end{tikzcd}$$

where $X' = X \times_\mu A$. We have

$$\omega^2_{X'/k} = n^{2m} \omega^2_{X/k}, \quad \chi(\mathcal{O}_{X'}) = n^{2m} \chi(\mathcal{O}_X).$$

We also have the following numerically equivalence on A:

$$\mu^* H \sim_{\mathrm{num}} n^2 H,$$

which yields

$$\tau^* L \sim_{\mathrm{num}} n^2 L'.$$

Here $L' = \nu^* H$. It follows that

$$L'^2 = n^{2m-4} \alpha, \quad \omega_{X'_n/k} L' = n^{2m-2} \beta.$$

Now choose $C_1, C_2 \in |L'|$ to be two general curves which intersect each other transversally. Here C_1 and C_2 might not be smooth, which could happen if $\mathrm{char}\, k > 0$. But since $|L'|$ is base point free, we can assume that C_1 and C_2 are both reduced, irreducible Gorenstein curves [Jo, Li2]. Moreover, we know C_1 and C_2 intersect at $n^{2m-4}\alpha$ points. After blowing up these points, we get a surface fibration $f : X'' \to \mathbb{P}^1$. It follows that

$$\omega_{X''/k}^2 = \omega_{X'/k}^2 - n^{2m-4}\alpha = n^{2m}\omega_{X/k}^2 - n^{2m-4}\alpha, \tag{1}$$

$$\chi(\mathcal{O}_{X''}) = \chi(\mathcal{O}_{X'}) = n^{2m}\chi(\mathcal{O}_X). \tag{2}$$

It is straightforward that f is relatively minimal. In order to apply Theorem 1.2, we need to compute the genus of the fiber, which is just $p_a(C_i)$. By the genus formula,

$$g = p_a(C_i) = 1 + \frac{\omega_{X'/k}L' + L'^2}{2} = 1 + \frac{n^{2m-2}\beta + n^{2m-4}\alpha}{2}. \tag{3}$$

Also note that f is a fibration over $\mathbb{P}^1$. We have

$$\omega_f = \omega_{X''/k} \otimes \mathcal{O}_{X''}(2F),$$

where F is a general fiber of f. Hence

$$\omega_f^2 = \omega_{X''/k}^2 + 8(g-1). \tag{4}$$

Moreover, it follows that

$$h^0(\omega_f) \geq h^0(\omega_{X''/k}) = h^0(\omega_{X'/k}) \geq \chi(\mathcal{O}_{X'}) - 1 = n^{2m}\chi(\mathcal{O}_X) - 1. \tag{5}$$

In [Pa], in order to get a singular fiber and apply the slope inequality for such a fibration, C_1 and C_2 were chosen from $|2L'|$ and $C_1 = D_1 + D_2$ was singular, where $D_1, D_2 \in |L'|$. Here, Theorem 1.2 works for any fibration, so we do not need this.

4.3 Proof of Theorem 1.5

Now we reach the end of the proof. Theorem 1.2 for f tells us that

$$\omega_f^2 \geq (4 - \frac{4}{g})h^0(\omega_f) + (4g-4), \tag{6}$$

Combine (1), (2), (3), (4), (5) and (6) together. We finally get

$$n^{2m}\omega_{X/k}^2 - O(n^{2m-4}) \geq (4 - \frac{4}{O(n^{2m-2})})n^{2m}\chi(\mathcal{O}_X) + O(n^{2m-2}).$$

Because $m \geq 2$, the proof is completed by letting $n \to \infty$.

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