Modular Heights of Quaternionic Shimura Curves

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1 Introduction

The goal of this paper is to prove a formula expressing the modular height of a quaternionic Shimura curve over a totally real number field in terms of the logarithmic derivative of the Dedekind zeta function of the totally real number field. Our proof is based on the work Yuan–Zhang–Zhang [YZZ] on the Gross–Zagier formula, and the work Yuan–Zhang [YZ] on the averaged Colmez conjecture. All these works are in turn inspired by the Pioneering work Gross–Zagier [GZ] and some philosophies of Kudla's program.

In the following, let us state the exact formula, compare it with other similar formulas, and explain our idea of proof.

1.1 Modular height of the Shimura curve

Let F be a totally real number field. Let Σ be a finite set of places of F containing all the archimedean places and having an odd cardinality $|\Sigma|$. Denote by Σ_f the subset of non-archimedean places in Σ. Let $\mathbb{B}$ be the totally definite incoherent quaternion algebra over the adele ring $\mathbb{A} = \mathbb{A}_F$ with ramification set Σ. Let $U \subset \mathbb{B}_f^\times$ be a *maximal* open compact subgroup.

Let X_U be the associated *Shimura curve* over F, which is a projective and smooth curve over F descended from the analytic quotient

$$X_{U,\sigma}(\mathbb{C}) = (B(\sigma)^\times \backslash \mathcal{H}^\pm \times \mathbb{B}_f^\times / U) \cup \{\text{cusps}\},$$

where $\sigma : F \to \mathbb{C}$ is any archimedean place of F and $B(\sigma)$ is the quaternion algebra over F with ramification set $\Sigma \setminus \{\sigma\}$. Note that X_U is defined as the corresponding coarse moduli scheme, which is a projective and smooth curve over F. See [YZZ, §1.2.1] for more details.

Let L_U be the *Hodge bundle* of X_U corresponding to modular forms of weight 2. It is $\mathbb{Q}$-line bundle over X_U, i.e. an element of $\mathrm{Pic}(X_U) \otimes_{\mathbb{Z}} \mathbb{Q}$, defined by

$$L_U = \omega_{X_U/F} \otimes \mathcal{O}_{X_U}\Big(\sum_{Q \in X_U(\overline{F})} (1 - e_Q^{-1}) Q\Big).$$

Here $\omega_{X_U/F}$ be the canonical bundle of X_U over F, and for each $Q \in X_U(\overline{F})$, the ramification index e_Q is described as follows. If Q is a cusp, then $e_Q = \infty$ and $1 - e_Q^{-1} = 1$. If Q is not a cusp, the connected component of Q in $X_{U,\sigma}(\mathbb{C})$ can be written as a quotient $\Gamma \backslash \mathcal{H}^*$ for a discrete group Γ, then e_Q is the ramification index of any preimage of Q under the map $\mathcal{H} \to \Gamma \backslash \mathcal{H}$. One can check that e_Q does not depend on the choice of the preimage, and that e_Q is Galois invariant, so L_U is indeed defined over F. See [YZZ, §3.1.3] for more details.

Let $\mathcal{X}_U$ be the *canonical integral model* of X_U over O_F, as reviewed in [YZ, §4.2]. If $|\Sigma| = 1$ (or equivalently $F = \mathbb{Q}$ and $\Sigma = \{\infty\}$), then X_U is a modular curve, $X_U \simeq \mathbb{P}^1_{\mathbb{Q}}$ via the j-function, and $\mathcal{X}_U \simeq \mathbb{P}^1_{\mathbb{Z}}$ under this identification. We refer to Deligne–Rapoport [DR] for a thorough theory of this situation. If $|\Sigma| > 1$, $\mathcal{X}_U$ is a projective, flat, normal and $\mathbb{Q}$-factorial arithmetic surface over O_F, defined as quotients of the canonical integral models of $\mathcal{X}_{U'}$ for sufficiently small open compact subgroups U' of U. We refer to Carayol [Ca] and Boutot–Zink [BZ] for integral models for sufficiently small level groups in this case.

Let $\mathcal{L}_U$ be the *canonical integral model* of L_U over $\mathcal{X}_U$, as reviewed in [YZ, §4.2]. As a $\mathbb{Q}$-line bundle over $\mathcal{X}_U$,

$$\mathcal{L}_U = \omega_{\mathcal{X}_U/O_F} \otimes \mathcal{O}_{\mathcal{X}_U}\Big(\sum_{Q\in X_U} (1-e_Q^{-1})\mathcal{Q}\Big).$$

Here $\omega_{\mathcal{X}_U/O_F}$ is the relative dualizing sheaf, the summation is through closed points Q of X_U, $\mathcal{Q}$ is the Zariski closure of Q in $\mathcal{X}_U$, and e_Q is the ramification index of any point of $X_U(\overline{F})$ corresponding to Q.

At any archimedean place $\sigma: F\to \mathbb{C}$, the *Petersson metric* of $\mathcal{L}_U$ is given by

$$\|f(\tau)d\tau\|_{\mathrm{Pet}} = 2\,\mathrm{Im}(\tau)|f(\tau)|,$$

where τ is the standard coordinate function on $\mathcal{H}\subset\mathbb{C}$, and $f(\tau)$ is any meromorphic modular form of weight 2 over $X_{U,\sigma}(\mathbb{C})$. Thus we have *the arithmetic Hodge bundle*

$$\overline{\mathcal{L}}_U = (\mathcal{L}_U, \{\|\cdot\|_\sigma\}_\sigma).$$

The *modular height* of X_U with respect to the arithmetic Hodge bundle $\overline{\mathcal{L}}_U$ is defined to be

$$h_{\overline{\mathcal{L}}_U}(X_U) = \frac{\widehat{\deg}(\hat{c}_1(\overline{\mathcal{L}}_U)^2)}{2\deg(L_U)}.$$

Here $\deg(L_U)$ is the degree over the generic fiber X_U, and the numerator is the arithmetic self-intersection number on the arithmetic surface $\mathcal{X}_U$ in the setting of Arakelov geometry. Note that if $|\Sigma|>1$, then $\overline{\mathcal{L}}_U$ is a hermitian $\mathbb{Q}$-line bundle over $\mathcal{X}_U$, and the self-intersection number essentially follows from the theory of Gillet–Soulé [GS]; if $|\Sigma|=1$, then the metric has a logarithmic singularity along the cusp, and the intersection number is defined in the framework of Bost [Bo] or Kühn [Kuh].

For any non-archimedean place v of F, denote by N_v the norm of v. Recall the Dedekind zeta function

$$\zeta_F(s) = \prod_{v\nmid\infty}(1-N_v^{-s})^{-1}.$$

The functional equation switches the values and derivatives of $\zeta_F(s)$ between -1 and 2. The goal of this paper is to prove the following formula.

Theorem 1.1 (modular height)**.**

$$h_{\overline{\mathcal{L}}_U}(X_U) = -\frac{\zeta_F'(-1)}{\zeta_F(-1)} - \frac{1}{2}[F:\mathbb{Q}] + \sum_{v\in\Sigma_f}\frac{3N_v-1}{4(N_v-1)}\log N_v.$$

If $F=\mathbb{Q}$ and $\Sigma=\{\infty\}$, the formula was proved by Bost (un-published) and Kühn (cf. [Kuh, Theorem 6.1]); if $F=\mathbb{Q}$ and $|\Sigma|>1$, the formula was proved by Kudla–Rapoport–Yang (cf. [KRY2, Theorem 1.0.5]).

Denote by h_F the class number of O_F. A classical formula of Vignéras [Vi] gives

$$\deg(L_U) = 4 \cdot h_F \cdot (-2)^{-[F:\mathbb{Q}]} \cdot \zeta_F(-1) \cdot \prod_{v \in \Sigma_f} (N_v - 1).$$

This is also an easy consequence of the formula in the remark right after [YZZ, Proposition 4.2]. Theorem 1.1 is an arithmetic version of this formula. It computes the arithmetic degree instead of the geometric degree, and the result is given by the logarithmic derivative at -1 instead of the value at -1.

The relation between these two formulas is similar to the relation between the Gross–Zagier formula and the Waldspurger formula (as fully explored in [YZZ]), and is also similar to the relation between the averaged Colmez conjecture and the class number formula (as treated in [YZZ]).

In Kudla's program, it is crucial to extend the (modular) generating series of CM cycles over a Shimura variety to a (modular) generating series of arithmetic cycles over a reasonable integral model. An idea of S. Zhang [Zh2, §3.5] to treat this problem is to apply his notion of admissible arithmetic extensions. This approach relies on concrete results on arithmetic intersection numbers, so our main formula fits this setting naturally. Inspired by S. Zhang's idea, Qiu [Qi] solved the problem for generating series of divisors over unitary Shimura varieties under some assumptions, and his argument is based on many computational results of this paper.

1.2 The case $F = \mathbb{Q}$ and other similar formulas

If $F = \mathbb{Q}$ and $\Sigma = \{\infty\}$, or equivalently if X_U is the usual modular curve, then the formula of Bost and Kühn [Kuh, Theorem 6.1] agrees with our formula by [Yu2, Theorem 5.3, Remark 5.4].

If $F = \mathbb{Q}$ and $|\Sigma| > 1$, the formula in [KRY2, Theorem 1.0.5] of Kudla–Rapoport–Yang is equivalent to

$$h_{\varpi_0}(X_U) = -\frac{\zeta_{\mathbb{Q}}'(-1)}{\zeta_{\mathbb{Q}}(-1)} - \frac{1}{2} + \sum_{p \in \Sigma_f} \frac{p+1}{4(p-1)} \log p.$$

This formula is compatible with our formula. In fact, the right-hand side of the formula differs from that of ours by $\frac{1}{2}\log d_{\mathbb{B}}$, and $h_{\overline{\mathcal{L}}_U}(X_U) = h_{\varpi_0}(X_U) + \frac{1}{2}\log d_{\mathbb{B}}$ by the explicit results on the Kodaira–Spencer map in [Yu2, Theorem 2.2, Remark 2.3].

There are many formulas of similar flavor in the literature. Besides the above mentioned works of Bost, Kühn and Kudla–Rapoport–Yang, Bruinier–Burgos–Kühn [BBK] proved a modular height formula for Hilbert modular surfaces, Hörmann [Ho] proved a modular height formula up to $\log \mathbb{Q}_{>0}$ for Shimura varieties of orthogonal types over $\mathbb{Q}$, and Bruinier–Howard [BH] recently proved a modular height formula for Shimura varieties of unitary types over $\mathbb{Q}$. The formulas of [Ho, BH] are based on the formulas of Bost, Kühn and Kudla–Rapoport–Yang.

In a slightly different direction, Freixas–Sankaran [FS] proved some other formulas for intersections of more general Chern classes over Hilbert modular surfaces. Finally, we refer to Maillot–Rössler [MR1, MR2] for far-reaching conjectures generalizing these formulas.

Our formula is primitive in that it involves Dedekind zeta functions of general totally real fields, while the above known formulas involve Dedekind zeta functions of $\mathbb{Q}$ and quadratic fields.

1.3 Modular height of a CM point

Our proof of Theorem 1.1 is inspired by the works [YZZ, YZ]. In the proof, we need to pick an auxiliary CM point, and the height of this point is also relevant to our treatment. Let us first review a formula in [YZ] which is related to our main theorem.

Let E be a totally imaginary quadratic extension over F. Assume that there is an embedding $\mathbb{A}_E \hookrightarrow \mathbb{B}$ of $\mathbb{A}$-algebras such that the image of $\widehat{O}_E^\times$ lies in the maximal compact subgroup U.

Let $P_U \in X_U(E^{\mathrm{ab}})$ be the CM point represented by $[\tau_0, 1]$ under the complex uniformization, where τ_0 is the unique fixed point of $E^\times$ in $\mathcal{H}$. Its modular height is defined by

$$h_{\overline{\mathcal{L}}_U}(P_U) \coloneqq \frac{1}{\deg(P_U)} \widehat{\deg}(\overline{\mathcal{L}}_U|_{\bar{P}_U}),$$

where $\bar{P}_U$ denotes the Zariski closure of the image of P_U in $\mathcal{X}_U$, and $\deg(P_U)$ is the degree of the field of definition of of P_U over F. By [YZ, Theorem 1.7], we have the following formula.

Theorem 1.2. *Assume that there is no non-archimedean place of F ramified in both E and $\mathbb{B}$. Then*

$$h_{\overline{\mathcal{L}}_U}(P_U) = -\frac{L_f'(0,\eta)}{L_f(0,\eta)} + \frac{1}{2} \log \frac{d_{\mathbb{B}}}{d_{E/F}}.$$

Here $d_{\mathbb{B}} = \prod_{v \in \Sigma_f} N_v$ is the absolute discriminant of $\mathbb{B}$, and $d_{E/F}$ is the norm of the relative discriminant of E/F.

Theorem 1.2 is one of the two steps in the proof of the averaged Colmez conjecture of [YZ]. The averaged Colmez conjecture was proved independently by Andreatta–Goren–Howard–Madapusi-Pera [AGHM], and plays a crucial role in the final solution of the André–Oort conjecture of Tsimerman [Ts].

In the case $F = \mathbb{Q}$ and $\Sigma = \{\infty\}$, Theorem 1.2 is equivalent to the classical Chowla–Selberg formula proved in [CS]. We refer to [Yu1, §3.3] for many equivalent forms of the Chowla–Selberg formula.

1.4 Kronecker's limit formula

Both the Bost–Kühn formula and the Chowla–Selberg formula are easy consequences of the more classical Kronecker limit formula.

In fact, by [Kuh, Prop. 5.2], the Kronecker limit formula asserts that

$$-\log|\Delta(\tau)^2\mathrm{Im}(\tau)^{12}| = 4\pi\lim_{s\to1}(E(\tau,s)-\varphi(s)),$$

where

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24},\quad q=e^{2\pi i\tau}$$

is the modular discriminant function,

$$E(\tau,s)=\frac{1}{2}\sum_{c,d\in\mathbb{Z},\ \gcd(c,d)=1}\frac{\mathrm{Im}(\tau)^s}{|c\tau+d|^{2s}}$$

is the classical non-holomorphic Eisenstein series, and

$$\frac{\pi}{3}\varphi(s)=\frac{1}{s-1}+2-2\log(4\pi)-24\zeta_{\mathbb{Q}}'(-1)+O(s-1).$$

In particular, $\Delta(\tau)$ induces a global section of $L_U^{\otimes 6}$ over the modular curve $X_U = X_0(1)$. Then we can use this section to compute $h_{\overline{\mathcal{L}}_U}(X_U)$ and $h_{\overline{\mathcal{L}}_U}(P_U)$.

Integrating $-\log|\Delta(\tau)^2\mathrm{Im}(\tau)^{12}|$ over $X_U(\mathbb{C})$ with respect to the Poincare measure $y^{-2}dxdy$, the Kronecker limit formula implies the Bost–Kühn formula. This is essentially the proof of Kühn [Kuh].

Averaging $-\log|\Delta(\tau)^2\mathrm{Im}(\tau)^{12}|$ over the Galois orbit of the CM point P_U, the Kronecker limit formula implies the Chowla–Selberg formula. This is essentially the proof in Weil [We].

In summary, in the case $|\Sigma|=1$, both Theorem 1.1 and Theorem 1.2 are consequences of the Kronecker limit formula.

On the other hand, there is no analogous formulation of the Kronecker limit formula over totally real fields, since there is no explicit modular form over a quaternionic Shimura curve to replace the classical modular discriminant function Δ. Hence, the above proof of the theorem does not work in the general case.

Our proofs of Theorem 1.1 and Theorem 1.2 are extensions of the treatment of [YZZ]. The original goal of [YZZ] is to prove the Gross–Zagier formula over Shimura curves, but the method was enhanced in [YZ] to prove Theorem 1.2, and now we can further enhance the method to prove Theorem 1.1. Note that our proof of Theorem 1.1 in the case $F=\mathbb{Q}$ is different from those of [Kuh, KRY2].

It is interesting that in both the classical proofs and our current proofs, Theorem 1.1 and Theorem 1.2 are always put in the same framework.

1.5 Idea of proof

Now we sketch our proof of Theorem 1.1. It is an extension of the proof of the Gross–Zagier formula in [YZZ] and the proof of the averaged Colmez conjecture in [YZ]. To have a setup compatible with those in [YZZ, YZ], we first choose a CM extension E over F as in Theorem 1.2, though E is irrelevant to the final statement of Theorem 1.1.

The degeneracy assumptions

Recall that the Gross–Zagier formula is an identity between the derivative of the Rankin–Selberg L-function of a Hilbert modular form and the height of a CM point on a modular abelian variety. This formula is proved by a comparison of a derivative series $\mathcal{P}rI'(0,g,\phi)$ with a geometric series $2Z(g,(1,1),\phi)$ parametrized by certain modified Schwartz function $\phi\in\overline{\mathcal{S}}(\mathbb{B}\times\mathbb{A}^\times)$. More precisely, we have proved that the difference

$$\mathcal{D}(g,\phi)=\mathcal{P}rI'(0,g,\phi)-2Z(g,(1,1),\phi),\qquad g\in\mathrm{GL}_2(\mathbb{A}_F)$$

is perpendicular to the relevant cusp form.

The matching for the "main terms" of $\mathcal{D}(g,\phi)$ eventually implies the Gross–Zagier formula in [YZZ]. In this process, many assumptions on the choice of ϕ in [YZZ, §5.2.1] are made to "annihilate" the "degenerate terms", which simplifies the calculations dramatically and forces the computational results to satisfy the conditions of an approximation argument. The "strictest" degeneracy assumptions involved are [YZZ, Assumption 5.3, Assumption 5.4]. The assumptions are not harmful for the Gross–Zagier formula, as proved in [YZZ, Theorem 5.7].

Nonetheless, if we allow the Schwartz function to be more general, the matching process will actually give us more formulas. In fact, after removing [YZZ, Assumption 5.3], we obtain a matching of some "degenerate terms", which eventually implies Theorem 1.2. This is the work of [YZ, Part II].

In the current paper, we remove both [YZZ, Assumption 5.3, Assumption 5.4] when considering the matching of the series $\mathcal{P}rI'(0,g,\phi)$ and $2Z(g,(1,1),\phi)$. Then we finally obtain an extra identity, which eventually implies Theorem 1.1. Our precise choice of the Schwartz functions is given in §3.2.

From [YZZ] to [YZ], and from [YZ] to the current paper, each step removes a degeneracy assumption, which causes two significant problems. The first problem is that more terms appear in the comparison, which incur far more involved local computations. This is eventually overcome by patience and carefulness. The second problem is how to obtain exact identity from the "partial matching" of the two series; i.e., the matching of "all but finitely many" terms of the two series. In [YZZ], this problem is solved by the method of approximation (cf. [YZZ, §1.5.10]). In [YZ], this problem is solved by the theory of pseudo-theta series (cf. [YZ, §6]), which is an extension of the method of approximation. In the current paper, the theory of pseudo-theta series is not sufficient for the comparison. Our solution is to introduce a new notion of *pseudo-Eisenstein series*, and generalize [YZ, Lemma 6.1], the key matching principle of pseudo-theta series, to include both pseudo-theta series and pseudo-Eisenstein series.

In the following, we review the derivative series $\mathcal{P}rI'(0,g,\phi)$ and the height series $Z(g,(1,1),\phi)$ and introduce some new ingredients of our proof.

Derivative series

By the reduced norm q, the incoherent quaternion algebra $\mathbb{B}$ is viewed as a quadratic space over $\mathbb{A}=\mathbb{A}_F$. Then we have a modified space $\overline{\mathcal{S}}(\mathbb{B}\times\mathbb{A}^\times)$ of Schwartz functions with a Weil representation r by $\mathrm{GL}_2(\mathbb{A}) \times \mathbb{B}^\times \times \mathbb{B}^\times$. For each $\phi \in \overline{\mathcal{S}}(\mathbb{B} \times \mathbb{A}^\times)$ invariant under an open compact subgroup $U \times U$ of $\mathbb{B}_f^\times \times \mathbb{B}_f^\times$, we have a mixed theta–Eisenstein series

$$I(s,g,\phi) = \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{\gamma \in P^1(F)\backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s \sum_{x_1 \in E} r(\gamma g)\phi(x_1,u),$$

where $\mu_U = F^\times \cap U$, and P^1 is the upper triangular subgroup of SL_2.

The derivative $I'(0,g,\phi)$ of $I(s,g,\phi)$ at $s=0$ is an automorphic form in $g \in \mathrm{GL}_2(\mathbb{A})$. Let $\mathcal{P}rI'(0,g,\phi)$ be the *holomorphic projection* of the derivative $I'(0,g,\phi)$. This holomorphic projection is just the orthogonal projection from the space of automorphic forms to the space of cuspidal and holomorphic automorphic forms of parallel weight two with respect to the Petersson inner product.

In §3, we decompose $\mathcal{P}rI'(0,g,\phi)$ into a sum of "local terms", and compute all the relevant local components. Most of the terms are computed in [YZZ, YZ]. However, as in §3.1, a new extra term $\mathcal{P}r'\mathcal{J}'(0,g,\phi)$ appears in the expression of $\mathcal{P}rI'(0,g,\phi)$. This term comes from the overly fast growth of $I'(0,g,\phi)$ in the computation of the holomorphic projection. It was zero under [YZZ, Assumption 5.4], but its non-vanishing is crucial to the treatment here. The extra term $\mathcal{P}r'\mathcal{J}'(0,g,\phi)$ is computed in Proposition 3.2, and its local component computed in Lemma 3.4(1) gives $\zeta_v'(2)/\zeta_v(2)$ at almost all places v. The sum over all places gives the global logarithmic derivative $\zeta_F'(2)/\zeta_F(2)$, which is the main term on the right-hand side of Theorem 1.1.

Height series

For any $\phi \in \overline{\mathcal{S}}(\mathbb{B} \times \mathbb{A}^\times)$ invariant under $U \times U$, we have a generating series of Hecke operators on the Shimura curve X_U:

$$Z(g,\phi)_U = Z_0(g,\phi) + w_U \sum_{a \in F^\times} \sum_{x \in U\backslash \mathbb{B}_f^\times/U} r(g)\phi(x, aq(x)^{-1}) Z(x)_U,$$

where $w_U = |\{\pm 1\} \cap U|$ and every $Z(x)_U$ is a divisor of $X_U \times X_U$ associated to the Hecke operator corresponding to the double coset UxU. The constant term $Z_0(g,\phi)$ does not play any essential role in this paper, and we denote by $Z_*(g,\phi)$ the sum of the other terms. By [YZZ, Theorem 3.17], this series is absolutely convergent and defines an automorphic form in $g \in \mathrm{GL}_2(\mathbb{A})$ with coefficients in $\mathrm{Pic}(X_U \times X_U)_{\mathbb{C}}$.

Recall that $P_U \in X_U(E^{\mathrm{ab}})$ is the CM point represented by $[\tau_0, 1]$ under the complex uniformization, where τ_0 is the unique fixed point of $E^\times$ in the upper half plane $\mathcal{H}$. More generally, we have a CM point $t = [\tau_0, t]$ for any $t \in E^\times(\mathbb{A}_f)$. Let $t^\circ = t - \xi_t$ be the divisor in $\mathrm{Pic}(X_{U,\bar F}) \otimes_{\mathbb{Z}} \mathbb{Q}$ of degree zero on every connected component. Here the normalized Hodge class $\xi_t = \dfrac{1}{\deg(L_{U,t})} L_{U,t}$, where $L_{U,t}$ is the restriction of the Hodge bundle L_U to the connected component of $X_{U,\bar F}$ containing t. Then we can form a height series

$$Z(g,(t_1,t_2),\phi) = \langle Z(g,\phi)_U t_1^\circ,\ t_2^\circ \rangle_{\mathrm{NT}},$$

where the right-hand side is the Neron–Tate height pairing.

In §4, we decompose the height series $Z(g,(t_1,t_2),\phi)$ into a sum of "local terms", and compute all the relevant local components. The starting point is the decomposition

$$Z(g,(t_1,t_2)) = \langle Z_*(g,\phi)t_1,t_2\rangle - \langle Z_*(g,\phi)\xi_{t_1},t_2\rangle + \langle Z_*(g,\phi)\xi_{t_1},\xi_{t_2}\rangle - \langle Z_*(g,\phi)t_1,\xi_{t_2}\rangle.$$

The first term is computed in [YZZ, YZ]. The remaining three terms are further computed in §4.5. These three terms are zero under [YZZ, Assumption 5.4], but their non-vanishing is crucial to the treatment here. In particular, by Proposition 4.7, the term $\langle Z_*(g,\phi)\xi_{t_1},\xi_{t_2}\rangle$ is equal to an Eisenstein series times $\langle\xi_{t_2},\xi_{t_2}\rangle$, and thus it is an easy multiple of $h_{\overline{\mathcal{L}}_U}(X_U)$. This gives the main term on the left-hand side of Theorem 1.1.

Pseudo-Eisenstein series

The notion of pseudo-Eisenstein series is parallel to that of pseudo-theta series of [YZ]. To illustrate the idea, we sketch the idea of both notions for SL_2, while those for GL_2, which are the ones we really need, can be introduced similarly.

Let (V,q) be a quadratic space over a totally real number field F, assumed to be even-dimensional for simplicity. Let $\phi\in\mathcal{S}(V(\mathbb{A}))$ be a Schwartz function. Then we have an action of $g\in\mathrm{SL}_2(\mathbb{A})$ on ϕ via the Weil representation.

Start with the theta series

$$\theta(g,\phi) = \sum_{x\in V} r(g)\phi(x),\qquad g\in\mathrm{SL}_2(\mathbb{A}).$$

Let S be a finite set of non-archimedean places of F. In $r(g)\phi(x) = r(g_S)\phi_S(x)r(g^S)\phi^S(x)$, if we replace $r(g_S)\phi_S(x)$ by a locally constant function $\phi'_S(g,x)$ of $(g,x)\in\mathrm{GL}_2(F_S)\times V(F_S)$, then we obtain a *pseudo-theta series*

$$A^{(S)}_{\phi'}(g) = \sum_{x\in V}\phi'_S(g,x)r(g)\phi^S(x),\qquad g\in\mathrm{SL}_2(\mathbb{A}).$$

Note that $A^{(S)}_{\phi'}$ is not automorphic in general. More general types of pseudo-theta series are introduced in [YZ, §6].

We say that the pseudo-theta series $A^{(S)}_{\phi'}(g)$ is *non-singular* if $\phi'_S(1,x)$ (for $g=1$) is actually a Schwarz function of $x\in V(F_S)$. In this case, we form a true theta series

$$\theta_{A^{(S)}}(g) = \sum_{x\in V} r(g)\phi'_S(1,x)r(g)\phi^S(x),\qquad g\in\mathrm{SL}_2(\mathbb{A}).$$

It is automorphic and approximates the original series in the sense that $A^{(S)}_{\phi'}(g) = \theta_{A^{(S)}}(g)$ as long as $g_S = 1$.

Now we start with the Siegel–Eisenstein series

$$E(s,g,\phi) = \sum_{\gamma\in P^1(F)\backslash\mathrm{SL}_2(F)}\delta(\gamma g)^s r(\gamma g)\phi(0),\quad g\in\mathrm{SL}_2(\mathbb{A}).$$

The non-constant part of $E(s,g,\phi)$ has a Fourier expansion

$$E_*(s,g,\phi) = \sum_{a\in F^\times} W_a(s,g,\phi),$$

where the Whittaker function is defined by

$$W_a(s,g,\phi) = \int_{\mathbb{A}} \delta(wn(b)g)^s\ r(wn(b)g)\phi(0)\psi(-ab)db, \quad a\in F.$$

We define the local Whittaker functions similarly. For our purpose, we only care about the behavior at $s=0$. Let S be a finite set of non-archimedean places of F. In $W_a(0,g,\phi) = W_{a,S}(0,g,\phi_S)W_a^S(0,g,\phi^S)$, if we replace $W_{a,S}(0,g,\phi_S)$ by a locally constant function $B_{a,S}(g)$ of $(a,g)\in F_S^\times \times \mathrm{SL}_2(F_S)$, then we obtain a *pseudo-Eisenstein series*

$$B_\phi^{(S)}(g) = \sum_{a\in F^\times} B_{a,S}(g) W_a^S(0,g,\phi^S), \qquad g\in \mathrm{SL}_2(\mathbb{A}).$$

Pseudo-Eisenstein series arise naturally in derivatives of Eisenstein series. In fact, the derivative of $E_*(s,g,\phi)$ at $s=0$ is

$$E_*'(0,g,\phi) = \sum_{a\in F^\times}\sum_v W_{a,v}'(0,g,\phi)W_a^v(0,g,\phi^v).$$

For every non-archimedean v, the "v-part"

$$\sum_{a\in F^\times} W_{a,v}'(0,g,\phi)W_a^v(0,g,\phi^v)$$

is a pseudo-Eisenstein series.

We say that the pseudo-Eisenstein series $B_\phi^{(S)}(g)$ is *non-singular* if for every $v\in S$, there exist $\phi_v^+\in\mathcal{S}(V_v^+)$ and $\phi_v^-\in\mathcal{S}(V_v^-)$ such that

$$B_{a,v}(1) = W_{a,v}(0,1,\phi_v^+) + W_{a,v}(0,1,\phi_v^-), \quad \forall a\in F_v^\times.$$

Here $\{V_v^+, V_v^-\}$ is the set of (one or two) quadratic spaces over F_v with the same dimension and the same discriminant as V_v. In this case, we form a linear combination of true Eisenstein series

$$E_{B^{(S)}}(s,g) = \sum_{\epsilon:S\to\{\pm\}} E(s,g,\phi_S^\epsilon\otimes\phi^S), \qquad g\in\mathrm{SL}_2(\mathbb{A}).$$

It approximates the original series in the sense that $B_{a,S}(g)$ is equal to the non-constant part of $E_{B^{(S)}}(0,g)$ as long as $g_S=1$.

The key result in this theory is Lemma 2.2, as an extension of [YZ, Lemma 6.1]. It asserts that if an automorphic form is equal to a finite linear combination of non-singular pseudo-theta series and non-singular pseudo-Eisenstein series, then it is actually equal to the finite linear combination of the corresponding theta series and Eisenstein series.

The comparison

Go back to the difference

$$\mathcal{D}(g,\phi) = \mathcal{P}rI'(0,g,\phi) - 2Z(g,(1,1),\phi), \qquad g\in \mathrm{GL}_2(\mathbb{A}_F).$$

By the computational result of §3 and §4, we eventually see that $\mathcal{D}(g,\phi)$ is a finite linear combination of non-singular pseudo-theta series and non-singular pseudo-Eisenstein series.

By Lemma 2.2, $\mathcal{D}(g,\phi)$ is actually equal to the finite linear combination of the corresponding theta series and Eisenstein series. Note that $\mathcal{D}(g,\phi)$ is cuspidal, so the linear combination of the corresponding constant terms is zero. This gives a nontrivial relation involving the major terms of Theorem 1.1. It suffices to take g to be a specific matrix to make the relation precise. Take $g=(g_v)_v\in \mathrm{GL}_2(\mathbb{A})$ with $g_v=1$ for $v\notin\Sigma_f$ and $g_v=w$ for $v\in\Sigma_f$. After explicit computation, the nontrivial relation becomes

$$d_0 \sum_{u\in\mu_U^2\backslash F^\times} r(g)\phi(0,u)=0.$$

Here d_0 is the difference of two sides of Theorem 1.1. This proves the theorem.

Note that if we take $g=1$, then the nontrivial relation becomes $0=0$, since $\phi_v(0,u)=0$ for any $v\in\Sigma_f$ by our choice $\phi_v=1_{O_{\mathbb{B}_v}^\times\times O_{F_v}^\times}$ in §3.2. As $r(w)\phi_v(0,u)\neq 0$, we choose g_v to be w for $v\in\Sigma_f$ instead. This serves the purpose, but incurs more computations about evaluating $g_v=w$ and about averaging of many local terms.

1.6 Notations and conventions

Most of the notations of this paper are compatible with those in [YZZ, YZ]. The basic notations are as in [YZZ, §1.6]. In particular, we normalize the character $\psi=\oplus_v\psi_v : F\backslash\mathbb{A}_F\to\mathbb{C}^\times$, based on which we introduce the Weil representation, and choose a precise Haar measure on each relevant algebraic group locally everywhere.

The following are all the conventions of this paper that are different from those of [YZZ, YZ], while only (3) is a major difference which brings extra computations.

(1) The Petersson metric on $\mathcal{L}_U$ is defined by $\|d\tau\|_{\mathrm{Pet}}=2\,\mathrm{Im}(\tau)$ in [YZ] and the current paper, while it is defined by $\|d\tau\|_{\mathrm{Pet}}=4\pi\,\mathrm{Im}(\tau)$ in [YZZ]. This discrepancy does not affect our applying results of [YZZ], since only the curvature form of the Petersson metric is crucial in [YZZ].

(2) In the current paper, $\zeta_F(s)$ denotes the usual Dedekind zeta function (without Gamma factors), and $L(s,\eta)$ denotes the completed L-function (with Gamma factors) of the quadratic character η. In [YZZ, YZ], both $\zeta_F(s)$ and $L(s,\eta)$ denote the completed L-functions (with Gamma factors).

(3) Our choice of (U,ϕ) in §3.2 is different from those in [YZZ, YZ] due to the dropping of the degeneracy assumptions. Moreover, [YZ] and the current paper assume that U is maximal compact, while [YZZ] does not. We will mention this difference and its effect from time to time.

(4) This paper and [YZZ] do not assume $|\Sigma| > 1$, while [YZ] assumes $|\Sigma| > 1$. Most results of [YZ] actually hold in the case $|\Sigma| = 1$.

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2 Pseudo-Eisenstein series

In [YZ, §6], the notion of pseudo-theta series is introduced, and its crucial property in [YZ, Lemma 6.1] is the key to get a clean identity from the matching of the major terms. The goal of this section is to introduce a notion of pseudo-Eisenstein series and extend [YZ, Lemma 6.1] to a result including both pseudo-theta series and pseudo-Eisenstein series.

Throughout this section, let F be a totally real number field, and $\mathbb{A}$ the adele ring of F. We will use the terminologies of [YZ, §6.1] freely.

2.1 Theta series and Eisenstein series

We will first recall the notations of theta series and Eisenstein series following [YZ, §6.1].

Theta series

Let (V, q) be a positive definite quadratic space over a totally real number field F. Let

$$\overline{\mathcal{S}}(V(\mathbb{A}) \times \mathbb{A}^\times) = \otimes_v \overline{\mathcal{S}}(V(F_v) \times F_v^\times)$$

be the space of Schwartz functions introduced in [YZZ, §2.1, §4.1]. Assume that $\dim V$ is even in the following, which is always satisfied in our application.

In [YZZ, §2.1.3], the Weil representation on the usual space $\mathcal{S}(V(\mathbb{A}))$ is extended to a representation of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GO}(V(\mathbb{A}))$ on $\overline{\mathcal{S}}(V(\mathbb{A}) \times \mathbb{A}^\times)$. Note that the actions of $\mathrm{GL}_2(\mathbb{A})$ and $\mathrm{GO}(V(\mathbb{A}))$ commute with each other. This extension is originally from Waldspurger [Wa].

Take any $\phi \in \overline{\mathcal{S}}(V(\mathbb{A}) \times \mathbb{A}^\times)$. There is the partial theta series

$$\theta(g, u, \phi) = \sum_{x \in V} r(g)\phi(x, u), \quad g \in \mathrm{GL}_2(\mathbb{A}), \ u \in \mathbb{A}^\times.$$

If $u \in F^\times$, it is invariant under the left multiplication of $\mathrm{SL}_2(F)$ on g.

To get an automorphic form on $\mathrm{GL}_2(\mathbb{A})$, we define

$$\theta(g,\phi)_K = \sum_{u\in\mu_K^2\backslash F^\times} \theta(g,u,\phi) = \sum_{u\in\mu_K^2\backslash F^\times}\sum_{x\in V} r(g)\phi(x,u), \quad g\in \mathrm{GL}_2(\mathbb{A}).$$

Here $\mu_K = F^\times \cap K$, and K is any open compact subgroup of $\mathrm{GO}(\mathbb{A}_f)$ such that ϕ_f is invariant under the action of K by the Weil representation. The summation is well-defined and absolutely convergent. The result $\theta(g,\phi)_K$ is an automorphic form in $g\in\mathrm{GL}_2(\mathbb{A})$, and $\theta(g,r(h)\phi)_K$ is an automorphic form in $(g,h)\in\mathrm{GL}_2(\mathbb{A})\times\mathrm{GO}(\mathbb{A})$. See [YZZ, §4.1.3] for more details.

Furthermore, if the infinite component ϕ_∞ is standard, i.e., for any archimedean place v,

$$\phi_v(x,u) = \begin{cases} e^{-2\pi u q(x)}, & u>0,\\ 0, & u<0,\end{cases}$$

then $\theta(g,\phi)_K$ is holomorphic of parallel weight $\frac{1}{2}\dim V$.

Eisenstein series

In the above setting of $\phi\in\overline{\mathcal{S}}(V(\mathbb{A})\times\mathbb{A}^\times)$ for a quadratic space (V,q) over F, we can define an Eisenstein series $E(s,g,\phi)$. Then $E(s,g,\phi)$ and $\theta(g,\phi)$ are related by the Siegel–Weil formula. On the other hand, we also have Eisenstein series associated to incoherent quadratic collections in the sense of Kudla [Kud]. For convenience, we introduce the notion of adelic quadratic spaces to include both cases by dropping the last condition of [Kud, Definition 2.1].

A collection $\{(\mathbb{V}_v,q_v)\}_v$ of quadratic spaces $(\mathbb{V}_v,q_v)$ over F_v indexed by the set of places v of F is called *adelic* if it satisfies the following conditions:

(1) There is a quadratic space (V,q_0) over F such that there is an isomorphism $(V(F_v),q_0)\to(\mathbb{V}_v,q)$ for almost all places v;

(2) For any place v of F, the quadratic spaces $(V(F_v),q_0)$ and $(\mathbb{V}_v,q)$ have the same dimension and the same discriminant.

In that case, we obtain a quadratic space

$$(\mathbb{V},q) \coloneqq \otimes_v(\mathbb{V}_v,q_v)$$

over $\mathbb{A}$. Here the restricted product makes sense by condition (1). We call $(\mathbb{V},q)$ an *adelic quadratic space over* $\mathbb{A}$. The dimension $\dim\mathbb{V}\in\mathbb{Z}$ and the quadratic character $\chi_{(\mathbb{V},q)}: F^\times\backslash\mathbb{A}^\times\to\mathbb{C}^\times$ are defined to be those of (V,q). Its Hasse invariant is defined to be

$$\epsilon(\mathbb{V},q)\coloneqq\prod_v \epsilon(\mathbb{V}_v,q_v).$$

We say that the adelic quadratic space $(\mathbb{V},q)$ is *coherent* (resp. *incoherent*) if $\epsilon(\mathbb{V},q)=1$ (resp. $\epsilon(\mathbb{V},q)=-1$). Note that $(\mathbb{V},q)$ is coherent if it is isomorphic to $(V(\mathbb{A}),q_0)$ for some quadratic space (V,q_0) over F.

Let $(\mathbb{V}, q)$ be an adelic quadratic space over $\mathbb{A}$ which is positive definite at all archimedean places. For simplicity, we still assume that $\dim \mathbb{V}$ is even. The Weil representation of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GO}(\mathbb{V})$ on $\overline{\mathcal{S}}(\mathbb{V} \times \mathbb{A}^\times)$ is defined by local products as in the coherent case.

Let $\phi \in \overline{\mathcal{S}}(\mathbb{V} \times \mathbb{A}^\times)$ be a Schwartz function. Recall the associated partial Siegel Eisenstein series

$$\begin{aligned} E(s,g,u,\phi) &= \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s r(\gamma g)\phi(0,u) \\ &= \sum_{\gamma \in P^1(F)\backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s r(\gamma g)\phi(0,u), \quad g \in \mathrm{GL}_2(\mathbb{A}),\ u \in \mathbb{A}^\times. \end{aligned}$$

Here P^1 (resp. P) denotes the algebraic subgroup of upper triangle matrices in SL_2 (resp. GL_2), and δ is the standard modulus function as in [YZZ, §1.6.6]. If $u \in F^\times$, it is invariant under the left multiplication of $\mathrm{SL}_2(F)$ on g, and it has a meromorphic continuation to $s \in \mathbb{C}$ and a functional equation with center $s = 1 - \frac{\dim V}{2}$.

To get an automorphic form on $\mathrm{GL}_2(\mathbb{A})$, we define

$$E(s,g,\phi)_K = \sum_{u \in \mu_K^2 \backslash F^\times} E(s,g,u,\phi), \quad g \in \mathrm{GL}_2(\mathbb{A}).$$

Here as before, $\mu_K = F^\times \cap K$, and K is any open compact subgroup of $\mathrm{GO}(\mathbb{A}_f)$ such that ϕ_f is invariant under the action of K by the Weil representation. It is easy to see that $E(s,g,\phi)_K$ is invariant under the left multiplication of $\mathrm{GL}_2(F)$ on g. See [YZZ, §4.1.4].

The Eisenstein series $E(s,g,u,\phi)$ has the standard Fourier expansion

$$E(s,g,u,\phi) = \delta(g)^s r(g)\phi(0,u) + \sum_{a \in F} W_a(s,g,u,\phi).$$

Here the Whittaker function is given by

$$W_a(s,g,u,\phi) = \int_{\mathbb{A}} \delta(wn(b)g)^s\ r(wn(b)g)\phi(0,u)\psi(-ab)db, \quad a \in F,\ u \in F^\times.$$

We also have the constant term

$$E_0(s,g,u,\phi) = \delta(g)^s r(g)\phi(0,u) + W_0(s,g,u).$$

For each place v of F and any $\phi_v \in \overline{\mathcal{S}}(\mathbb{V}_v \times F_v^\times)$, we also introduce the local Whittaker function

$$W_{a,v}(s,g,u,\phi_v) = \int_{F_v} \delta(wn(b)g)^s\ r(wn(b)g)\phi_v(0,u)\psi_v(-ab)db, \quad a \in F_v,\ u \in F_v^\times.$$

If $(\mathbb{V}, q)$ is coherent, then we can express $E(0,g,\phi)$ and $E(0,g,u,\phi)$ in terms of the theta series by the Siegel–Weil formula (in most convergent cases).

If $(\mathbb{V}, q)$ is incoherent, then there is no theta series available. However, we can still express $W_{a,v}(0,g,u,\phi_v)$ in terms of certain average of the Schwartz function ϕ_v. See the local Siegel–Weil formula in [YZZ, Theorem 2.2]. See also the examples of incoherent Eisenstein series (for SL_2 or $\widetilde{\mathrm{SL}}_2$) in [YZZ, §2.5].

2.2 Pseudo-Eisenstein series

Here we introduce the notion of pseudo-Eisenstein series, which is parallel to the notion of pseudo-theta series in [YZ, §6.2]. Note that the term "pseudo-Eisenstein series" is also used in the literature as an unrelated terminology.

Definition

Let (V, q) be an even-dimensional quadratic space over a totally real number field F, positive definite at all archimedean places. Let S be a fixed finite set of non-archimedean place of F, and

$$\phi^S = \otimes_{w\notin S}\phi_w \in \overline{\mathcal{S}}(V(\mathbb{A}^S) \times \mathbb{A}^{S,\times})$$

be a Schwartz function with standard archimedean components. A *pseudo-Eisenstein series* is a series of the form

$$B_\phi^{(S)}(g) = \sum_{u\in\mu^2\backslash F^\times} \sum_{a\in F^\times} B_{a,S}(g,u) W_a^S(0,g,u,\phi^S), \qquad g \in \mathrm{GL}_2(\mathbb{A}).$$

We explain the notations as follows:

- $W_a^S(0,g,u,\phi^S) = \prod_{w\notin S} W_{a,w}(0,g,u,\phi_w)$ is the product of the local Whittaker functions defined before.
- $B_{a,S}(g,u) = \prod_{v\in S} B_{a,v}(g,u)$ is the product of the local terms.
- For any $v \in S$, the function

  $$B_{\bullet,v}(\bullet,\bullet) : F_v^\times \times \mathrm{GL}_2(F_v) \times F_v^\times \to \mathbb{C}$$

  is locally constant. It is smooth in g in the sense that there is an open compact subgroup K_v of $\mathrm{GL}_2(F_v)$ such that

  $$B_{a,v}(g\kappa,u) = B_{a,v}(g,u), \quad \forall (a,g,u) \in F_v^\times \times \mathrm{GL}_2(F_v) \times F_v^\times,\ \kappa \in K_v.$$

  It is compactly supported in u in the sense that there is a compact subset D_g of $F_v^\times$ depending on g (but independent of a) such that $B_{a,v}(g,u) = 0$ for any (a,g,u) with $u \notin D_g$.
- μ is a subgroup of $O_F^\times$ of finite index which acts trivially on the variable u of $B_{a,v}(g,u)$ and $W_{a,w}(0,g,u,\phi_w)$ for every non-archimedean $w \notin S$ and $v \in S$.
- For any $g \in \mathrm{GL}_2(\mathbb{A})$, the double sum is absolutely convergent.

Note that $B_\phi^{(S)}(g)$ does not have a "constant term" in the sense that the summation is over $a \in F^\times$.

Example 2.1. Let $(\mathbb{V}, q)$ be an adelic quadratic space over $\mathbb{A}$ which is positive definite at infinity, and $\phi \in \overline{\mathcal{S}}(\mathbb{V} \times \mathbb{A}^\times)$ be a Schwartz function which is standard at infinity. Consider the non-constant part

$$E_*(s, g, \phi) = \sum_{u \in \mu_K^2 \backslash F^\times} \sum_{a \in F^\times} W_a(s, g, u, \phi).$$

Its derivative at $s = 0$ is

$$E'_*(0, g, \phi) = \sum_{u \in \mu_K^2 \backslash F^\times} \sum_{a \in F^\times} \sum_{v} W'_{a,v}(0, g, u, \phi) W_a^v(0, g, u, \phi^v).$$

For every non-archimedean v, the "v-part"

$$\sum_{u \in \mu_K^2 \backslash F^\times} \sum_{a \in F^\times} W'_{a,v}(0, g, u, \phi) W_a^v(0, g, u, \phi^v)$$

is a pseudo-Eisenstein series if it is absolutely convergent. For archimedean v, the "v-part" is not a pseudo-Eisenstein series by our definition, but a holomorphic projection will convert it to a multiple of $E_*(0, g, \phi)$.

Non-singular pseudo-Eisenstein series

Let $B_\phi^{(S)}(g)$ be the pseudo-Eisenstein series associated to (V, q) as above. For every $v \in S$, there are one or two quadratic spaces over F_v up to isomorphism with the same dimension and the same discriminant as $(V(F_v), q)$. Order them by (V_v^+, q^+) and (V_v^-, q^-) so that their Hasse invariants $\epsilon(V_v^+, q^+) = 1$ and $\epsilon(V_v^-, q^-) = -1$. If there is only one such space, which happens when V is isomorphic to the 2-dimensional hyperbolic space over F_v, ignore the notation V_v^-.

The pseudo-Eisenstein series $B_\phi^{(S)}(g)$ is called *non-singular* if for every $v \in S$, there exist $\phi_v^+ \in \overline{\mathcal{S}}(V_v^+ \times F_v^\times)$ and $\phi_v^- \in \overline{\mathcal{S}}(V_v^- \times F_v^\times)$ such that

$$B_{a,v}(1, u) = W_{a,v}(0, 1, u, \phi_v^+) + W_{a,v}(0, 1, u, \phi_v^-), \quad \forall (a, u) \in F_v^\times \times F_v^\times.$$

Note that the equality is only for $g_v = 1$. Once this is true, replacing $B_{a,v}(g, u)$ by $W_{a,v}(0, g, u, \phi_v^+) + W_{a,v}(0, g, u, \phi_v^-)$ in $B_\phi^{(S)}(g)$, we see that

$$B_\phi^{(S)}(g) = \sum_{\epsilon: S \to \{\pm\}} E_*(0, g, \phi_S^\epsilon \otimes \phi^S), \qquad \forall g \in 1_S \mathrm{GL}_2(\mathbb{A}^S).$$

Here $\phi_S^\epsilon = \otimes_{v \in S} \phi_v^{\epsilon(v)}$ is the Schwartz function associated to the adelic quadratic space $V_S^\epsilon \otimes V(\mathbb{A}^v)$ with $V_S^\epsilon = \otimes_{v \in S} V_v^{\epsilon(v)}$, and $E_*(0, g, \phi_S^\epsilon \otimes \phi^S)$ denotes the non-constant part of the Eisenstein series $E_*(0, g, \phi_S^\epsilon \otimes \phi^S)$. If V_v^- does not exist, take the convention that $\epsilon(v) = +$ for every ϵ.

This is the counterpart of the approximation formula for pseudo-theta series in [YZ, §6.2]. Hence, it is convenience to denote

$$E_B(g) = E_{B_\phi^{(S)}}(g) = \sum_{\epsilon: S \to \{\pm\}} E(0, g, \phi_S^\epsilon \otimes \phi^S).$$

It is called *the Eisenstein series associated to* $B_\phi^{(S)}(g)$.

One can also formulate the terminology of pseudo-Eisenstein series for SL_2 based on Schwartz functions in $\mathcal{S}(V(\mathbb{A}))$ instead of $\overline{\mathcal{S}}(V(\mathbb{A})\times\mathbb{A}^\times)$. It can be done based on principal series of SL_2, which is really what an Eisenstein series needs. However, we stick with the current formulation because it fits our application.

Key lemma

The following is a generalization of [YZ, Lemma 6.1(1)] to a sum of pseudo-theta series and pseudo-Eisenstein series. There is also a generalization of [YZ, Lemma 6.1(2)], but we omit it due to the complexity of the statement.

Lemma 2.2. *Let* $\{A_\ell^{(S_\ell)}\}_\ell$ *be a finite set of non-singular pseudo-theta series sitting on vector spaces* $V_{\ell,0}\subset V_{\ell,1}\subset V_\ell$. *Let* $\{B_j^{(S'_j)}\}_j$ *be a finite set of non-singular pseudo-theta series sitting on quadratic spaces* V'_j. *Assume that the sum*

$$f(g)=\sum_\ell A_\ell^{(S_\ell)}(g)+\sum_j B_j^{(S'_j)}(g)$$

is automorphic for $g\in\mathrm{GL}_2(\mathbb{A})$. *Then*

$$f(g)=\sum_{\ell\in L_{0,1}}\theta_{A_\ell,1}(g)+\sum_j E_{B_j}(g)$$

Here $L_{0,1}$ *is the set of* ℓ *such that* $V_{\ell,1}=V_\ell$ *or equivalently* $A_\ell^{(S_\ell)}$ *is non-degenerate.*

Proof. The proof is similar to that of [YZ, Lemma 6.1(1)], by taking extra care of the pseudo-Eisenstein series. In fact, in the equation $f-\sum_\ell A_\ell^{(S_\ell)}-\sum_j B_j^{(S'_j)}=0$, replace each $A_\ell^{(S_\ell)}$ by its corresponding combinations of theta series as in the proof of [YZ, Lemma 6.1(1)], and replace each $B_j^{(S'_j)}$ by the associated difference $E_{B_j}-E_{B_j,0}$. Here $E_{B_j,0}$ denotes the constant term of the Eisenstein series E_{B_j}.

We claim that the constant term $E_{B_j,0}$ can be approximated by a finite linear combination of products of ρ_∞, δ and automorphic forms as in the case of pseudo-theta series. It suffices to treat a general Eisenstein series $E(0,g,\phi)_K$ associated to an adelic quadratic space $\mathbb{V}$ over $\mathbb{A}$ of dimension d. Then the constant term

$$E_0(0,g,\phi)_K=\sum_{u\in\mu_U^2\backslash F^\times}r(g)\phi(0,u)+\sum_{u\in\mu_U^2\backslash F^\times}W_0(0,g,u,\phi).$$

The first term on the right-hand side is already a pseudo-theta series. Then an approximation as in equation (6.2.1) of [YZ, §6.2] gives

$$\sum_{u\in\mu_U^2\backslash F^\times}r(g)\phi(0,u)=\rho_\infty(g)^{\frac d2}\delta(g)^{\frac d2}\sum_{u\in\mu_U^2\backslash F^\times}\phi(0,\det(g)^{-1}u),\quad g\in 1_S\mathrm{GL}_2(\mathbb{A}^S).$$

Here S is a finite set of non-archimedean places of F. Note that the summation on the right-hand side is automorphic in $g \in \mathrm{GL}_2(\mathbb{A})$.

For the second term on the right-hand side, note that for almost all v, $W_{0,v}(0,g,u,\phi_v)$ is a multiple of $\delta(g_v)^{2-d} r(g)\phi_v(0,u)$, as a basic result of intertwining operators of principal series. Consequently, we have a similar approximation

$$\sum_{u \in \mu_U^2 \backslash F^\times} W_0(0,g,u,\phi) = \rho_\infty(g)^{\frac{d}{2}} \delta(g)^{2-\frac{d}{2}} \sum_{u \in \mu_U^2 \backslash F^\times} W_0(0,1,\det(g)^{-1}u,\phi), \quad g \in 1_S \mathrm{GL}_2(\mathbb{A}^S).$$

Finally, we can replace $E_{B_j,0}$ by the corresponding approximations. After recollecting these theta series according to the powers of $\rho_\infty(g)$ and $\delta(g)$, we end up with an equation of the following form:

$$\sum_{(k,k')} \rho_\infty(g)^k \delta(g)^{k'} f_{k,k'}(g) = 0, \qquad \forall g \in 1_S \mathrm{GL}_2(\mathbb{A}^S).$$

Here S is some finite set of non-archimedean places, and $f_{k,k'}$ is some automorphic form on $\mathrm{GL}_2(\mathbb{A})$ coming from combinations of f, the theta series, and the Eisenstein series. In particular,

$$f_{0,0} = f - \sum_{\ell \in L_{0,1}} \theta_{A_\ell,1} - \sum_j E_{B_j}$$

is the term we care about. Note that for an index (k,k') appearing in the summation, if $k' = 0$, then we also have $k = 0$. The rest of the proof is the same as that of [YZ, Lemma 6.1]. □

2.3 Example by local quaternion algebras

In the case of quaternion algebras, we are going to figure out some important class of functions $B_{a,v}(1,u)$ which make the pseudo-Eisenstein series non-singular.

Let v be a non-archimedean place of F. Let $(M_2(F_v),q)$ (resp. (D_v,q)) be the matrix algebra (resp. the unique quaternion division algebra) over F_v with the reduced norm. Consider the map

$$\mathcal{W}_v : \overline{\mathcal{S}}(M_2(F_v) \times F_v^\times) \oplus \overline{\mathcal{S}}(D_v \times F_v^\times) \longrightarrow C^\infty(F_v^\times \times F_v^\times)$$

given by

$$(\phi^+,\phi^-) \longmapsto W_{a,v}(0,1,u,\phi^+) + W_{a,v}(0,1,u,\phi^-).$$

Here $C^\infty(F_v^\times \times F_v^\times)$ denotes the space of locally constant functions with complex values, and the last expression is viewed as a function of $(a,u) \in F_v^\times \times F_v^\times$.

Lemma 2.3. *The following are true:*

(1) *For any pair (ϕ^+,ϕ^-) as above, the sum $r(g)\phi^+(0,u) + r(g)\phi^-(0,u)$ as a function of $(g,u) \in \mathrm{GL}_2(F_v) \times F_v^\times$ is completely determined by the image Ψ of (ϕ^+,ϕ^-) in $C^\infty(F_v^\times \times F_v^\times)$. In particular,*

$$r(w)\phi^+(0,u) + r(w)\phi^-(0,u) = \int_{F_v} \Psi(a,u)da, \qquad u \in F_v^\times.$$

(2) *Let* $\Psi \in C^\infty(F_v^\times \times F_v^\times)$ *be a linear combination of the function* $1_{O_{F_v}\times O_{F_v^\times}}$ *and a locally constant and compactly supported functions on* $F_v^\times \times F_v^\times$*. Then* Ψ *has a preimage* (ϕ^+, ϕ^-) *satisfying*

$$\phi^+(0,u) + \phi^-(0,u) = 0, \quad \forall u \in F_v^\times.$$

Proof. We first prove (1). Note that $f(g,u) = r(g)\phi^+(0,u) + r(g)\phi^-(0,u)$ is determined by its restriction to $\mathrm{SL}_2(F_v) \times F_v^\times$. For fixed u, it is a principal series of $g \in \mathrm{SL}_2(F_v)$. Then this is a classical result closely related to Kirillov models and has nothing to do with Weil representations. In fact, we need to recover $f(g,u)$ from

$$\Psi(a,u) = \int_{F_v} f(wn(b),u)\psi(-ab)db, \quad a \in F, \ u \in F^\times.$$

Observe that $\Psi(a,u)$ as a function of $a \in F_v$ is the Fourier transform of $f(wn(b),u)$ as a function of $b \in F_v$. Thus we can recover $f(wn(b),u)$ by the Fourier inversion formula. Then $f(m(a)n(b')wn(b),u)$ can be recovered for any $a \in F_v^\times$ and $b' \in F_v$. But the set $m(a)n(b')wn(b)$ is dense in $\mathrm{SL}_2(F_v)$, as can be seen from the Bruhat decomposition. This determines all values of $f(g,u)$. In particular, the Fourier inversion formula gives

$$f(w,u) = \int_{F_v} \Psi(a,u)da.$$

This proves (1).

The proof of (2) is immediately reduced to two cases:

(a) Ψ is a locally constant and compactly supported function on $F_v^\times \times F_v^\times$;

(b) $\Psi = 1_{O_{F_v}\times O_{F_v^\times}}$.

As preparation, recall that the local Siegel-Weil formula in [YZZ, Proposition 2.9(2)] gives

$$W_{a,v}(0,1,u,\phi) = \epsilon(B_v)\ |a|_v \int_{B_v^1} \phi(hx_a,u)dh, \quad a,u \in F_v^\times.$$

Here (B_v,ϕ) can be either the pair $(M_2(F_v),\phi^+)$ or (D_v,ϕ^-), and $x_a \in B_v$ is any element satisfying $uq(x_a) = a$.

Now we treat case (a). Note that D_v^1 is compact. We will actually find a preimage of the form $(0,\phi^-)$, where ϕ^- is invariant under the action of D_v^1. In fact, the local Siegel–Weil formula gives

$$\phi^-(x,u) = -\frac{1}{\mathrm{vol}(D_v^1)|uq(x)|_v} W_{uq(x),v}(0,1,u,\phi^-) = -\frac{1}{\mathrm{vol}(D_v^1)|uq(x)|_v}\Psi(uq(x),u).$$

It is a Schwartz function since $\Psi(a,u)$ is assumed to be compactly supported in a. It is also clear that $\phi^-(0,u) = 0$ for any $u \in F_v^\times$.

For case (b), the local Siegel–Weil formula gives

$$W_{a,v}(0,1,u,1_{O_{D_v}\times O_{F_v}^\times}) = -|d_v|^{\frac{3}{2}} N_v^{-1}(1+N_v^{-1}) \cdot |a|_v \cdot 1_{O_{F_v}\times O_{F_v^\times}}(a,u).$$

Here O_{D_v} denotes the maximal order of D_v, and $\mathrm{vol}(D_v^1) = |d_v|^{\frac{3}{2}} N_v^{-1}(1+N_v^{-1})$ as normalized in [YZZ, §1.6.2].

On the other hand,

$$W_{a,v}(0,1,u,1_{M_2(O_{F_v})\times O_{F_v}^\times}) = |a|_v \cdot \mathrm{vol}(\mathrm{SL}_2(O_{F_v})) \cdot |\mathrm{SL}_2(O_{F_v})\backslash M_2(O_{F_v})(a)| \cdot 1_{O_{F_v}\times O_{F_v^\times}}(a,u).$$

Here $M_2(O_{F_v})(a)$ denotes matrices in $M_2(O_{F_v})$ of determinant a. Set $r = v(a) \geq 0$, and denote

$$M_2(O_{F_v})_r = \{x \in M_2(O_{F_v})_r : v(\det(x)) = r\}.$$

Then we have

$$\mathrm{SL}_2(O_{F_v})\backslash M_2(O_{F_v})(a) = \mathrm{GL}_2(O_{F_v})\backslash M_2(O_{F_v})_r.$$

The last coset corresponds exactly to the classical Hecke correspondence $T(p_v^r)$, and its order is just $1+N_v+\cdots+N_v^r$. Combine $\mathrm{vol}(\mathrm{SL}_2(O_k)) = |d_v|^{\frac{3}{2}}(1-N_v^{-2})$ as normalized in [YZZ, §1.6.2]. We end up with

$$W_{a,v}(0,1,u,1_{M_2(O_{F_v})\times O_{F_v}^\times}) = |d_v|^{\frac{3}{2}} N_v^{-1}(1+N_v^{-1}) \cdot (N_v - |a|_v) \cdot 1_{O_{F_v}\times O_{F_v^\times}}(a,u).$$

The linear combination of these two expressions gives a preimage

$$\phi^+ = |d_v|^{-\frac{3}{2}}(1+N_v^{-1})^{-1} \cdot 1_{M_2(O_{F_v})\times O_{F_v}^\times}, \qquad \phi^- = -|d_v|^{-\frac{3}{2}}(1+N_v^{-1})^{-1} \cdot 1_{O_{D_v}\times O_{F_v}^\times}.$$

It is clear that $\phi^+(0,u) + \phi^-(0,u) = 0$ for any $u \in F_v^\times$ in this case. □

Remark 2.4. In part (2), the result $\phi^+(0,u) + \phi^-(0,u) = 0$ in case (b) is not as random as what our computational proof suggests. In fact, we claim that for any image

$$\Psi(a,u) = W_{a,v}(0,1,u,\phi^+) + W_{a,v}(0,1,u,\phi^-),$$

if Ψ can be extended to a locally constant and compactly supported function on $F_v \times F_v^\times$ (instead of the more restrictive $F_v^\times \times F_v^\times$), then $\phi^+(0,u)+\phi^-(0,u) = 0$. For a proof, for $b \in F_v^\times$, set

$$g = n(b)m(-b)wn(b) = \begin{pmatrix} 1 & \\ b^{-1} & 1 \end{pmatrix}.$$

The right hand side goes to 1 as the valuation $v(b) \to -\infty$. We have

$$r(g)\phi^+(0,u) + r(g)\phi^-(0,u) = |b|_v^2 \cdot \big(r(wn(b))\phi^+(0,u) + r(wn(b))\phi^-(0,u)\big).$$

Note that $r(wn(b))\phi^+(0,u) + r(wn(b))\phi^-(0,u)$ is the Fourier transform of $\Psi(a,u)$, so it is also a locally constant and compactly supported function in $b \in F_v$. In particular, it is zero if $v(b)$ is sufficiently negative. This proves $\phi^+(0,u) + \phi^-(0,u) = 0$.

3 Derivative series

The goal of this section is to study the holomorphic projection of the derivative of some mixed Eisenstein–theta series. This section is based on [YZ, §7] and [YZZ, §6], but the situation is more complicated since we do not have [YZ, Assumption 7.1] or equivalently [YZZ, Assumption 5.4].

3.1 Derivative series

Let F be a totally real field, and E be a totally imaginary quadratic extension of F. Denote by $\mathbb{A} = \mathbb{A}_F$ the ring of adeles of F. Let $\mathbb{B}$ be a totally definite incoherent quaternion algebra over $\mathbb{A}$ with an embedding $E_{\mathbb{A}} \to \mathbb{B}$ of $\mathbb{A}$-algebras.

Fix a Schwartz function $\phi \in \overline{\mathcal{S}}(\mathbb{B} \times \mathbb{A}^\times)$ invariant under $U \times U$ for some open compact subgroup U of $\mathbb{B}_f^\times$. Start with the mixed theta-Eisenstein series

$$I(s, g, \phi)_U = \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{\gamma \in P^1(F) \backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s \sum_{x_1 \in E} r(\gamma g) \phi(x_1, u), \quad g \in \mathrm{GL}_2(\mathbb{A}).$$

It was first introduced in [YZZ, §5.1.1]. If $\phi = \phi_1 \otimes \phi_2$ with respect to an orthogonal decomposition $\mathbb{B} = E_{\mathbb{A}} + E_{\mathbb{A}}\mathfrak{j}$, then

$$I(s, g, \phi)_U = \sum_{u \in \mu_U^2 \backslash F^\times} \theta(g, u, \phi_1)\ E(s, g, u, \phi_2),$$

where for any $g \in \mathrm{GL}_2(\mathbb{A})$, the theta series and the Eisenstein series are given by

$$\begin{aligned} \theta(g, u, \phi_1) &= \sum_{x_1 \in E} r(g) \phi_1(x_1, u), \\ E(s, g, u, \phi_2) &= \sum_{\gamma \in P^1(F) \backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s r(\gamma g) \phi_2(0, u). \end{aligned}$$

The derivative series $\mathcal{P}r I'(0, g, \phi)$ is the holomorphic projection of the derivative $I'(0, g, \phi)$ of $I(s, g, \phi)$. We will start with some general results about the holomorphic projection.

Holomorphic projection

Recall that the holomorphic projection is the orthogonal projection

$$\mathcal{P}r : \mathcal{A}(\mathrm{GL}_2(\mathbb{A}), \omega) \longrightarrow \mathcal{A}_0^{(2)}(\mathrm{GL}_2(\mathbb{A}), \omega)$$

with respect to the Petersson inner product. Here $\omega : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ is a Hecke character with trivial archimedean components, $\mathcal{A}(\mathrm{GL}_2(\mathbb{A}), \omega)$ is the space of automorphic forms of central character ω, and $\mathcal{A}_0^{(2)}(\mathrm{GL}_2(\mathbb{A}), \omega)$ is the subspace of holomorphic cusp forms of parallel weight two. It induces a projection

$$\mathcal{P}r : \bigoplus_{\omega} \mathcal{A}(\mathrm{GL}_2(\mathbb{A}), \omega) \longrightarrow \bigoplus_{\omega} \mathcal{A}_0^{(2)}(\mathrm{GL}_2(\mathbb{A}), \omega).$$

As in [YZ, §7.1], by decomposing $I'(0, g, \phi)$ into a finite direct sum of automorphic forms with (distinct) central characters, we see that $\mathcal{P}r I'(0, g, \phi)$ lies in $\oplus_\omega \mathcal{A}(\mathrm{GL}_2(\mathbb{A}), \omega)$. Thus the holomorphic projection $\mathcal{P}r I'(0, g, \phi)$ is a well-defined holomorphic cusp form of parallel weight two in $g \in \mathrm{GL}_2(\mathbb{A})$. We are still going to apply the formula in [YZZ, Proposition 6.12] to compute $\mathcal{P}r I'(0, g, \phi)$.

To recall [YZZ, Proposition 6.12], we start with the operator $\mathcal{P}r'$ defined right after the proposition. For convenience, we first introduce the corresponding operator $\mathcal{P}r'_\psi$ for Whittaker functions. For any (Whittaker) function $\alpha : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$ with $\alpha(n(b)g) = \psi(b)\alpha(g)$ for any $b \in \mathbb{R}$ and $g \in \mathrm{GL}_2(\mathbb{R})$, define

$$(\mathcal{P}r'_\psi\alpha)(g) := 4\pi W^{(2)}(g) \cdot \widetilde{\lim}_{s\to 0} \int_{Z(\mathbb{R})N(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})} \delta(h)^s \alpha(h)\overline{W^{(2)}(h)}dh,$$

if the right-hand side is convergent. Here $W^{(2)}(g)$ is the standard Whittaker function of weight two as in [YZZ, §4.1.1], and $\widetilde{\lim}_{s\to 0}$ is the constant term in the Laurent expansion at $s = 0$. The definition extends to global Whittaker functions $\alpha : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ by

$$(\mathcal{P}r'_\psi\alpha)(g) = (4\pi)^{[F:\mathbb{Q}]} W^{(2)}_\infty(g_\infty) \cdot \widetilde{\lim}_{s\to 0} \int_{Z(F_\infty)N(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \delta(h)^s \alpha(g_f h)\overline{W^{(2)}(h)}dh$$

if it is convergent.

For any function $f : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$, we first take the Whittaker function

$$f_\psi(g) = \int_{N(F)\backslash N(\mathbb{A})} f(n(b)g)\psi(-b)db,$$

and set

$$(\mathcal{P}r' f)(g) = \sum_{a\in F^\times} (\mathcal{P}r'_\psi f_\psi)(d^*(a)g),$$

if both are convergent in suitable sense.

Finally, [YZZ, Proposition 6.12] asserts that if f is an automorphic form satisfying certain growth condition, then

$$\mathcal{P}r f = \mathcal{P}r' f.$$

In other words, the above formula really computes the holomorphic projection of f.

Go back to $\mathcal{P}r I'(0, g, \phi)$. In our previous works, [YZ, Assumption 7.1] or [YZZ, Assumption 5.4] makes $I'(0, g, \phi)$ satisfy the growth condition of [YZZ, Proposition 6.12], but here we do not make the assumption her, and we will see that the growth condition is not satisfied. Then the final result has an extra term contributed by the growth of $I'(0, g, \phi)$, as remarked in [YZZ, §6.4.3].

To track the growth of $I'(0, g, \phi)$, we are going to apply [YZZ, Lemma 6.13]. Then we recall the absolute constant term

$$I_{00}(s, g, \phi) = \sum_{u\in\mu_U^2\backslash F^\times} I_{00}(s, g, u, \phi),$$

where

$$I_{00}(s, g, u, \phi) = \theta_0(g, u, \phi_1)E_0(s, g, u, \phi_2).$$

Let $\mathcal{J}(s, g, u, \phi)$ be the Eisenstein series formed by $I_{00}(s, g, u, \phi)$:

$$\mathcal{J}(s, g, u, \phi) = \sum_{\gamma\in P^1(F)\backslash \mathrm{SL}_2(F)} I_{00}(s, \gamma g, u, \phi).$$

Denote

$$\mathcal{J}(s,g,\phi)_U = \sum_{u\in\mu_U^2\backslash F^\times} \mathcal{J}(s,g,u,\phi),$$

which will also be abbreviated as $\mathcal{J}(s,g,\phi)$.

By [YZZ, Lemma 6.13], the difference

$$I'(0,g,\phi)_U - I'_{00}(0,g,\phi)_U$$

satisfies the growth condition of [YZZ, Proposition 6.12]. Note that the original statement is about the twisting of the difference by some character χ, but a similar proof by decomposing the difference in terms of central characters works for the current situation. A similar argument proves that

$$\mathcal{J}'(0,g,\phi)_U - I'_{00}(0,g,\phi)_U$$

also satisfies the growth condition. As a consequence,

$$I'(0,g,\phi)_U - \mathcal{J}'(0,g,\phi)_U$$

satisfies the growth condition.

Therefore, we have

$$\begin{aligned}\mathcal{P}r(I'(0,g,\phi)) &= \mathcal{P}r(I'(0,g,\phi) - \mathcal{J}'(0,g,\phi))\\ &= \mathcal{P}r'(I'(0,g,\phi) - \mathcal{J}'(0,g,\phi)) = \mathcal{P}r'(I'(0,g,\phi)) - \mathcal{P}r'(\mathcal{J}'(0,g,\phi)),\end{aligned}$$

where the operator $\mathcal{P}r'$ is defined by the algorithm as recalled above. The term $\mathcal{P}r'(I'(0,g,\phi))$ is computed exactly as in [YZ, Theorem 7.2].

For its importance, we summarize the result as follows:

Theorem 3.1. *Assume that ϕ is standard at infinity. Then*

$$\mathcal{P}rI'(0,g,\phi)_U = \mathcal{P}r'I'(0,g,\phi)_U - \mathcal{P}r'\mathcal{J}'(0,g,\phi)_U,$$

where $\mathcal{P}r'I'(0,g,\phi)_U$ has the same expression as that of $\mathcal{P}rI'(0,g,\phi)_U$ in [YZ, Theorem 7.2]. Namely,

$$\begin{aligned}\mathcal{P}r'I'(0,g,\phi)_U = &-\sum_{v|\infty} \overline{I'}(0,g,\phi)(v) - \sum_{v\nmid\infty \text{ nonsplit}} I'(0,g,\phi)(v)\\ &- c_1 \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in E^\times} r(g)\phi(y,u) - \sum_{v\nmid\infty}\sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in E^\times} c_{\phi_v}(g,y,u)\, r(g)\phi^v(y,u)\\ &+ \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in E^\times} \left(2\log\delta_f(g_f) + \log|uq(y)|_f\right)\, r(g)\phi(y,u),\end{aligned}$$

and the right-hand side is explained in the following.

(1) *For any archimedean v,*

$$\overline{I}'(0,g,\phi)(v) = 2\fint_{C_U} \overline{\mathcal{K}}_\phi^{(v)}(g,(t,t))dt,$$

$$\overline{\mathcal{K}}_\phi^{(v)}(g,(t_1,t_2)) = w_U \sum_{a\in F^\times} \widetilde{\lim}_{s\to 0} \sum_{y\in\mu_U\backslash(B(v)_+^\times - E^\times)} r(g,(t_1,t_2))\phi(y)_a\ k_{v,s}(y),$$

$$k_{v,s}(y) = \frac{\Gamma(s+1)}{2(4\pi)^s}\int_1^\infty \frac{1}{t(1-\lambda(y)t)^{s+1}}dt,$$

where $\lambda(y) = q(y_2)/q(y)$ is viewed as an element of F_v.

(2) *For any non-archimedean v which is nonsplit in E,*

$$I'(0,g,\phi)(v) = 2\fint_{C_U} \mathcal{K}_\phi^{(v)}(g,(t,t))dt,$$

$$\mathcal{K}_\phi^{(v)}(g,(t_1,t_2)) = \sum_{u\in\mu_U^2\backslash F^\times} \sum_{y\in B(v)-E} k_{r(t_1,t_2)\phi_v}(g,y,u) r(g,(t_1,t_2))\phi^v(y,u),$$

$$k_{\phi_v}(g,y,u) = \frac{L(1,\eta_v)}{\mathrm{vol}(E_v^1)} r(g)\phi_{1,v}(y_1,u) W_{uq(y_2),v}^{\circ}{}'(0,g,u,\phi_{2,v}), \quad y_2 \neq 0.$$

Here the last identity holds under the relation $\phi_v = \phi_{1,v}\otimes\phi_{2,v}$, and the definition extends by linearity to general ϕ_v.

(3) *The constant*

$$c_1 = 2\frac{L_f'(0,\eta)}{L_f(0,\eta)} + \log|d_E/d_F|.$$

(4) *Under the relation $\phi_v = \phi_{1,v}\otimes\phi_{2,v}$,*

$$c_{\phi_v}(g,y,u) = r_E(g)\phi_{1,v}(y,u) W_{0,v}^{\circ}{}'(0,g,u,\phi_{2,v}) + \log\delta(g_v) r(g)\phi_v(y,u).$$

The definition extends by linearity to general ϕ_v.

Recall that $W_{a,v}^\circ(s,g,u,\phi_{2,v})$ in the theorem is normalized as in [YZ, §7.1]. Namely, for $a\in F_v^\times$, define

$$W_{a,v}^\circ(s,g,u,\phi_{2,v}) \quad = \quad \gamma_{u,v}^{-1} W_{a,v}(s,g,u,\phi_{2,v}).$$

Here $\gamma_{u,v}$ is the Weil index of $(E_v\mathfrak{j}_v, uq)$, where $\mathbb{B}_v = E_v + E_v\mathfrak{j}_v$ is an orthogonal decomposition. For $a=0$, define

$$W_{0,v}^\circ(s,g,u,\phi_{2,v}) \quad = \quad \gamma_{u,v}^{-1}\frac{L(s+1,\eta_v)}{L(s,\eta_v)}|D_v|^{-\frac{1}{2}}|d_v|^{-\frac{1}{2}} W_{0,v}(s,g,u,\phi_{2,v}).$$

Contribution of the Eisenstein series

Now we compute the term $\mathcal{P}r'\mathcal{J}'(0,g,\phi)$.

For any $\phi\in\overline{\mathcal{S}}(\mathbb{B}\times\mathbb{A}^\times)$ of the form $\phi=\phi_1\otimes\phi_2$,

$$I'_{00}(0,g,u,\phi)=\log\delta(g)\ r(g)\phi(0,u)+r_E(g)\phi_1(0,u)W'_0(0,g,u,\phi_2).$$

Recall that the computation in [YZZ, Proposition 6.7] or that in [YZ, §7.1, p. 586] gives

$$W'_0(0,g,u,\phi_2)=-c_0r(g)\phi_2(0,u)-\sum_v r(g^v)\phi_2^v(0,u)W^\circ_{0,v}{}'(0,g_v,u,\phi_{2,v})$$

with the constant

$$c_0=\frac{d}{ds}|_{s=0}\left(\log\frac{L(s,\eta)}{L(s+1,\eta)}\right)=2\frac{L'(0,\eta)}{L(0,\eta)}+\log|d_E/d_F|.$$

Here $L(s,\eta)$ is the completed L-function with gamma factors, and we have used the functional equation

$$L(1-s,\eta)=|d_E/d_F|^{s-\frac{1}{2}}L(s,\eta).$$

It follows that

$$\begin{aligned}
&I'_{00}(0,g,u,\phi)\\
=\ &\log\delta(g)\cdot r(g)\phi(0,u)-c_0r(g)\phi(0,u)-\sum_v r(g^v)\phi^v(0,u)\cdot r_E(g_v)\phi_{1,v}(0,u)W^\circ_{0,v}{}'(0,g_v,u,\phi_{2,v})\\
=\ &2\log\delta(g)\cdot r(g)\phi(0,u)-c_0r(g)\phi(0,u)-\sum_v r(g^v)\phi^v(0,u)c_{\phi_v}(g,0,u),
\end{aligned}$$

where

$$c_{\phi_v}(g,y,u)=r_E(g)\phi_{1,v}(y,u)W^\circ_{0,v}{}'(0,g,u,\phi_{2,v})+\log\delta(g_v)r(g)\phi_v(y,u).$$

Here the sum on v is actually a finite sum, and $c_{\phi_v}(g,y,u)=0$ for any archimedean v by [YZ, Lemma 7.6].

One checks that $c_{\phi_v}(g,0,u)$ is a principal series in the sense that

$$c_{\phi_v}(m(a)n(b)g,0,u)=|a|_v^2\ c_{\phi_v}(g,0,u),\quad a\in F_v^\times,\ b\in F_v.$$

This is a consequence of the basic fact

$$W^\circ_{0,v}(s,m(a)n(b)g,u)=|a|_v^{1-s}\eta_v(a)W^\circ_{0,v}(s,g,u)$$

and the result

$$W^\circ_{0,v}(0,g,u)=r(g)\phi_{2,v}(0,u).$$

of [YZZ, Proposition 6.1].

Then we introduce Eisenstein series

$$\begin{aligned}
E(s,g,u,\phi)&=\sum_{\gamma\in P^1(F)\backslash \mathrm{SL}_2(F)}\delta(\gamma g)^s r(\gamma g)\phi(0,u),\\
C(s,g,u,\phi)(v)&=\sum_{\gamma\in P^1(F)\backslash \mathrm{SL}_2(F)}\delta(\gamma g)^s c_{\phi_v}(\gamma g,0,u)\ r(\gamma g^v)\phi^v(0,u),
\end{aligned}$$

and

$$E(s,g,\phi)_U = \sum_{u\in\mu_U^2\backslash F^\times} E(s,g,u,\phi),$$
$$C(s,g,\phi)_U(v) = \sum_{u\in\mu_U^2\backslash F^\times} C(s,g,u,\phi)(v).$$

Denote also

$$C(s,g,u,\phi) = \sum_{v\nmid\infty} C(s,g,u,\phi)(v)$$
$$C(s,g,\phi)_U = \sum_{v\nmid\infty} C(s,g,u,\phi)(v).$$

Note that both summations have only finitely many nonzero terms. We will usually supress the sub-index U in $E(s,g,\phi)_U$ and $C(s,g,\phi)_U$.

Consequently, we can write

$$\mathcal{J}'(0,g,\phi) = 2E'(0,g,\phi) - c_0E(0,g,\phi) - C(0,g,\phi).$$

Applying the formula for $\mathcal{P}r'$, we have the following expression.

Proposition 3.2. *Assume that ϕ is standard at infinity. Then*

$$\mathcal{P}r'\mathcal{J}'(0,g,\phi) = -(c_0 + (1+\log 4)[F:\mathbb{Q}])E_*(0,g,\phi) - C_*(0,g,\phi) + 2\sum_{v\nmid\infty} E'(0,g,\phi)(v).$$

Here E_* and C_* are the non-constant parts of the Eisenstein series E and C. And for any $v\nmid\infty$,

$$E'(0,g,\phi)(v) = \sum_{u\in\mu_U^2\backslash F^\times} E'(0,g,u,\phi)(v),$$

where

$$E'(0,g,u,\phi)(v) = \sum_{a\in F^\times} W_a^v(0,g,a^{-1}u,\phi^v)\left(W'_{a,v}(0,g,a^{-1}u,\phi_v) - \frac{1}{2}\log|a|_v\cdot W_{a,v}(0,g,a^{-1}u,\phi_v)\right).$$

Proof. By linearity,

$$\mathcal{P}r'\mathcal{J}'(0,g,\phi) = 2\mathcal{P}r'E'(0,g,\phi) - c_0\mathcal{P}r'E(0,g,\phi) - \mathcal{P}r'C(0,g,\phi)$$

Since the Whittaker function of $E(0,g,\phi)$ is already holomorphic, the holomorphic projection doesn't change it. We have

$$\mathcal{P}r'E(0,g,\phi) = E_*(0,g,\phi).$$

Similarly,

$$\mathcal{P}r'C(0,g,\phi) = C_*(0,g,\phi).$$

For $\mathcal{P}r'E'(0,g,\phi)$, start with the Whittaker function

$$W_1'(0,g,u,\phi) = \sum_{u\in\mu_U^2\backslash F^\times} W_1'(0,g,u,\phi) = \sum_{u\in\mu_U^2\backslash F^\times}\sum_v W_{1,v}'(0,g,u,\phi_v)W_1^v(0,g,u,\phi^v).$$

Then it amounts to apply $\mathcal{P}r'_\psi$ to $W'_{1,v}(0,g,u,\phi_v)W_1^v(0,g,u,\phi)$ for each place v of F.

If $v|\infty$, then

$$\mathcal{P}r'_\psi W'_{1,v}(0,g,u,\phi_v) = c_3 W_{1,v}(0,g,u,\phi_v)$$

for some constant c_3. It follows that

$$\mathcal{P}r'_\psi\left(W'_{1,v}(0,g,u,\phi_v)W_1^v(0,g,u,\phi^v)\right) = c_3 W_1(0,g,u,\phi).$$

Recovering its contribution to the whole series, we get

$$c_3 E_*(0,g,\phi) = c_3\sum_{u\in\mu_U^2\backslash F^\times} E_*(0,g,u,\phi).$$

Furthermore, the constant $c_3 = -\frac{1}{2}(1+\log 4)$ is computed in Lemma 3.3 below.

If $v\nmid\infty$, then $\mathcal{P}r'_\psi$ does not change $W'_{1,v}(0,g,u,\phi_v)W_1^v(0,g,u,\phi^v)$ since it is already holomorphic. However, when getting back to the whole series, its contribution is

$$\sum_{a\in F^\times} W'_{1,v}(0,d^*(a)g,u,\phi_v)W_1^v(0,d^*(a)g,u,\phi^v).$$

Apply the basic result

$$W_{1,v'}(s,d^*(a)g,u,\phi_{v'}) = |a|_{v'}^{-\frac{s}{2}} W_{a,v'}(s,g,a^{-1}u,\phi_{v'}),$$

which can be verified using $wn(b)d^*(a) = d(a)wn(a^{-1}b)$. We have

$$W_1^v(0,d^*(a)g,u,\phi^v) = W_a^v(0,g,a^{-1}u,\phi^v),$$

and

$$W'_{1,v}(0,d^*(a)g,u,\phi_v) = W'_{a,v}(0,g,a^{-1}u,\phi_v) - \frac{1}{2}\log|a|_v\cdot W_{a,v}(0,g,a^{-1}u,\phi_v).$$

Then the result follows. □

3.2 Choice of the Schwartz function

To make further explicit local computations, we need to specify the Schwartz functions. We will see that our choice is slightly different from that of [YZ, §7.2].

Start with the setup of Theorem 1.1 and Theorem 1.2. Let F be a totally real field, and E be a totally imaginary quadratic extension of F. Let $\mathbb{B}$ be a totally definite incoherent quaternion algebra over $\mathbb{A} = \mathbb{A}_F$ with an embedding $E_\mathbb{A}\to\mathbb{B}$ of $\mathbb{A}$-algebras. Let $U = \prod_{v\nmid\infty} U_v$ be a maximal open compact subgroup of $\mathbb{B}_f^\times$ containing (the image of) $\widehat{O}_E^\times = \prod_{v\nmid\infty} O_{E_v}^\times$.

As in Theorem 1.2, assume that there is no non-archimedean place of F ramified in E and $\mathbb{B}$ simultaneously. For fixed $\mathbb{B}$, it is easy to find such E.

Note that we have already assumed that U_v is maximal at any $v \nmid \infty$. Denote by $O_{\mathbb{B}_v}$ the O_{F_v}-subalgebra of $\mathbb{B}_v$ generated by U_v. Then $O_{\mathbb{B}_v}$ is a maximal order of $\mathbb{B}_v$, and $U_v = O_{\mathbb{B}_v}^\times$ is the group of invertible elements. Furthermore, the inclusion $O_{E_v}^\times \subset U_v$ induces $O_{E_v} \subset O_{\mathbb{B}_v}$.

As for the Schwartz function $\phi = \otimes_v \phi_v$, we make the following choices:

(1) If v is archimedean, set ϕ_v to be the standard Gaussian as in [YZZ, §4.1.1].

(2) If v is non-archimedean and split in $\mathbb{B}$, set ϕ_v to be the standard characteristic function $1_{O_{\mathbb{B}_v} \times O_{F_v}^\times}$.

(3) If v is non-archimedean and nonsplit in $\mathbb{B}$, set ϕ_v to be the characteristic function $1_{O_{\mathbb{B}_v}^\times \times O_{F_v}^\times}$ (instead of the standard $1_{O_{\mathbb{B}_v} \times O_{F_v}^\times}$).

By definition, ϕ is invariant under both the left action and the right action of U.

Note that [YZ, §7.2] assumes that there is a set S_2 consisting of two places of F split in E such that ϕ_v takes a specific degenerate from for $v \in S_2$. We do not make this assumption here, since this assumption exactly kills the terms we need for our main theorem.

For any $v \nmid \infty$, fix an element $\mathfrak{j}_v \in O_{\mathbb{B}_v}$ orthogonal to E_v such that $v(q(\mathfrak{j}_v))$ is non-negative and minimal; i.e., $v(q(\mathfrak{j}_v)) \in \{0,1\}$. Then $v(q(\mathfrak{j}_v)) = 1$ if and only if $\mathbb{B}_v$ is nonsplit (and thus E_v/F_v is inert by assumption). The existence of $\mathfrak{j}_v$ is basic and verified in [YZ, §7.2].

For any non-archimedean place v nonsplit in E, let $B(v)$ be the nearby quaternion algebra over F. Fix an embedding $E \to B(v)$ and isomorphisms $B(v)_{v'} \simeq \mathbb{B}_{v'}$ for any $v' \neq v$, which are assumed to be compatible with the embedding $E_{\mathbb{A}} \to \mathbb{B}$. At v, we also take an element $j_v \in B(v)_v$ orthogonal to E_v, such that $v(q(j_v))$ is non-negative and minimal as above. We remark that this set $\{\mathfrak{j}_{v'} : v' \neq v\} \cup \{j_v\}$ is not required to be the localizations of a single element of $B(v)$.

3.3 Explicit local derivatives

Recall that we have defined the Eisenstein series

$$E(s,g,u,\phi) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s r(\gamma g)\phi(0,u).$$

Note that this Eisenstein series uses the whole Schwartz function ϕ and thus have weight two, comparing to the Eisenstein series $E(s,g,u,\phi_2)$ in the definition of the derivative series, which only uses ϕ_2 and thus have weight one. In the section, we will abbreviate

$$E(s,g,u) = E(s,g,u,\phi).$$

We have the usual Fourier expansion:

$$E(s,g,u) = E_0(s,g,u) + \sum_{a \in F^\times} W_a(s,g,u)$$

with

$$
\begin{aligned}
E_0(s,g,u) &= \delta(g)^s r(g)\phi(0,u) + W_0(s,g,u),\\
W_a(s,g,u) &= \int_{\mathbb{A}} \delta(wn(b)g)^s r(wn(b)g)\phi(0,u)\psi(-ab)db, \quad a \in F.
\end{aligned}
$$

We also introduce the local Whittaker function

$$
W_{a,v}(s,g,u) = \int_{F_v} \delta(wn(b)g)^s r(wn(b)g)\phi(0,u)\psi(-ab)db, \quad a \in F_v,\ u \in F_v^\times,\ g \in \mathrm{GL}_2(F_v).
$$

Local holomorphic projection: archimedean place

Recall that ϕ at any archimedean place is the standard Gaussian as in [YZZ, §4.1.1].

Lemma 3.3. *Let v be an archimedean place.*

(1) For any $a \in F_v$ with $a > 0$,

$$
\begin{aligned}
W_{1,v}(0,d^*(a),u) &= -4\pi^2 a e^{-2\pi a} 1_{F_{v,+}^\times}(u),\\
W_{1,v}'(0,d^*(a),u) &= -\left(\frac{\pi}{2}e^{-2\pi a} + 2\pi^2(\log\pi + \gamma - 1)ae^{-2\pi a}\right)1_{F_{v,+}^\times}(u).
\end{aligned}
$$

Here γ is Euler's constant.

(2) The holomorphic projection

$$
\mathcal{P}r_\psi' W_{1,v}'(0,g,u) = -\frac{1}{2}(1+\log 4)W_{1,v}(0,g,u).
$$

Proof. We first check (1). By $wn(b)d^*(a) = d(a)wn(a^{-1}b)$, it is easy to get

$$
W_{1,v}(s,d^*(a),u) = a^{-\frac{s}{2}}W_{a,v}(s,1,a^{-1}u).
$$

It is reduced to compute

$$
W_{a,v}(s,1,u) = \int_{F_v} \delta(wn(b))^s r(wn(b))\phi(0,u)\psi(-ab)db, \quad a > 0.
$$

Assume $u > 0$; otherwise, the above vanishes.

The process is parallel to [YZZ, Proposition 2.11] and also uses the technique of [KRY1]. In fact, from the proof of [YZZ, Proposition 2.11] for the case $d = 4$,

$$
W_{a,v}(s,1,u) = -\frac{2\pi^{s+2}}{\Gamma(\frac{s}{2}+2)\Gamma(\frac{s}{2})}e^{-2\pi a}\int_0^\infty e^{-2\pi t}(t+2a)^{\frac{s}{2}+1}t^{\frac{s}{2}-1}dt.
$$

To see its behavior at $s = 0$, write

$$
W_{a,v}(s,1,u) = -e^{-2\pi a}\frac{\pi^{s+2}s}{\Gamma(\frac{s}{2}+2)\Gamma(\frac{s}{2}+1)}\int_0^\infty e^{-2\pi t}(t+2a)^{\frac{s}{2}+1}t^{\frac{s}{2}-1}dt.
$$

Then the product before the integral has a simple zero at $s=0$.

Since

$$\begin{aligned}&\int_0^\infty e^{-2\pi t}(t+2a)^{\frac{s}{2}+1}t^{\frac{s}{2}-1}dt-(2a)^{\frac{s}{2}+1}(2\pi)^{-\frac{s}{2}}\Gamma(\frac{s}{2})\\=&\int_0^\infty e^{-2\pi t}\frac{(t+2a)^{\frac{s}{2}+1}-(2a)^{\frac{s}{2}+1}}{t}t^{\frac{s}{2}}dt=\frac{1}{2\pi}+O(s),\end{aligned}$$

we get

$$\begin{aligned}-W_{a,v}(s,1,u)=&e^{-2\pi a}\frac{\pi^{s+2}s}{\Gamma(\frac{s}{2}+2)\Gamma(\frac{s}{2}+1)}\left(\frac{1}{2\pi}+(2a)^{\frac{s}{2}+1}(2\pi)^{-\frac{s}{2}}\Gamma(\frac{s}{2})\right)+O(s^2)\\=&e^{-2\pi a}\frac{\pi^{s+2}}{\Gamma(\frac{s}{2}+2)\Gamma(\frac{s}{2}+1)}\left(\frac{1}{2\pi}s+2(2a)^{\frac{s}{2}+1}(2\pi)^{-\frac{s}{2}}\Gamma(\frac{s}{2}+1)\right)+O(s^2).\end{aligned}$$

It follows that

$$\begin{aligned}-W_{a,v}(0,1,u)=&\ 4\pi^2ae^{-2\pi a},\\-W_{a,v}'(0,1,u)=&\ \frac{\pi}{2}e^{-2\pi a}+2\pi^2(\log\pi+\gamma-1)ae^{-2\pi a}+2\pi^2ae^{-2\pi a}\log a,\\-W_{1,v}'(0,d^*(a),u)=&\frac{\pi}{2}e^{-2\pi a}+2\pi^2(\log\pi+\gamma-1)ae^{-2\pi a}.\end{aligned}$$

Now we compute the holomorphic projection

$$\mathcal{P}r_\psi'W_{1,v}'(0,g,u)=4\pi W^{(2)}(g)\cdot\widetilde{\lim}_{s\to0}\int_{Z(\mathbb{R})N(\mathbb{R})\backslash\mathrm{GL}_2(\mathbb{R})}\delta(h)^sW_{1,v}'(0,h,u)\overline{W^{(2)}(h)}dh.$$

By the Iwasawa decomposition,

$$\begin{aligned}\mathcal{P}r_\psi'W_{1,v}'(0,g,u)=&4\pi W^{(2)}(g)\widetilde{\lim}_{s\to0}\int_0^\infty y^se^{-2\pi y}W_{1,v}'(0,d^*(y),u)\frac{dy}{y}\\=&-4\pi W^{(2)}(g)\widetilde{\lim}_{s\to0}\int_0^\infty y^se^{-2\pi y}\left(\frac{\pi}{2}e^{-2\pi y}+2\pi^2(\log\pi+\gamma-1)ye^{-2\pi y}\right)\frac{dy}{y}.\end{aligned}$$

The integral above is computed by

$$\begin{aligned}&\int_0^\infty y^se^{-2\pi y}\left(\frac{\pi}{2}e^{-2\pi y}+2\pi^2(\log\pi+\gamma-1)ye^{-2\pi y}\right)\frac{dy}{y}\\=&\frac{\pi}{2}\int_0^\infty y^se^{-4\pi y}\frac{dy}{y}+2\pi^2(\log\pi+\gamma-1)\int_0^\infty y^{s+1}e^{-4\pi y}\frac{dy}{y}\\=&\frac{\pi}{2}(4\pi)^{-s}\Gamma(s)+2\pi^2(\log\pi+\gamma-1)(4\pi)^{-s-1}\Gamma(s+1).\end{aligned}$$

Its constant term is equal to

$$\frac{\pi}{2}(-\log(4\pi)-\gamma)+2\pi^2(\log\pi+\gamma-1)(4\pi)^{-1}=-\frac{1}{2}\pi(1+\log4).$$

Hence,

$$\mathcal{P}r'_\psi W'_{1,v}(0,g,u) = 2\pi^2(1+\log 4)W^{(2)}(g).$$

This holds for $u>0$. By the result of (1),

$$W_{1,v}(0,g,u) = -4\pi^2 W^{(2)}(g)1_{F^\times_{v,+}}(u).$$

Then (2) follows. □

Derivative of Whittaker functions: non-archimedean place

Recall that for a non-archimedean place v, we have $\phi_v = 1_{O_{\mathbb{B}_v}\times O^\times_{F_v}}$ if v is split in $\mathbb{B}$, and $\phi_v = 1_{O^\times_{\mathbb{B}_v}\times O^\times_{F_v}}$ if v is nonsplit in $\mathbb{B}$.

Lemma 3.4. *Let v be a non-archimedean place of F, and let $a\in F_v^\times$.*

(1) Let v be a non-archimedean place split in $\mathbb{B}$. Then $W_{a,v}(s,1,u)$ is nonzero only if $u\in O^\times_{F_v}$ and $v(a)\ge -v(d_v)$. In the case $u\in O^\times_{F_v}$ and $a\in O_{F_v}$,

$$\begin{aligned} W_{a,v}(s,1,u) &= |d_v|^{s+\frac{3}{2}}\frac{(1-N_v^{-(s+2)})(1-N_v^{-(v(a)+1)(s+1)})}{1-N_v^{-(s+1)}} \\ &\quad + |d_v|^{\frac{5}{2}}\frac{(1-N_v^{-s})(1-|d_v|^{s-1})}{1-N_v^{-(s-1)}}. \end{aligned}$$

Therefore, for $u\in O^\times_{F_v}$ and $a\in O_{F_v}$,

$$\begin{aligned} & W'_{a,v}(0,1,u) - \frac{1}{2}\log|a|_v W_{a,v}(0,1,u) \\ = \ & \left(-\zeta_v'(2)/\zeta_v(2)+\log|d_v|\right)W_{a,v}(0,1,u) \\ & +|d_v|^{\frac{3}{2}}\frac{1+N_v^{-1}}{2(1-N_v^{-1})}\left((r+2)N_v^{-(r+1)} - rN_v^{-(r+2)} - (r+2)N_v^{-1}+r\right)\log N_v \\ & +|d_v|^{\frac{3}{2}}\frac{1-|d_v|}{N_v-1}\log N_v. \end{aligned}$$

Here $r=v(a)$.

(2) Let v be a non-archimedean place nonsplit in $\mathbb{B}$. Then $W_{a,v}(s,1,u)$ is nonzero only if $v(a)\ge -v(d_v)$ and $u\in O^\times_{F_v}$, and it is constant (depending on s) for $(a,u)\in p_v\times O^\times_{F_v}$. Moreover, for any $u\in F_v^\times$,

$$\int_{F_v}\left(W'_{a,v}(0,1,u) - \frac{1}{2}\log|a|_v W_{a,v}(0,1,u)\right)da = 0.$$

Proof. The calculation is rather involved due to the non-triviality of d_v. To simplify the calculation, we move between two different types of methods. We divide the process into three steps due to the ramifications of v in $\mathbb{B}$ and over $\mathbb{Q}$.

Step 1. unramified case: v is split in $\mathbb{B}$ and $|d_v| = 1$. Apply the formula

$$W_{a,v}(s,1,u) = \int_{F_v} \delta(wn(b))^s \ r(wn(b))\phi_v(0,u)\psi_v(-ab)db.$$

Note that ϕ_v is invariant under the action of $\mathrm{GL}_2(O_{F_v})$ (as symplectic similitudes), as $\mathrm{GL}_2(O_{F_v})$ is generated by $w, m(a), n(b)$ with all $a \in O_{F_v}^\times, b \in O_{F_v}$. Thus the Iwasawa decomposition gives

$$r(g)\phi_v(0,u) = \delta(g)^2 \ 1_{O_{F_v}^\times}(u), \quad g \in \mathrm{SL}_2(F_v).$$

Notice

$$\delta(wn(b)) = \begin{cases} 1 & \text{if } b \in O_{F_v}, \\ |b|^{-1} & \text{otherwise.} \end{cases}$$

Assuming $u \in O_{F_v}^\times$ (so that $W_{a,v}(s,1,u) \neq 0$), we have

$$\begin{aligned} W_{a,v}(s,1,u) &= \int_{F_v} \delta(wn(b))^{s+2}\psi_v(-ab)db \\ &= \int_{O_{F_v}} \psi_v(-ab)db + \int_{F_v - O_{F_v}} |b|^{-(s+2)}\psi_v(-ab)db. \end{aligned}$$

Write the domain $F_v - O_{F_v}$ of the second integral as a disjoint union of $p_v^{-n} - p_v^{-(n-1)}$ for $n \geq 1$. We have

$$\begin{aligned} W_{a,v}(s,1,u) &= \int_{O_{F_v}} \psi_v(-ab)db + \sum_{n=1}^{\infty} \int_{p_v^{-n} - p_v^{-(n-1)}} N_v^{-n(s+2)}\psi_v(-ab)db \\ &= \int_{O_{F_v}} \psi_v(-ab)db + \sum_{n=1}^{\infty} \int_{p_v^{-n}} N_v^{-n(s+2)}\psi_v(-ab)db - \sum_{n=1}^{\infty} \int_{p_v^{-(n-1)}} N_v^{-n(s+2)}\psi_v(-ab)db \\ &= (1 - N_v^{-(s+2)}) \sum_{n=0}^{\infty} N_v^{-n(s+2)} \int_{p_v^{-n}} \psi_v(-ab)db. \end{aligned}$$

It is nonzero only if $a \in O_{F_v}$. In that case,

$$W_{a,v}(s,1,u) = (1 - N_v^{-(s+2)}) \sum_{n=0}^{v(a)} N_v^{-n(s+2)} N_v^n = \frac{(1 - N_v^{-(s+2)})(1 - N_v^{-(v(a)+1)(s+1)})}{1 - N_v^{-(s+1)}}.$$

It follows that

$$W_{a,v}(0,1,u) = \frac{(1 - N_v^{-2})(1 - N_v^{-(v(a)+1)})}{1 - N_v^{-1}} = (1 + N_v^{-1})(1 - N_v^{-(v(a)+1)})$$

and

$$\begin{aligned} & W'_{a,v}(0,1,u) - \frac{1}{2}\log|a|_v W_{a,v}(0,1,u) \\ =\ & W_{a,v}(0,1,u)\Big(\frac{W'_{a,v}(0,1,u)}{W_{a,v}(0,1,u)} - \frac{1}{2}\log|a|_v\Big) \\ =\ & W_{a,v}(0,1,u)\Big(\frac{N_v^{-2}}{1 - N_v^{-2}} + \frac{(v(a)+1)N_v^{-(v(a)+1)}}{1 - N_v^{-(v(a)+1)}} - \frac{N_v^{-1}}{1 - N_v^{-1}} + \frac{1}{2}v(a)\Big)\log N_v \\ =\ & \frac{N_v^{-2}\log N_v}{1 - N_v^{-2}} W_{a,v}(0,1,u) + \frac{1 - N_v^{-2}}{2(1 - N_v^{-1})^2}\Big((r+2)N_v^{-(r+1)} - rN_v^{-(r+2)} - (r+2)N_v^{-1} + r\Big)\log N_v. \end{aligned}$$

This proves part (1) under $|d_v| = 1$.

Step 2. A general formula: By the proof of [YZZ, Proposition 6.10(1)], we have

$$W_{a,v}(s,1,u,\phi_v) = \gamma(\mathbb{B}_v, uq)|d_v|^{\frac{1}{2}}(1 - N_v^{-s}) \sum_{n=0}^{\infty} N_v^{-n(s-1)} \int_{D_n(a)} \phi_v(x,u) d_u x,$$

where $d_u x$ is the self-dual measure of $(\mathbb{B}_v, uq)$, and

$$D_n(a) = \{x \in \mathbb{B}_v : uq(x) \in a + p_v^n d_v^{-1}\}$$

is a subset of $\mathbb{B}_v$. The local Weil index $\gamma(\mathbb{B}_v, uq) = \pm 1$ coincides with the Hasse invariant of $\mathbb{B}_v$. Note that the quadratic space $(\mathbb{B}_v, uq)$ here is different from the quadratic space $(E_v \mathfrak{j}_v, uq)$ in [YZZ, Proposition 6.10(1)], but the proof is similar.

It is easy to see that $W_{a,v}(s,1,u) \neq 0$ only if $u \in O_{F_v}^\times$ and $v(a) \geq -v(d_v)$. In the following, we always assume $u \in O_{F_v}^\times$ and $v(a) \geq 0$.

Step 3. matrix case: v is split in $\mathbb{B}$ and d_v is arbitrary. By the normalization of $\psi_v : F_v \to \mathbb{C}^\times$ in [YZZ, §1.6.1], the characteristic function $1_{O_{F_v}}$ is not self-dual under ψ_v if $|d_v| \neq 1$. Consequently, ϕ_v is not invariant under the action of $\mathrm{GL}_2(O_{F_v})$. Then the method of Step 1 does not work in this case, and we are going to use the formula in Step 2.

We have

$$W_{a,v}(s,1,u) = |d_v|^{\frac{1}{2}}(1 - N_v^{-s}) \sum_{n=0}^{\infty} N_v^{-n(s-1)} \mathrm{vol}(D_n(a) \cap O_{\mathbb{B}_v})$$

with

$$D_n(a) \cap O_{\mathbb{B}_v} = \{x \in O_{\mathbb{B}_v} : uq(x) \in a + p_v^n d_v^{-1}\}.$$

By $\mathrm{vol}(O_{\mathbb{B}_v}) = |d_v|^2$, we write

$$W_{a,v}(s,1,u) = |d_v|^{\frac{5}{2}}(1 - N_v^{-s}) \sum_{n=0}^{\infty} N_v^{-n(s-1)} \frac{\mathrm{vol}(D_n(a) \cap O_{\mathbb{B}_v})}{\mathrm{vol}(O_{\mathbb{B}_v})}.$$

We use this expression because it holds for any Haar measure on $\mathbb{B}_v$. Split the summation according to $n < v(d_v)$ and $n \geq v(d_v)$. It gives

$$W_{a,v}(s,1,u) = W_{a,v}(s,1,u)_{n<v(d_v)} + W_{a,v}(s,1,u)_{n\geq v(d_v)}$$

accordingly. In the following we compute the terms on the right-hand side separately.

By the assumption $a \in O_{F_v}$, for any $n < v(d_v)$, we have

$$D_n(a) = \{x \in \mathbb{B}_v : q(x) \in p_v^n d_v^{-1}\} \supset O_{\mathbb{B}_v}.$$

It follows that

$$W_{a,v}(s,1,u)_{n<v(d_v)} = |d_v|^{\frac{5}{2}}(1 - N_v^{-s}) \sum_{n=0}^{v(d_v)-1} N_v^{-n(s-1)} = |d_v|^{\frac{5}{2}} \frac{(1 - N_v^{-s})(1 - |d_v|^{s-1})}{1 - N_v^{-(s-1)}}.$$

A direct calculation of $W_{a,v}(s,1,u)_{n\geq v(d_v)}$ is quite involved, so we compare it with the unramified case instead. For clarification, denote

$$D_n(a)^\circ = \{x \in \mathbb{B}_v : uq(x) \in a + p_v^n\},$$

which is equal to the set $D_n(a)$ in the unramified case in Step 1. For $n \geq v(d_v)$, the substitution $n \mapsto n + v(d_v)$ gives

$$W_{a,v}(s,1,u)_{n\geq v(d_v)} = |d_v|^{\frac{5}{2}}|d_v|^{s-1}(1-N_v^{-s})\sum_{n=0}^{\infty} N_v^{-n(s-1)}\frac{\mathrm{vol}(D_n(a)^\circ \cap O_{\mathbb{B}_v})}{\mathrm{vol}(O_{\mathbb{B}_v})}.$$

This is equal to $W_{a,v}(s,1,u)$ in the case $|d_v| = 1$ considered in Step 1. In other words, the result of Step 1 gives the combinatorial equality

$$(1-N_v^{-s})\sum_{n=0}^{\infty} N_v^{-n(s-1)}\frac{\mathrm{vol}(D_n(a)^\circ \cap O_{\mathbb{B}_v})}{\mathrm{vol}(O_{\mathbb{B}_v})} = \frac{(1-N_v^{-(s+2)})(1-N_v^{-(v(a)+1)(s+1)})}{1-N_v^{-(s+1)}}$$

in the current setting. Hence, we have

$$W_{a,v}(s,1,u)_{n\geq v(d_v)} \quad = \quad |d_v|^{s+\frac{3}{2}}\frac{(1-N_v^{-(s+2)})(1-N_v^{-(v(a)+1)(s+1)})}{1-N_v^{-(s+1)}}.$$

Now we have a formula for $W_{a,v}(s,1,u)$, and some elementary computations finish the proof of part (1) of the lemma.

Step 4. division case: v is nonsplit in $\mathbb{B}$. Then the formula in Step 2 becomes

$$W_{a,v}(s,1,u) = -|d_v|^{\frac{1}{2}}(1-N_v^{-s})\sum_{n=0}^{\infty} N_v^{-n(s-1)}\mathrm{vol}(D_n(a)\cap O_{\mathbb{B}_v}^\times),$$

where

$$D_n(a)\cap O_{\mathbb{B}_v}^\times = \{x \in O_{\mathbb{B}_v}^\times : uq(x) \in a + p_v^n d_v^{-1}\}.$$

Here we have assumed $u \in O_{F_v}^\times$; otherwise, $W_{a,v}(s,1,u) = 0$. Assume $v(a) \geq -v(d_v)$ by the same reason.

If $v(a) > 0$, the condition $uq(x) \in a + p_v^n d_v^{-1}$ is equivalent to $n \leq v(d_v)$. In this case $D_n(a)\cap O_{\mathbb{B}_v}^\times = O_{\mathbb{B}_v}^\times$. Therefore,

$$\begin{aligned} W_{a,v}(s,1,u) &= -|d_v|^{\frac{1}{2}}(1-N_v^{-s})\sum_{n=0}^{v(d_v)} N_v^{-n(s-1)}\mathrm{vol}(O_{\mathbb{B}_v}^\times) \\ &= -|d_v|^{\frac{1}{2}}(1-N_v^{-s})\frac{1-N_v^{(v(d_v)+1)(1-s)}}{1-N_v^{1-s}}\mathrm{vol}(O_{\mathbb{B}_v}^\times). \end{aligned}$$

This proves the first assertion in (2).

It remains to verify the formula

$$\int_{F_v}\left(W'_{a,v}(0,1,u) - \frac{1}{2}\log|a|_v W_{a,v}(0,1,u)\right)da = 0.$$

We first check that

$$\log|a|_v W_{a,v}(0,1,u)=0, \qquad \forall a\in F_v^\times.$$

In fact, it suffices to check $W_{a,v}(0,1,u)=0$ if $v(a)\neq 0$. This is an easy consequence of the local Siegel–Weil formula in [YZZ, Theorem 2.2] or [YZZ, Proposition 2.9]. Alternatively, we can verify it by the type of computation here. Since we already know the vanishing for $v(a)>0$ from the above computation, it remains to check the case $-v(d_v)\leq v(a)<0$. In this case, the condition $uq(x)-a\in p_v^n d_v^{-1}$ with $x\in O_{\mathbb{B}_v}^\times$ is equivalent to $a\in p_v^n d_v^{-1}$. It holds only if $n\leq v(ad_v)$. Under this condition $D_n(a)\cap O_{\mathbb{B}_v}^\times = O_{\mathbb{B}_v}^\times$. It follows that

$$\begin{aligned} W_{a,v}(s,1,u) &= -|d_v|^{\frac12}(1-N_v^{-s})\sum_{n=0}^{v(ad_v)} N_v^{-n(s-1)}\mathrm{vol}(O_{\mathbb{B}_v}^\times) \\ &= -|d_v|^{\frac12}(1-N_v^{-s})\frac{1-N_v^{(v(ad_v)+1)(1-s)}}{1-N_v^{1-s}}\mathrm{vol}(O_{\mathbb{B}_v}^\times). \end{aligned}$$

Thus $W_{a,v}(0,1,u)=0$.

It is reduced to prove

$$\int_{F_v} W'_{a,v}(0,1,u)da=0.$$

We are going to compute

$$\int_{F_v} W_{a,v}(s,1,u)da = -|d_v|^{\frac12}(1-N_v^{-s})\sum_{n=0}^{\infty} N_v^{-n(s-1)}\int_{F_v}\mathrm{vol}(D_n(a)\cap O_{\mathbb{B}_v}^\times)da.$$

Use a Fuibini type of result to change the order of the last integral. We have

$$\int_{F_v}\mathrm{vol}(D_n(a)\cap O_{\mathbb{B}_v}^\times)da = \iint_{uq(x)-a\in p_v^n d_v^{-1}} dxda = \int_{O_{\mathbb{B}_v}^\times}\int_{uq(x)+p_v^n d_v^{-1}} dadx = \mathrm{vol}(O_{\mathbb{B}_v}^\times)|d_v|^{-\frac12}N_v^{-n}.$$

Hence,

$$\int_{F_v} W_{a,v}(s,1,u)da = -|d_v|^{\frac12}(1-N_v^{-s})\sum_{n=0}^{\infty} N_v^{-n(s-1)}\mathrm{vol}(O_{\mathbb{B}_v}^\times)|d_v|^{-\frac12}N_v^{-n} = -\mathrm{vol}(O_{\mathbb{B}_v}^\times).$$

Taking derivative at $s=0$, we get the desired result. The proof is complete. □

Derivative of intertwining operators

Recall that for $\phi_v=\phi_{1,v}\otimes\phi_{2,v}$, we have

$$c_{\phi_v}(g,y,u) = r_E(g)\phi_{1,v}(y,u)W_{0,v}^{\circ}{}'(0,g,u,\phi_{2,v}) + \log\delta(g_v)r(g)\phi_v(y,u),$$

where the normalization

$$W_{0,v}^\circ(s,g,u,\phi_{2,v}) = \gamma_{u,v}^{-1}|D_v|^{-\frac12}|d_v|^{-\frac12}\frac{L(s+1,\eta_v)}{L(s,\eta_v)}W_{0,v}(s,g,u,\phi_{2,v}).$$

Here $\gamma_{u,v}$ is the Weil index of $(E_v\mathfrak{j}_v, uq)$. The following result is a variant of [YZ, Lemma 7.6].

Lemma 3.5. *For any non-archimedean place v nonsplit in $\mathbb{B}$,*

$$r(w)\phi_v(0,u) = -|d_v|^2 N_v^{-1}(1-N_v^{-2})1_{O_{F_v}^\times}(u),$$

and

$$c_{\phi_v}(w,0,u) = \begin{cases} \dfrac{(1-N_v)\log N_v}{1+N_v} r(w)\phi_v(0,u) & \text{if } 2 \mid v(d_v), \\ 0 & \text{if } 2 \nmid v(d_v). \end{cases}$$

Proof. Note that v is inert in E by assumption, and

$$\phi_v = 1_{O_{\mathbb{B}_v}^\times \times O_{F_v}^\times}, \qquad \phi_{1,v} = 1_{O_{E_v}^\times \times O_{F_v}^\times}, \qquad \phi_{2,v} = 1_{O_{E_v}\mathfrak{j}_v \times O_{F_v}^\times}.$$

We need to compute

$$c_{\phi_v}(w,0,u) = r(w)\phi_{1,v}(0,u) W_{0,v}^{\circ}{}'(0,w,u,\phi_{2,v}).$$

It is easy to have

$$r(w)\phi_{1,v}(0,u) = \gamma(E_v,uq)1_{O_{F_v}^\times}(u)\int_{O_{E_v}^\times} dx = \gamma(E_v,uq)|d_v|(1-N_v^{-2})1_{O_{F_v}^\times}(u),$$

$$r(w)\phi_{2,v}(x_2,u) = \gamma(E_v\mathfrak{j}_v,uq)|d_v q(\mathfrak{j}_v)| \cdot 1_{d_v^{-1}q(\mathfrak{j}_v)^{-1}O_{E_v}\mathfrak{j}_v}(x_2)1_{O_{F_v}^\times}(u).$$

This proves the first result, as $v(q(\mathfrak{j}_v)) = 1$ by assumption, and

$$\gamma(E_v,uq)\gamma(E_v\mathfrak{j}_v,uq) = \gamma(\mathbb{B}_v,uq) = -1.$$

Now we prove the second identity. From the definition

$$W_{0,v}(s,g,u,\phi_{2,v}) = \int_{F_v} \delta(wn(b)g)^s r(wn(b)g)\phi_{2,v}(0,u)db,$$

we have

$$W_{0,v}(s,w,u,\phi_{2,v}) = W_{0,v}(s,1,u,r(w)\phi_{2,v}).$$

Its computation is similar to that of [YZ, Lemma 7.6]. In fact, we still have

$$W_{0,v}(s,1,u,r(w)\phi_{2,v}) = \gamma(E_v\mathfrak{j}_v,uq)|d_v|^{\frac{1}{2}}(1-N_v^{-s})\sum_{n=0}^{\infty} N_v^{-ns+n}\int_{D_n} r(w)\phi_{2,v}(x_2,u)d_u x_2,$$

where

$$D_n = \{x_2 \in E_v\mathfrak{j}_v : uq(x_2) \in p_v^n d_v^{-1}\}$$

and the measure $d_u x_2$ gives $\mathrm{vol}(O_{E_v}\mathfrak{j}_v) = |d_v u q(\mathfrak{j}_v)|$. It follows that

$$W_{0,v}^{\circ}(s,w,u,\phi_{2,v}) = \frac{1-N_v^{-2s}}{1+N_v^{-(s+1)}}\sum_{n=0}^{\infty} N_v^{-ns+n}\int_{D_n} r(w)\phi_{2,v}(x_2,u)d_u x_2.$$

Apply the formula of $r(w)\phi_{2,v}(x_2,u)$ in the above. Assume $u \in O_{F_v}^\times$ in the following. Note that for any $n \geq 0$,

$$D_n = p_v^{[\frac{n-v(d_v)}{2}]} O_{E_v} \mathfrak{j}_v \ \subset \ d_v^{-1} q(\mathfrak{j}_v)^{-1} O_{E_v} \mathfrak{j}_v.$$

We have

$$W_{0,v}^\circ(s,w,u,\phi_{2,v}) = \gamma(E_v\mathfrak{j}_v, uq)|d_v q(\mathfrak{j}_v)| \frac{1-N_v^{-2s}}{1+N_v^{-(s+1)}} \sum_{n=0}^\infty N_v^{-ns+n} \mathrm{vol}(D_n).$$

The summation is equal to

$$\sum_{n=0}^\infty N_v^{-ns+n} N_v^{-2[\frac{n-v(d_v)}{2}]} |d_v q(\mathfrak{j}_v)| = \begin{cases} N_v^{-1}(1+N_v^{1-s})(1-N_v^{-2s})^{-1} & \text{if } 2 \mid v(d_v), \\ (1+N_v^{-1-s})(1-N_v^{-2s})^{-1} & \text{if } 2 \nmid v(d_v). \end{cases}$$

Hence,

$$W_{0,v}^\circ(s,w,u,\phi_{2,v}) = \gamma(E_v\mathfrak{j}_v, uq)|d_v q(\mathfrak{j}_v)| \cdot \begin{cases} N_v^{-1}(1+N_v^{1-s})(1+N_v^{-1-s})^{-1} & \text{if } 2 \mid v(d_v), \\ 1 & \text{if } 2 \nmid v(d_v). \end{cases}$$

Then

$$W_{0,v}^\circ{}'(0,w,u,\phi_{2,v}) = \gamma(E_v\mathfrak{j}_v, uq)|d_v q(\mathfrak{j}_v)| \cdot \begin{cases} \dfrac{(1-N_v)\log N_v}{1+N_v} & \text{if } 2 \mid v(d_v), \\ 0 & \text{if } 2 \nmid v(d_v). \end{cases}$$

This finishes the proof.

□

An average formula

Let v be a non-archimedean place nonsplit in $\mathbb{B}$. Recall that

$$\bar{k}_{\phi_v}(y,u) = k_{\phi_v}(1,y,u) - m_{\phi_v}(y,u)\log N_v$$

extends to a Schwartz function in $\overline{\mathcal{S}}(B(v)_v \times F_v^\times)$. This is a combination of [YZ, Lemma 7.4] and [YZ, Lemma 8.7]. In the following, we compute the action of w on this Schwartz function.

Lemma 3.6. *For any non-archimedean place v nonsplit in $\mathbb{B}$,*

$$r(w)\bar{k}_{\phi_v}(0,u) = -r(w)\phi_v(0,u) \cdot \begin{cases} \left(\dfrac{N_v}{N_v+1} + \dfrac{v(d_v)}{2}\right)\log N_v, & 2 \mid v(d_v); \\ \dfrac{v(d_v)+1}{2}\log N_v, & 2 \nmid v(d_v). \end{cases}$$

Proof. Write $y = y_1 + y_2$ according to $B(v)_v = E_v + E_v j_v$ as usual. Note that E_v is unramified over F_v, and $v(q(j_v)) = 0$. By [YZ, Lemma 8.7],

$$m_{\phi_v}(y,u) = \phi_v(y_1,u)1_{O_{E_v}j_v}(y_2)\cdot\frac{1}{2}v(q(y_2)).$$

The function

$$k_{\phi_v}(1,y,u) = \frac{L(1,\eta_v)}{\mathrm{vol}(E_v^1)}\phi_{1,v}(y_1,u)W^{\circ}_{uq(y_2),v}{}'(0,1,u,\phi_{2,v}).$$

is computed in [YZ, Lemma 7.4]. Here $\mathrm{vol}(E_v^1) = |d_v|^{\frac{1}{2}}$ in the current case. From the proof of [YZ, Lemma 7.4] (written in [YZ, p. 596]), which has also computed $W^{\circ}_{a,v}{}'(0,1,u)$ for $-v(d_v) \le v(a) < 0$, we have

$$k_{\phi_v}(1,y,u) = (\log N_v)\phi_{1,v}(y_1,u)\cdot\begin{cases} 0, & v(q(y_2)) < -v(d_v);\\ \dfrac{N_v|q(y_2)|^{-1}-|d_v|}{N_v^2-1}, & -v(d_v)\le v(q(y_2)) < 0;\\ \left(\dfrac{N_v-|d_v|}{N_v^2-1}+\dfrac{1}{2}v(q(y_2))\right), & v(q(y_2))\ge 0.\end{cases}$$

It follows that

$$\bar{k}_{\phi_v}(y,u) = (\log N_v)\phi_{1,v}(y_1,u)\phi'_{2,v}(y_2,u),$$

where $\phi'_{2,v}\in\overline{\mathcal{S}}(E_v j_v\times F_v^{\times})$ is given by

$$\phi'_{2,v}(y_2,u) = 1_{O_{F_v}^{\times}}(u)\cdot\begin{cases} 0, & v(q(y_2)) < -v(d_v);\\ \dfrac{N_v|q(y_2)|^{-1}-|d_v|}{N_v^2-1}, & -v(d_v)\le v(q(y_2)) < 0;\\ \dfrac{N_v-|d_v|}{N_v^2-1}, & v(q(y_2))\ge 0.\end{cases}$$

Now we compute

$$r(w)\phi'_{2,v}(0,u) = \gamma(E_v j_v, uq)\int_{E_v j_v}\phi'_{2,v}(y_2,u)d_u y_2.$$

Assume $u\in O_{F_v}^{\times}$. Note that E_v is unramified over F_v, $v(q(j_v)) = 0$, and $\mathrm{vol}(O_{E_v}j_v) = |d_v|$. Writing $v(q(y_2)) = 2i$ for $i\in\mathbb{Z}$, we have

$$r(w)\phi'_{2,v}(0,u) = \gamma(E_v j_v, uq)\left(\sum_{-v(d_v)\le 2i<0}\frac{N_v N_v^{2i}-|d_v|}{N_v^2-1}\mathrm{vol}(\varpi_v^i O_{E_v}^{\times}j_v)+\frac{N_v-|d_v|}{N_v^2-1}\mathrm{vol}(O_{E_v}j_v)\right),$$

where $\varpi_v\in O_{F_v}$ is a uniformizer. It follows that

$$r(w)\phi'_{2,v}(0,u) = \gamma(E_v j_v, uq)|d_v|\left(\sum_{-v(d_v)\le 2i<0}\frac{N_v N_v^{2i}-|d_v|}{N_v^2-1}(1-N_v^{-2})N_v^{-2i}+\frac{N_v-|d_v|}{N_v^2-1}\right).$$

An elementary computation gives

$$r(w)\phi'_{2,v}(0,u) = \gamma(E_v j_v, uq)\cdot|d_v|\cdot 1_{O_{F_v}^\times}(u)\cdot\begin{cases}\dfrac{1}{N_v+1}+\dfrac{v(d_v)}{2N_v}, & 2\mid v(d_v);\\ \dfrac{v(d_v)+1}{2N_v}, & 2\nmid v(d_v).\end{cases}$$

Note that

$$r(w)\phi_{2,v}(0,u) = \gamma(E_v\mathfrak{j}_v, uq)\cdot N_v^{-1}|d_v|\cdot 1_{O_{F_v}^\times}(u).$$

It remains to check $\gamma(E_v j_v, uq) = -\gamma(E_v\mathfrak{j}_v, uq)$ for the Weil indexes. This follows from

$$\gamma(E_v, uq)\gamma(E_v\mathfrak{j}_v, uq) = \gamma(\mathbb{B}_v, uq) = \gamma(\mathbb{B}_v, q) = -1,$$

$$\gamma(E_v, uq)\gamma(E_v j_v, uq) = \gamma(B(v)_v, uq) = \gamma(B(v)_v, q) = 1.$$

□

4 Height series

The goal of this section is to decompose the height series $Z(g,(t_1,t_2))_U$ into a sum of pseudo-theta series and pseudo-Eisenstein series, and compute some related terms. This is mainly treated in [YZZ, YZ], but we do need to compute some extra terms for the purpose here.

Let F be a totally real number field, and $\mathbb{B}$ be a totally definite incoherent quaternion algebra over F with ramification set Σ. For any open compact subgroup U of $\mathbb{B}_f^\times$, we have a Shimura curve X_U, which is a projective and smooth curve over F. For any embedding $v: F\hookrightarrow\mathbb{C}$, it has the usual uniformization

$$X_{U,v}(\mathbb{C}) = B(v)^\times\backslash\mathcal{H}^\pm\times\mathbb{B}_f^\times/U\cup\{\text{cusps}\}.$$

Here $B(v)$ denotes the nearby quaternion algebra, i.e., the unique quaternion algebra over F with ramification set $\Sigma\setminus\{v\}$.

We first recall the generating series in [YZZ, §3.4.5]. For any $\phi\in\overline{\mathcal{S}}(\mathbb{B}\times\mathbb{A}^\times)$ invariant under $K = U\times U$, form a generating series

$$Z(g,\phi)_U = Z_0(g,\phi)_U + Z_*(g,\phi)_U,\quad g\in\mathrm{GL}_2(\mathbb{A}),$$

where

$$\begin{aligned} Z_0(g,\phi)_U &= -\sum_{\alpha\in F_+^\times\backslash\mathbb{A}_f^\times/q(U)}\ \sum_{u\in\mu_U^2\backslash F^\times} E_0(\alpha^{-1}u, r(g)\phi)\ L_{K,\alpha},\\ Z_*(g,\phi)_U &= w_U\sum_{a\in F^\times}\ \sum_{x\in U\backslash\mathbb{B}_f^\times/U} r(g)\phi(x, aq(x)^{-1})\ Z(x)_U.\end{aligned}$$

Here $\mu_U = F^\times\cap U$, and $w_U = |\{1,-1\}\cap U|$ is equal to 1 or 2. For any $x\in\mathbb{B}_f^\times$, $Z(x)_U$ notes the Hecke correspondence on X_U determined by the double coset UxU.

Let E/F be a totally imaginary quadratic extension, with a fixed embedding $E_{\mathbb{A}} \hookrightarrow \mathbb{B}$ over $\mathbb{A}$. Recall from [YZZ, §3.5.1, §5.1.2] and [YZ, §8.1] that we have a height series

$$Z(g,(t_1,t_2),\phi)_U = \langle Z(g,\phi)_U\ t_1^\circ,\ t_2^\circ\rangle_{\mathrm{NT}}, \quad t_1,t_2 \in E^\times(\mathbb{A}_f).$$

Here $Z(g,\phi)_U$ acts on t_1° as correspondences, and the Neron–Tate height over F is defined as in [YZZ, §7.1.2]. By the modularity in [YZZ, Theorem 3.17], $Z(g,(t_1,t_2),\phi)_U$ is an automorphic form in $g \in \mathrm{GL}_2(\mathbb{A})$. By [YZZ, Lemma 3.19], it is actually a cusp form. In particular, the constant term $Z_0(g,\phi)$ of the generating function plays no role here.

4.1 Weakly admissible extensions

In order to decompose the height series in terms of the arithmetic Hodge index theorem of Faltings–Hriljac, the notion of admissible extensions are used in [YZ, YZZ]. However, there is a minor mistake involving misconceptions about admissible extensions in [YZ, YZZ]. In fact, the Green's function is not admissible, but only weakly admissible in the current sense. As we will see, this mistake does not affect the main results of [YZ, YZZ], but it does affect the results here. In the following, we review the admissibility notion as described in [YZZ, §7.1-7.2], introduce the weak admissibility notion in the mean time, and then point out the mistake and the correction.

Weakly admissible extensions

Resume the terminology in [YZZ, §7.1.3-§7.1.4]. We will review some terminology of [YZZ, §7.1.5] and make some additional definitions in the following.

Let X be a projective and smooth curve over a number field F. By taking the definitions over every connected component, we can assume that X is connected, but we do not assume that it is geometrically connected. Denote by F' the algebraic closure of F in the function field $F(X)$, so that X is geometrically connected over F'.

Let $\mathcal{X}$ be a projective, flat, and regular integral model of X over O_F. Note that $\mathcal{X}$ is also a scheme over $O_{F'}$. Fix an arithmetic divisor class $\hat{\xi} \in \widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{Q}}$ whose generic fiber has degree 1 on X over F'.

Let $\widehat{D} = (D, g_D)$ be an arithmetic divisor on $\mathcal{X}$. We can always write $D = H + V$ where H is the horizontal part of D, and V is the vertical part of D. The arithmetic divisor $\widehat{D}$ is called *$\hat{\xi}$-admissible* if the following conditions hold:

(1) The difference $\widehat{D} - \deg D \cdot \hat{\xi}$ is flat over $\mathcal{X}$;

(2) The intersection number $(V \cdot \hat{\xi})_{v'} = 0$ for any non-archimedean place v' of F';

(3) The integral $\int_{X_{\sigma'}(\mathbb{C})} g_D c_1(\hat{\xi}) = 0$ for any embedding $\sigma' : F' \to \mathbb{C}$.

The arithmetic divisor $\widehat{D}$ is called *weakly $\hat{\xi}$-admissible* if it satisfies conditions (1) and (2).

A $\hat{\xi}$-*admissible extension* (resp. *weakly* $\hat{\xi}$-*admissible extension*) of a divisor D_0 over X is a $\hat{\xi}$-admissible (resp. *weakly* $\hat{\xi}$-*admissible*) arithmetic divisor $\widehat{D} = (D, g_D)$ over $\mathcal{X}$ such that the generic fiber $D_F = D_0$ over X.

The notion $\hat{\xi}$-*admissible* is introduced in [YZZ, §7.1.5], while the notion *weakly* $\hat{\xi}$-*admissible* is a new one added here. Note that $\hat{\xi}$-admissible extension exists and is unique. On the other hand, without condition (3), condition (1) only determines the Green's function up to constant functions over $X_{\sigma'}(\mathbb{C})$.

Nonetheless, in our calculation over the Shimura curve, we do have a fixed choice of Green's functions as follows. To illustrate the idea, we will specify a symmetric and smooth function $g : X(\mathbb{C}) \times X(\mathbb{C}) \setminus \Delta \to \mathbb{R}$ such that for any $P \in X(\mathbb{C})$, the 1-variable function $g(P, \cdot)$ is a Green's function for the divisor P over $X(\mathbb{C})$ with curvature form equal to $c_1(\hat{\xi})$. Then for any divisor D over X, we take the Green's function $g_D = g(D, \cdot)$.

Let D_1, D_2 be two divisors over $X_{\overline{F}}$. Then D_1, D_2 are realized as divisors over X_L for a finite extension L of F. Assume that L is unramified over any place of F at which $\mathcal{X}$ has bad reduction, so that $\mathcal{X}_{O_L}$ is still regular. By abuse of notations, still denote the pull-back of $\hat{\xi}$ to $\mathcal{X}_{O_L}$ by $\hat{\xi}$.

For $i = 1, 2$, let $\widehat{D}_i = (\overline{D}_i + V_i, g_i)$ be a weakly $\hat{\xi}$-admissible extension of D_i over $\mathcal{X}_{O_L}$. Here $\overline{D}_i$ is the Zariski closure of D_i in $\mathcal{X}_{O_L}$, V_i is the (uniquely determined) vertical divisor over $\mathcal{X}_{O_L}$, and $g_i = g_{D_i}$ is a Green's function over $X_L(\mathbb{C})$ determined by the 2-variable function g above. As in [YZZ, §7.1.6], it will be convenient to denote

$$\langle D_1, D_2 \rangle := -\frac{1}{[L : F]} \widehat{D}_1 \cdot \widehat{D}_2.$$

The definition is independent of the choice of L. We will have a decomposition $\langle \cdot, \cdot \rangle = -i - j$ in the following.

We first have equalities

$$\widehat{D}_1 \cdot \widehat{D}_2 = (\overline{D}_1, g_1) \cdot (\overline{D}_2 + V_2, g_2) = (\overline{D}_1, g_1) \cdot \overline{D}_2 + \overline{D}_1 \cdot V_2 + \int_{X_L(\mathbb{C})} g_2\, c_1(D_1, g_1).$$

Here the first equality holds by $V_1 \cdot \widehat{D}_2 = 0$, a consequence of condition (1) for $\widehat{D}_2$ and condition (2) for V_1.

So we can write

$$\langle D_1, D_2 \rangle = -i(D_1, D_2) - j(D_1, D_2)$$

with

$$i(D_1, D_2) := \frac{1}{[L : F]} (\overline{D}_1, g_1) \cdot \overline{D}_2$$

and

$$j(D_1, D_2) := \frac{1}{[L : F]} \overline{D}_1 \cdot V_2 + \frac{1}{[L : F]} \int_{X_L(\mathbb{C})} g_2\, c_1(D_1, g_1).$$

We further have a decomposition according to places v of F by

$$j(D_1, D_2) = \sum_{v \nmid \infty} j_v(D_1, D_2) \log N_v$$

with

$$j_v(D_1,D_2) := \begin{cases} \dfrac{1}{[L:F]}(\overline{D}_1\cdot V_2)_v, & v\nmid\infty, \\ \dfrac{1}{[L:F]}\displaystyle\int_{X_v(\mathbb{C})} g_2\, c_1(D_1,g_1), & v\mid\infty. \end{cases}$$

Here we take the convention $\log N_v = 1$ for archimedean v. The local intersection numbers make sense by viewing $\mathcal{X}_{O_L}$ as a scheme over O_F.

Note that $j_v(D_1,D_2)$ for archimedean v does not necessarily vanish if $\widehat{D}_2$ is not $\hat{\xi}$-admissible but only weakly $\hat{\xi}$-admissible. This is different from [YZZ, §7.1.7], where it considers the $\hat{\xi}$-admissible case, and thus $j_v(D_1,D_2)=0$ for archimedean v.

If D_1,D_2 have disjoint supports over $X_{\overline{F}}$, we can also decompose

$$i(D_1,D_2)=\sum_v i_v(D_1,D_2)\log N_v$$

with

$$i_v(D_1,D_2) := \begin{cases} \dfrac{1}{[L:F]}(\overline{D}_1\cdot \overline{D}_2)_v, & v\nmid\infty, \\ \dfrac{1}{[L:F]} g_1(D_{2,v}(\mathbb{C})), & v\mid\infty. \end{cases}$$

Each of the pairings i,j,i_v,j_v is symmetric as long as it is defined. In fact, for non-archimedean v, this is automatic for i_v, and this holds for j_v by $(\overline{D}_1\cdot V_2)_v = -(V_1\cdot V_2)_v$. For archimedean v, $g_1(D_{2,v}(\mathbb{C})) = g_2(D_{1,v}(\mathbb{C}))$ as they come from the same symmetric 2-variable function g, so i_v is symmetric, which implies the symmetry of j_v by Stokes' formula.

As in [YZZ, §7.1.7], we can also introduce the pairings $i_{\bar{v}}$ and $j_{\bar{v}}$, and write i_v and j_v respectively as averages of $i_{\bar{v}}$ and $j_{\bar{v}}$ over the Galois group $\mathrm{Gal}(\overline{F}/F)$.

The mistake in [YZZ, YZ] is that the arithmetic extensions used to compute the height pairing are not $\hat{\xi}$-admissible, but only weakly $\hat{\xi}$-admissible extension. This will incur $j_v(D_1,D_2)$ for archimedean v. In the following, we first review the Green's function, compute this extra term, and then decompose the height series by taking into account of the integration term. We will see that the extra term does not affect the main results of [YZZ, YZ], but do affect the main result of this paper.

Integral of the Green's function

Return to the situation that F is totally real, and X_U is a Shimura curve over F. Fix an archimedean place v of F. The Green's function g over $X_{U,v}(\mathbb{C})$ is defined in [YZZ, §8.1.1] following the original idea of Gross–Zagier [GZ]. Let us recall it briefly.

For any two points $z_1,z_2\in\mathcal{H}$, the hyperbolic cosine of the hyperbolic distance between them is given by

$$d(z_1,z_2) = 1+\frac{|z_1-z_2|^2}{2\mathrm{Im}(z_1)\mathrm{Im}(z_2)}.$$

It is invariant under the action of $\mathrm{GL}_2(\mathbb{R})$. For any $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$, denote

$$m_s(z_1,z_2)=Q_s(d(z_1,z_2)),$$

where

$$Q_s(t) = \int_0^\infty \left(t + \sqrt{t^2-1}\cosh u\right)^{-1-s} du$$

is the Legendre function of the second kind.

Denote by $B = B(v)$ the nearby quaternion algebra. For any two distinct points of

$$X_{U,v}(\mathbb{C}) = B_+^\times \backslash \mathcal{H} \times B^\times(\mathbb{A}_f)/U$$

represented by $(z_1, \beta_1), (z_2, \beta_2) \in \mathcal{H} \times B_{\mathbb{A}_f}^\times$, we denote

$$g_s((z_1,\beta_1),(z_2,\beta_2)) := \sum_{\gamma \in \mu_U \backslash B_+^\times} m_s(z_1, \gamma z_2)\ 1_U(\beta_1^{-1}\gamma\beta_2).$$

It converges for $\mathrm{Re}(s) > 0$ and has meromorphic continuation to $s = 0$ with a simple pole.

The Green's function $g : X_{U,v}(\mathbb{C})^2 \smallsetminus \Delta \to \mathbb{R}$ is defined by

$$g((z_1,\beta_1),(z_2,\beta_2)) := \widetilde{\lim}_{s\to 0}\ g_s((z_1,\beta_1),(z_2,\beta_2)).$$

Here $\widetilde{\lim}_{s\to 0}$ denotes the constant term at $s = 0$ of the Laurent expansion of $g_s((z_1,\beta_1),(z_2,\beta_2))$. In particular, for a fixed point $P = (z_1, \beta_1) \in X_{U,v}(\mathbb{C})$, we can view $g(P, \cdot)$ as a function over $X_{U,v}(\mathbb{C})$ with logarithmic singularity at P.

The first part of the following result is a classical one in the computation of Selberg's trace formula, which is a special case of [OT, Proposition 6.3.1(3)]. The second part of the following result is essentially a special case of [OT, Proposition 3.1.2], and our proof is a variant of that of the loc. cit..

Lemma 4.1. *Let v be an archimedean place of F and $P \in X_{U,v}(\mathbb{C})$ be a point.*

(1) The residue $\mathrm{Res}_{s=0} g_s(P,Q)$ is nonzero only if Q lies in the same connected component as P. In that case,

$$\mathrm{Res}_{s=0} g_s(P,Q) = \frac{1}{\kappa_U^\circ},$$

where κ_U° denotes the degree of L_U on a connected component of $X_{U,v}(\mathbb{C})$.

(2) The integral

$$\int_{X_{U,v}(\mathbb{C})} g(P,\cdot) c_1(\overline{\mathcal{L}}_U) = -1.$$

Proof. We will prove that for $\mathrm{Re}(s) > 0$,

$$\int_{X_{U,v}(\mathbb{C})} g_s(P,\cdot) c_1(\overline{\mathcal{L}}_U) = \frac{1}{s(s+1)}.$$

This implies (2) by taking the constant term. It also implies (1). In fact, the differential equation of the Legendre function transfers to a functional equation

$$\Delta g_s(P,\cdot) = s(s+1) g_s(P,\cdot).$$

This implies $\Delta(\mathrm{Res}_{s=0} g_s(P,\cdot)) = 0$, since $g_s(P,\cdot)$ has at most a simple pole at $s = 0$. It follows that $\mathrm{Res}_{s=0} g_s(P,\cdot)$ is constant on the connected component of P. Then the integration of $g_s(P,\cdot)$ determines the constant.

Now prove the formula for the integration of $g_s(P,\cdot)$. Denote $P = (z_1, \beta_1)$ as above. As in [YZZ, §8.1.1], the function $g_s(P,\cdot)$ is nonzero only over the connected component of $X_{U,v}(\mathbb{C})$ containing P. This connected component is isomorphic to $\Gamma\backslash\mathcal{H}$ with $\Gamma = B_+^\times \cap \beta_1 U \beta_1^{-1}$, and the embedding $\Gamma\backslash\mathcal{H} \to X_{U,v}(\mathbb{C})$ is given by $z \mapsto (z, \beta_1)$. Then the induced function $g_s(P,\cdot)$ on $\Gamma\backslash\mathcal{H}$ is given by

$$g_s(P,z) = g_s((z_1,\beta_1),(z,\beta_1)) = \sum_{\gamma\in\mu_U\backslash B_+^\times} m_s(z_1,\gamma z)\ 1_U(\beta_1^{-1}\gamma\beta_1) = \sum_{\gamma\in\mu_U\backslash\Gamma} m_s(z_1,\gamma z).$$

It follows that

$$\int_{X_{U,v}(\mathbb{C})} g_s(P,\cdot) c_1(\overline{\mathcal{L}}_U) = \int_{\Gamma\backslash\mathcal{H}} \sum_{\gamma\in\mu_U\backslash\Gamma} m_s(z_1,\gamma z) c_1(\overline{\mathcal{L}}_U).$$

Note that the stabilizer of $\mathcal{H}$ in Γ is exactly $\Gamma \cap F^\times = \mu_U$. Moreover, as in the proof of [YZZ, Lemma 3.1], $c_1(\overline{\mathcal{L}}_U)$ is represented by the standard volume form $\dfrac{dx\wedge dy}{2\pi y^2}$ over $\mathcal{H}$. Therefore, the integral is further equal to

$$\int_{\mathcal{H}} m_s(z_1,z)\frac{dxdy}{2\pi y^2},$$

where $z = x+iy$ is as in the convention.

Note that $m_s(\gamma z_1, \gamma z) = m_s(z_1,z)$ for any $\gamma \in \mathrm{SL}_2(\mathbb{R})$, the above integral is independent of z_1. It follows that we can assume $z_1 = i$. This gives

$$m_s(i,z) = Q_s(d(i,z))$$

with

$$d(i,z) = 1 + \frac{|i-z|^2}{2\mathrm{Im}(i)\mathrm{Im}(z)} = \frac{x^2+y^2+1}{2y}.$$

Then the integral becomes

$$\int_{\mathcal{H}} Q_s\Big(\frac{x^2+y^2+1}{2y}\Big)\frac{dxdy}{2\pi y^2}.$$

We need to prove that this integral is equal to $\dfrac{1}{s(s+1)}$. The remaining part is purely analysis.

Denote by $\mathbb{D} = \{z' \in \mathbb{C} : |z'| < 1\}$ the standard open unit disc. Under the standard isomorphism $\mathcal{H} \to \mathbb{D}$ given by $z' = \dfrac{z-i}{z+i}$ and $z = i\dfrac{1+z'}{1-z'}$, the integral becomes

$$\int_{\mathbb{D}} Q_s\Big(\frac{1+|z'|^2}{1-|z'|^2}\Big)\frac{4dx'dy'}{2\pi(1-|z'|^2)^2}.$$

Here $z' = x' + iy'$ as usual. In terms of the polar coordinate $z' = re^{i\theta}$, the integral becomes

$$\int_0^1 Q_s\Big(\frac{1+r^2}{1-r^2}\Big)\frac{4rdr}{(1-r^2)^2} = \int_1^\infty Q_s(t)dt.$$

Recall from [GZ, §II.2] that the Legendre function $Q_s(t)$ satisfies the differential equation

$$\Big((1-t^2)\frac{d^2}{dt^2} - 2t\frac{d}{dt} + s(s+1)\Big)Q_s = 0.$$

This gives

$$s(s+1)Q_s = \frac{d}{dt}\Big((t^2-1)\frac{d}{dt}Q_s\Big).$$

As a consequence, the original integral is equal to

$$\frac{1}{s(s+1)}\Big((t^2-1)\frac{d}{dt}Q_s\Big)\Big|_1^\infty.$$

By [GZ, II, (2.6)], we can express Q_s by

$$Q_s(t) = \frac{2^s\Gamma(s+1)^2}{\Gamma(2s+2)}\Big(\frac{1}{t+1}\Big)^{s+1}F\Big(s+1, s+1; 2s+2; \frac{2}{t+1}\Big).$$

Here the hypergeometric function

$$F(a,b;c;t) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$. If c is not a negative integer, the series $F(a,b;c;t)$ is absolutely convergent for $|t| < 1$, and satisfies the functional equation

$$\frac{d}{dt}F(a,b;c;t) = \frac{ab}{c}F(a+1, b+1; c+1; t).$$

This gives

$$\begin{aligned}(t^2-1)\frac{d}{dt}Q_s(t) = &-(s+1)\frac{2^s\Gamma(s+1)^2}{\Gamma(2s+2)}\frac{t-1}{(t+1)^{s+1}}F\Big(s+1, s+1; 2s+2; \frac{2}{t+1}\Big) \\ &- \frac{2^{s+1}\Gamma(s+2)^2}{\Gamma(2s+3)}\frac{t-1}{(t+1)^{s+2}}F\Big(s+2, s+2; 2s+3; \frac{2}{t+1}\Big).\end{aligned}$$

It follows that for $\mathrm{Re}(s) > 0$, the function $(t^2-1)\frac{d}{dt}Q_s(t)$ converges to 0 as $t \to \infty$.

By [AAR, Theorem 2.1.3], as $t \to 1^+$,

$$\Big(1 - \frac{2}{t+1}\Big)F\Big(s+1, s+1; 2s+2; \frac{2}{t+1}\Big) \longrightarrow 0$$

and

$$\Big(1 - \frac{2}{t+1}\Big)F\Big(s+2, s+2; 2s+3; \frac{2}{t+1}\Big) \longrightarrow \frac{\Gamma(2s+3)}{\Gamma(s+2)^2}.$$

Therefore,

$$\lim_{t\to 1^+}(t^2-1)\frac{d}{dt}Q_s(t) = -\frac{2^{s+1}\Gamma(s+2)^2}{\Gamma(2s+3)}\frac{1}{2^{s+1}}\frac{\Gamma(2s+3)}{\Gamma(s+2)^2} = -1.$$

This finishes the proof. □

4.2 Decomposition of the height series

The goal of this subsection to decompose the height series

$$Z(g,(t_1,t_2),\phi)_U = \langle Z(g,\phi)_U\ t_1^\circ,\ t_2^\circ\rangle_{\mathrm{NT}}, \quad t_1,t_2 \in E^\times(\mathbb{A}_f).$$

This was treated in [YZZ, §7.1-7.2] in terms of the arithmetic Hodge index theorem and admissible extensions. But as mentioned above, there is a minor mistake caused by the fact that the Green's function is only weakly admissible, so we will present the correct result here. We will still follow the idea of [YZZ, YZ], but we will also take into account the extra term caused by weak admissibility.

For the purpose here, U is a maximal open compact subgroup of $\mathbb{B}_f^\times$ containing $\widehat{O}_E^\times$ as in §3.2. As in [YZ, §4.2], we have a canonical arithmetic model $(\mathcal{X}_U, \overline{\mathcal{L}}_U)$ of (X_U, L_U) over O_F. Note that $\mathcal{X}_U$ is smooth outside Σ, but not necessarily regular everywhere. However, by [YZ, Corollary 4.6(2)], the base change $\mathcal{X}_{U,O_M}$ is $\mathbb{Q}$-factorial for any finite extension M of F unramified over Σ_f, so intersection theory is still well-defined for Weil divisors over $\mathcal{X}_{U,O_M}$.

Recall that κ_U° is the degree of L_U over any connected component of $X_{U,\overline{F}}$. Denote

$$\xi = (\kappa_U^\circ)^{-1} L_U \in \mathrm{Pic}(X_U)_{\mathbb{Q}}, \quad \hat{\xi} = (\kappa_U^\circ)^{-1}\overline{\mathcal{L}}_U \in \widehat{\mathrm{Pic}}(\mathcal{X}_U)_{\mathbb{Q}}.$$

For any finite extension M of F unramified over Σ_f, we can pull $\hat{\xi}$ back to the base change $\mathcal{X}_{U,O_M}$. Still denote the pull-back by $\hat{\xi}$ by abuse of notations. Then we have the notion of weakly $\hat{\xi}$-admissible extensions of divisors over $\mathcal{X}_{U,O_M}$.

In particular, for any CM point $[\beta] \in \mathrm{CM}_U = E^\times\backslash\mathbb{B}_f^\times/U$ represented by $\beta \in \mathbb{B}_f^\times$, it is defined over the abelian extension $H(\beta)$ of E determined by the open compact subgroup $\beta U\beta^{-1} \cap E^\times(\mathbb{A}_f)$ of $E^\times(\mathbb{A}_f)$ via the class field theory. By assumption, U_v is maximal at any $v \in \Sigma_f$, so $\beta_v U_v \beta_v^{-1} \cap E_v^\times = U_v \cap E_v^\times = O_{E_v}^\times$. It follows that the extension $H(\beta)$ of E is unramified over Σ_f. By this, we obtain a weakly $\hat{\xi}$-admissible extension

$$\hat{\beta} = (\overline{P}_\beta + V_\beta, g(P_\beta,\cdot))$$

of P_β over $\mathcal{X}_{U,O_{H(\beta)}}$. Here P_β is the point of $X_U(H(\beta))$ corresponding to $[\beta]$, $\overline{P}_\beta$ is the Zariski closure in $\mathcal{X}_{U,O_{H(\beta)}}$, V_β is a vertical divisor over $\mathcal{X}_{U,O_{H(\beta)}}$, and $g(P_\beta,\cdot)$ is the Green's function reviewed above.

Note that the weakly $\hat{\xi}$-admissible extension $\hat{\beta}$ is unique, as the Green's function is already chosen. Moreover, the base change of $\hat{\beta}$ by any extension M of $H(\beta)$ unramified over Σ_f is still weakly a $\hat{\xi}$-admissible extension, which we still denote by $\hat{\beta}$ by abuse of notations.

Finally, consider

$$Z(g,(t_1,t_2),\phi)_U = \langle Z_*(g,\phi)_U(t_1-\xi_{t_1}),\ t_2-\xi_{t_2}\rangle_{\mathrm{NT}}, \quad t_1,t_2 \in E^\times(\mathbb{A}_f).$$

Then the arithmetic Hodge index theorem of Faltings and Hriljac (cf. [YZZ, Theorem 7.4]) gives

$$Z(g,(t_1,t_2),\phi)_U = -((Z_*(g,\phi)_U t_1)^\wedge - (Z_*(g,\phi)_U\xi_{t_1})^\wedge)\cdot(\hat{t}_2 - \hat{\xi}_{t_2}).$$

We understand that the arithmetic intersection on the right-hand side involves base changes by finite extensions of F unramified over Σ_f to realize t_1 as a rational point, and the intersection numbers should be normalized by the degrees of the base changes. The extension $\hat{\xi}_{t_2}$ of ξ_{t_2} is given by the corresponding connected component of $\hat{\xi}$ (over suitable base changes of X_U). The extension $(Z_*(g,\phi)_U\xi_{t_1})^\wedge$ of $Z_*(g,\phi)_U\xi_{t_1}$ is defined similarly, as $Z_*(g,\phi)_U\xi_{t_1}$ is a linear combination of connected components of ξ. The weakly $\hat{\xi}$-admissible extension $\hat{t}_2$ of t_2 is introduced above. The weakly $\hat{\xi}$-admissible extension $(Z_*(g,\phi)_U t_1)^\wedge$ of $Z_*(g,\phi)_U t_1$ is defined similarly, as $Z_*(g,\phi)_U t_1$ is a linear combination of CM points of the form $[\beta] \in \mathrm{CM}_U$.

Take the notational convention

$$\langle D, D'\rangle := -\widehat{D}\cdot\widehat{D}',$$

where D, D' are the divisor classes involved above, and $\widehat{D}, \widehat{D}'$ are the arithmetic extensions introduced above. The right-hand side involves a normalizing factor again if a base change is taken. Then the decomposition is written as

$$Z(g,(t_1,t_2))_U = \langle Z_*(g,\phi)_U t_1, t_2\rangle - \langle Z_*(g,\phi)_U t_1, \xi_{t_2}\rangle - \langle Z_*(g,\phi)_U \xi_{t_1}, t_2\rangle + \langle Z_*(g,\phi)_U \xi_{t_1}, \xi_{t_2}\rangle.$$

Now we summarize the result term by term in the following.

Theorem 4.2. *For any $t_1, t_2 \in C_U$,*

$$Z(g,(t_1,t_2))_U = \langle Z_*(g,\phi)_U t_1, t_2\rangle - \langle Z_*(g,\phi)_U t_1, \xi_{t_2}\rangle - \langle Z_*(g,\phi)_U \xi_{t_1}, t_2\rangle + \langle Z_*(g,\phi)_U \xi_{t_1}, \xi_{t_2}\rangle,$$

where the first term on the right-hand side has the expression

$$\begin{aligned}\langle Z_*(g,\phi)_U t_1, t_2\rangle = &-\sum_{v \text{ nonsplit}} (\log N_v) \fint_{C_U} \mathcal{M}_\phi^{(v)}(g,(tt_1,tt_2))dt \\ &- \sum_{v\nmid\infty} \mathcal{N}_\phi^{(v)}(g,(t_1,t_2))\log N_v - \sum_{v\nmid\infty} j_v(Z_*(g,\phi)t_1,t_2)\log N_v \\ &- \frac{i_0(t_2,t_2)}{[E^\times \cap U : \mu_U]}\Omega_\phi(g,(t_1,t_2)) \\ &- \frac{1}{2}[F:\mathbb{Q}]E_*(0,g,r(t_1,t_2)\phi).\end{aligned}$$

Here the first three lines on the right hand side are the same as the formula of $Z(g,(t_1,t_2),\phi))_U$ in [YZ, Theorem 8.6]. Namely, they are explained in the following.

(1) The modified arithmetic self-intersection number

$$i_0(t_2,t_2) = i(t_2,t_2) - \sum_v i_v(t_2,t_2)\log N_v,$$

where the local term

$$i_v(t_2,t_2) = \fint_{C_U} i_{\bar{v}}(tt_2,tt_2)dt$$

uses the extended definition of $i_{\bar{v}}$, defined in [YZ, §8.2] by case-by-case formulas according to the type of the place v.

(2) The pseudo-theta series

$$\Omega_\phi(g,(t_1,t_2)) = \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in E^\times} r(g,(t_1,t_2))\phi(y,u).$$

(3) For any place v non-split in E,

$$\begin{aligned}\mathcal{M}_\phi^{(v)}(g,(t_1,t_2)) &= w_U\sum_{a\in F^\times}\widetilde{\lim}_{s\to0}\sum_{y\in\mu_U\backslash(B_+^\times-E^\times)} r(g,(t_1,t_2))\phi(y)_a m_s(y), \quad v|\infty,\\ \mathcal{M}_\phi^{(v)}(g,(t_1,t_2)) &= \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in B-E} r(g,(t_1,t_2))\phi^v(y,u)\ m_{r(g,(t_1,t_2))\phi_v}(y,u), \quad v\nmid\infty.\end{aligned}$$

(4) For any non-archimedean v,

$$\mathcal{N}_\phi^{(v)}(g,(t_1,t_2)) = \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in E^\times} r(g,(t_1,t_2))\phi^v(y,u)\ r(t_1,t_2)n_{r(g)\phi_v}(y,u).$$

Proof. This is computed in [YZ, Theorem 8.6], except that we will have an extra term coming from the weak admissibility. In fact, we first write

$$\langle Z_*(g)t_1,t_2\rangle = -i(Z_*(g)t_1,t_2) - j(Z_*(g)t_1,t_2).$$

Then we write

$$j(Z_*(g)t_1,t_2) = \sum_v j_v(Z_*(g)t_1,t_2),$$

where the sum is over all places v of F instead of just non-archimedean places. The extra terms are $j_v(Z_*(g)t_1,t_2)$ for archimedean v, while the other terms are computed in the proof of [YZ, Theorem 8.6].

If v is archimedean, by definition

$$j_v(Z_*(g)t_1,t_2) = \int_{X_{U,v}(\mathbb{C})} g(t_2,\cdot)c_1((Z_*(g)t_1)^\wedge).$$

Note that only the part of $c_1((Z_*(g)t_1)^\wedge)$ supported on the connected components of t_2 contributes to the integral. Recall the terminology for the connected components of $Z_*(g)$ in [YZZ, §4.3.1]. Then we only need to consider the component $Z_*(g)_{q(t_1^{-1}t_2)}$ of $Z_*(g)$. By the weak admissibility,

$$c_1((Z_*(g)_{q(t_1^{-1}t_2)}t_1)^\wedge) = \deg(Z_*(g)_{q(t_1^{-1}t_2)})\,c_1(\hat{\xi}_{t_2}).$$

By [YZZ, Proposition 4.2],

$$\deg(Z_*(g)_{q(t_1^{-1}t_2)}) = -\frac{1}{2}\kappa_U^\circ E_*(0,g,r(t_1,t_2)\phi).$$

It follows that

$$j_v(Z_*(g)t_1,t_2)=-\frac{1}{2}\kappa_U^\circ E_*(0,g,r(t_1,t_2)\phi)\int_{X_{U,v}(\mathbb{C})} g(t_2,\cdot)c_1(\hat{\xi}_{t_2}).$$

By Lemma 4.1,

$$\int_{X_{U,v}(\mathbb{C})} g(t_2,\cdot)c_1(\overline{\mathcal{L}}_U)=-1.$$

Hence,

$$j_v(Z_*(g)t_1,t_2)=\frac{1}{2}E_*(0,g,r(t_1,t_2)\phi).$$

This finishes the proof. □

Remark 4.3. The extra term $-\frac{1}{2}[F:\mathbb{Q}]E_*(0,g,r(t_1,t_2)\phi)$ in Theorem 4.2 appears due to the weak admissibility. This term is a priori missed in [YZZ, YZ]. However, it does not affect the main results of [YZZ, YZ], since both articles assume [YZZ, Assumption 5.4], under which the extra term vanishes.

4.3 Comparison at archimedean place

Let v be an archimedean place of F. Recall that in Theorem 4.2, $Z(g,(t_1,t_2))_U$ has a v-component

$$\mathcal{M}_\phi^{(v)}(g,(t_1,t_2))=w_U\sum_{a\in F^\times}\widetilde{\lim}_{s\to 0}\sum_{y\in\mu_U\backslash(B_+^\times-E^\times)} r(g,(t_1,t_2))\phi(y)_a m_s(y).$$

On the other hand, recall that in Theorem 3.1, $\mathcal{P}r'I'(0,g,\phi)_U$ has a v-component

$$\overline{\mathcal{K}}_\phi^{(v)}(g,(t_1,t_2))=w_U\sum_{a\in F^\times}\widetilde{\lim}_{s\to 0}\sum_{y\in\mu_U\backslash(B(v)_+^\times-E^\times)} r(g,(t_1,t_2))\phi(y)_a\ k_{v,s}(y),$$

where

$$k_{v,s}(y)=\frac{\Gamma(s+1)}{2(4\pi)^s}\int_1^\infty\frac{1}{t(1-\lambda(y)t)^{s+1}}dt,\qquad \lambda(y)=q(y_2)/q(y).$$

The goal of this subsection is to compute their difference. The final result is as follows.

Proposition 4.4. *For any $t_1,t_2\in C_U$,*

$$\mathcal{K}_\phi^{(v)}(g,(t_1,t_2))-\overline{\mathcal{M}}_\phi^{(v)}(g,(t_1,t_2))=\frac{1}{2}(\gamma+\log(4\pi)-1)E_*(0,g,r(t_1,t_2)\phi).$$

Here γ is Euler's constant.

Proof. This is computed as in [YZZ, Proposition 8.1], but we need some extra work to take care of contribution from the residue of the Green's function, which is missed in the loc. cit..

As in the calculation of Gross–Zagier [GZ],

$$\int_1^\infty \frac{1}{t(1-\lambda t)^{s+1}}dt = \frac{2\Gamma(2s+2)}{\Gamma(s+1)\Gamma(s+2)}Q_s(1-2\lambda) + O_s(|\lambda|^{-s-2}).$$

Moreover, the error term $O_s(|\lambda|^{-s-2})$ vanishes at $s=0$. This is a combination of the equations in the first line and in the 12th line of [GZ, p. 304], by noting that the left-hand sides of those two equations are equal. It follows that

$$k_{v,s}(y) = \frac{\Gamma(2s+2)}{(4\pi)^s\Gamma(s+2)}Q_s(1-2\lambda(y)) + O_s(|\lambda(y)|^{-s-2}).$$

Denote

$$\mathcal{M}_\phi^{(v)}(s,g,(t_1,t_2)) = w_U \sum_{a\in F^\times}\sum_{y\in\mu_U\backslash(B_+^\times - E^\times)} r(g,(t_1,t_2))\phi(y)_a m_s(y).$$

Then we have

$$\overline{\mathcal{M}}_\phi^{(v)}(g,(t_1,t_2)) = \widetilde{\lim}_{s\to0}\overline{\mathcal{M}}_\phi^{(v)}(s,g,(t_1,t_2))$$

and

$$\overline{\mathcal{K}}_\phi^{(v)}(g,(t_1,t_2)) = \widetilde{\lim}_{s\to0}\frac{\Gamma(2s+2)}{(4\pi)^s\Gamma(s+2)}\overline{\mathcal{M}}_\phi^{(v)}(s,g,(t_1,t_2)).$$

Then

$$\overline{\mathcal{K}}_\phi^{(v)}(g,(t_1,t_2)) - \overline{\mathcal{M}}_\phi^{(v)}(g,(t_1,t_2)) = \widetilde{\lim}_{s\to0}\left(\frac{\Gamma(2s+2)}{(4\pi)^s\Gamma(s+2)}-1\right)\overline{\mathcal{M}}_\phi^{(v)}(s,g,(t_1,t_2)).$$

Note that $\dfrac{\Gamma(2s+2)}{(4\pi)^s\Gamma(s+2)}-1$ vanishes at $s=0$, and its derivative at $s=0$ is given by

$$\Gamma'(2) - \log(4\pi) = 1-\gamma-\log(4\pi).$$

We will see that the series $\overline{\mathcal{M}}_\phi^{(v)}(s,g,(t_1,t_2))$ has a simple pole at $s=0$, coming from the pole of g_s as in Lemma 4.1. Hence,

$$\overline{\mathcal{K}}_\phi^{(v)}(g,(t_1,t_2)) - \overline{\mathcal{M}}_\phi^{(v)}(g,(t_1,t_2)) = (1-\gamma-\log(4\pi))\,\mathrm{Res}_{s=0}\overline{\mathcal{M}}_\phi^{(v)}(s,g,(t_1,t_2)).$$

We can see the simple pole of $\overline{\mathcal{M}}_\phi^{(v)}(s,g,(t_1,t_2))$ and compute the residue as follows. By a simple transformation as in [YZZ, Proposition 8.1], we have

$$\mathcal{M}_\phi^{(v)}(s,g,(t_1,t_2)) = \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(g)\phi(x)_a\ g_s(t_1x,t_2) = g_s(Z_*(g,\phi)t_1,t_2).$$

By Lemma 4.1, we have

$$\mathrm{Res}_{s=0}\mathcal{M}_{\phi}^{(v)}(s,g,(t_1,t_2)) = \frac{1}{\kappa_U^\circ}\deg(Z_*(g,\phi)_{q(t_1^{-1}t_2)}).$$

Here $Z_*(g,\phi)_{q(t_1^{-1}t_2)}t_1$ is the part of $Z_*(g,\phi)t_1$ that lies in the same connected component as t_2. See [YZZ, §4.3.1] for the connected components of $Z_*(g,\phi)$. In particular, [YZZ, Proposition 4.2] gives

$$\deg(Z_*(g,\phi)_{q(t_1^{-1}t_2)}) = -\frac{1}{2}\kappa_U^\circ E_*(0,g,r(t_1,t_2)\phi).$$

This finishes the proof. □

Remark 4.5. Note that [YZZ, Proposition 8.1] asserts $\mathcal{M}_{\phi}^{(v)}(g,(t_1,t_2)) = \overline{\mathcal{K}}_{\phi}^{(v)}(g,(t_1,t_2))$, which a priori wrong by Proposition 4.4. However, it holds under [YZZ, Assumption 5.4], so the correction does not affect the main results of [YZZ, YZ]. The situation is similar to Remark 4.3.

4.4 The j-part by bad reduction

If v is a non-archimedean place of F split in $\mathbb{B}$, then the j-part $j_v(Z_*(g,\phi)t_1,t_2) = 0$ automatically. This is a trivial consequence of the fact that $\mathcal{X}_U$ is smooth above v. In the following, assume that v is a non-archimedean place nonsplit in $\mathbb{B}$ and inert in E. Note that U_v is maximal and $\phi_v = 1_{O_{\mathbb{B}_v}^\times \times O_{F_v}^\times}$. The j-part $j_v(Z_*(g,\phi)t_1,t_2)$ is treated briefly in [YZZ] and [YZ, Lemma 8.9]. For the purpose here, we need some extra information.

Lemma 4.6. *Let v be a non-archimedean place nonsplit in $\mathbb{B}$ and inert in E. Then the j-part $j_{\bar{v}}(Z_*(g,\phi)_U t_1,t_2)$ is a non-singular pseudo-theta series of the form*

$$\sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in B(v)-\{0\}} r(g,(t_1,t_2))\phi^v(y,u)\ r(t_1,t_2)l_{r(g)\phi_v}(y,u).$$

Furthermore,

$$\int_{B(v)_v} l_{\phi_v}(y,u)dy = \frac{1}{4}|d_v|^2 N_v^{-1}(1-N_v^{-1})^2\cdot 1_{O_{F_v}^\times}(u),$$

and thus

$$r(w)l_{\phi_v}(0,u) = -\frac{N_v-1}{4(N_v+1)}r(w)\phi_v(0,u).$$

Proof. Note that the first part of the lemma is exactly [YZ, Lemma 8.9]. In the following, we first recall the formula of $l_{\phi_v}(g,y,u)$ in [YZ, Lemma 8.9], and then compute its average by a more careful analysis of the p-adic uniformization.

Denote by $B = B(v)$ the nearby quaternion algebra over F. We need the p-adic uniformization of Čerednik–Drinfe'ld (cf. [BC]) over $\mathbb{Q}$ and that of Boutot–Zink [BZ] over totally real field, which gives an isomorphism

$$\widehat{\mathcal{X}}_U \times_{\mathrm{Spf}\,O_{F_v}} \mathrm{Spf}\,O_{F_v^{\mathrm{ur}}} = B^\times\backslash(\widehat{\Omega}\times_{\mathrm{Spf}\,O_{F_v}}\mathrm{Spf}\,O_{F_v^{\mathrm{ur}}})\times\mathbb{B}_f^\times/U.$$

Here $\widehat{\mathcal{X}}_U$ denotes the formal completion of $\mathcal{X}_U$ along the special fiber above v, F_v^{ur} denotes the completion of the maximal unramified extension of F_v. In particular, $\widehat{\Omega}$ is Deligne's integral model of Drinfe'ld (rigid-analytic) upper half plane Ω over O_{F_v}. The group $B_v^\times \cong \mathrm{GL}_2(F_v)$ acts on $\widehat{\Omega}$ by the linear transformation, and on $\mathbb{B}_v^\times/U_v \cong \mathbb{Z}$ via translation by $v \circ q = v \circ \det$.

By the definition in [YZZ, §7.1.7],

$$j_{\overline{v}}(Z_*(g)t_1, t_2) = \overline{Z_*(g)t_1} \cdot V_{t_2}.$$

Here $\overline{Z_*(g)t_1}$ is the Zariski closure in $\mathcal{X}_{U,O_{F_v^{\mathrm{ur}}}}$, and V_{t_2} is the unique vertical divisor on $\mathcal{X}_{U,O_{F_v^{\mathrm{ur}}}}$, supported on the geometrically connected component of t_2 in $\mathcal{X}_{U,O_{F_v^{\mathrm{ur}}}}$, satisfying the following properties:

(1) $(V_{t_2} + \bar{t}_2) \cdot C = \hat{\xi} \cdot C$ for any vertical divisor C of $\mathcal{X}_{U,O_{F_v^{\mathrm{ur}}}}$;

(2) $V_{t_2} \cdot \hat{\xi} = 0$.

Write $V_1 = \sum_i a_i W_i$ (for $t_2 = 1$), where $\{W_i\}_i$ is the set of irreducible components of the special fiber of $\mathcal{X}_{U,O_{F_v^{\mathrm{ur}}}}$ lying in the same connected component as 1. Let $\widetilde{W}_i$ be an irreducible component of the special fiber of $\widehat{\Omega} \times_{\mathrm{Spf}\, O_{F_v}} \mathrm{Spf}\, O_{F_v^{\mathrm{ur}}}$ lifting W_i. Write $\widetilde{V} = \sum_i a_i \widetilde{W}_i$, viewed as a vertical divisor of $\widehat{\Omega} \times_{\mathrm{Spf}\, O_{F_v}} \mathrm{Spf}\, O_{F_v^{\mathrm{ur}}}$.

Via the p-adic uniformization, the proof of [YZ, Lemma 8.9] actually gives

$$j_{\overline{v}}(Z_*(g)t_1, t_2) = \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{\gamma \in B^\times} r(g, (t_1, t_2))\phi^v(\gamma, u)\ r(t_1, t_2) l_{\phi_v}(g, \gamma, u),$$

where

$$l_{\phi_v}(g, \gamma, u) = \sum_{x \in \mathbb{B}_v^\times/U_v} r(g)\phi_v(x, uq(\gamma)/q(x)) 1_{O_{F_v}^\times}(q(x)/q(\gamma))\ (\gamma^{-1} z_0 \cdot \widetilde{V}).$$

Here $z_0 \in \Omega(E_v)$ is the unique point in $\Omega(\mathbb{C}_v)$ fixed by $E_v^\times$. Moreover, we also have

$$l_{\phi_v}(1, \gamma, u) = (\gamma^{-1} z_0 \cdot \widetilde{V}) \cdot 1_{O_{F_v}^\times}(q(\gamma)) \cdot 1_{O_{F_v}^\times}(u).$$

Assume that $u \in O_{F_v}^\times$ in the following.

Recall that the irreducible components of the special fiber of $\widehat{\Omega}$ are indexed by

$$\mathrm{GL}_2(F_v)/F_v^\times \mathrm{GL}_2(O_{F_v}).$$

Denote by $\alpha_i F_v^\times \mathrm{GL}_2(O_{F_v})$ the coset representing the component $\widetilde{W}_i$. Then we have

$$(\gamma^{-1} z_0 \cdot \widetilde{W}_i) = 1_{\alpha_i F_v^\times \mathrm{GL}_2(O_{F_v})}(\gamma^{-1}) = 1_{F_v^\times \mathrm{GL}_2(O_{F_v})\alpha_i^{-1}}(\gamma).$$

By $\widetilde{V} = \sum_i a_i \widetilde{W}_i$, we have

$$\int_{B_v} l_{\phi_v}(1, \gamma, u) d\gamma = \sum_i a_i \int_{B_{v,0}} (\gamma^{-1} z_0 \cdot \widetilde{W}_i) d\gamma = \sum_i a_i\ \mathrm{vol}(B_{v,0} \cap F_v^\times \mathrm{GL}_2(O_{F_v})\alpha_i^{-1}).$$

Here

$$B_{v,0} = \{\gamma \in B_v : q(\gamma) \in O_{F_v}^\times\}.$$

For any $\alpha \in B_v^\times$, it is easy to have

$$B_{v,0} \cap F_v^\times \mathrm{GL}_2(O_{F_v})\alpha^{-1} = \begin{cases} \varnothing, & 2 \nmid v(q(\alpha)); \\ \mathrm{GL}_2(O_{F_v})\varpi_v^{v(q(\alpha))/2}\alpha^{-1}, & 2 \mid v(q(\alpha)). \end{cases}$$

Here ϖ_v is a uniformizer of F_v. Note that the self-dual measure on $B_v = M_2(F_v)$ gives

$$\mathrm{vol}(\mathrm{GL}_2(O_{F_v})) = |\mathrm{GL}_2(O_{F_v}/p_v)| \cdot \mathrm{vol}(1 + p_v M_2(O_{F_v})) = |d_v|^2(1 - N_v^{-1})(1 - N_v^{-2}).$$

As a consequence

$$\int_{B_v} l_{\phi_v}(1, \gamma, u) d\gamma = |d_v|^2(1 - N_v^{-1})(1 - N_v^{-2}) \sum_{i: 2 | v(q(\alpha_i))} a_i.$$

For convenience, denote by S_0 (resp. S_1) the set of i such that $2 \mid v(q(\alpha_i))$ (resp. $2 \nmid v(q(\alpha_i))$). Then $S = S_0 \cup S_1$ is the set of all indexes i. Denote

$$A_0 = \sum_{i \in S_0} a_i, \qquad A_1 = \sum_{i \in S_1} a_i.$$

We need to compute A_0. We are going to prove the following equation:

$$A_0 + A_1 = 0, \qquad A_0 - A_1 = \frac{1}{2(N_v + 1)}.$$

The relations give

$$A_0 = \frac{1}{4(N_v + 1)}, \qquad \int_{B_v} l_{\phi_v}(1, \gamma, u) d\gamma = \frac{1}{4} |d_v|^2 N_v^{-1} (1 - N_v^{-1})^2.$$

Then the last equality of the lemma follows from Lemma 3.5.

It remains to prove the two equations of A_0 and A_1. We need the following intersection results:

(1) The orders $|S_0|$ and $|S_1|$ are equal. In fact, for any $x_v \in \mathbb{B}_v{}^\times$ with $2 \nmid v(q(x_v))$ the Hecke action $Z(x_v)_U$ corresponding to $U_v x_v U_v = x_v U_v$ is an automorphism of $\mathcal{X}_{U, O_{F_v^{\mathrm{ur}}}}$ and switches S_0 with S_1. Denote $n = |S_0| = |S_1|$ in the following, so $2n = |S|$.

(2) $W_i \cdot W_i = -(N_v + 1)$ for any $i \in S$. In fact, by the construction of $\widehat{\Omega}$, any irreducible component of the special fiber of $\widehat{\Omega} \times_{\mathrm{Spf}\, O_{F_v}} \mathrm{Spf}\, O_{F_v^{\mathrm{ur}}}$ is isomorphic to $\mathbb{P}^1$, and any irreducible component intersects with exactly $N_v + 1$ other components. Both properties are inherited by the quotient process. Then

$$W_i \cdot W_i = -W_i \cdot \sum_{j \in S,\ j \neq i} W_j = -(N_v + 1).$$

(3) Fix $r = 0, 1$. Then $W_i \cdot W_j = 0$ for $i, j \in S_r$ with $i \neq j$. In fact, for two different lattices Λ, Λ' of F_v^2 (corresponding to W_i and W_j), the relation $\varpi_v \Lambda \subset \Lambda' \subset \Lambda$ implies that $v(\det(\Lambda))$ and $v(\det(\Lambda'))$ have different parities. Here $\det(\Lambda)$ denotes the transition matrix of the determinant of an O_{F_v}-basis of Λ to the standard basis of F_v^2.

(4) $W_i \cdot \hat{\xi} = 1/(2n)$ for any $i \in S$. Note that $\hat{\xi} \cdot \sum_{i \in S} W_i = 1$, since ξ has degree one on every connected component of $X_{U, F_v^{\mathrm{ur}}}$. Then it suffices to prove that $W_i \cdot \hat{\xi}$ is independent of i. Consider the quotient map $\mathcal{X}_{U_v U'^v} \to \mathcal{X}_U$ and its base change to $O_{F_v^{\mathrm{ur}}}$ for a sufficiently small $U'^v \subset U^v$. By the projection formula, it suffices to prove that $W_i' \cdot \mathcal{L}'$ is independent of i. Here $\mathcal{L}'$ is the relative dualizing sheaf of $\mathcal{X}_{U_v U'^v, O_{F_v^{\mathrm{ur}}}}$, and W_i' is an irreducible component of the special fiber of $\mathcal{X}_{U_v U'^v, O_{F_v^{\mathrm{ur}}}}$ lifting W_i. We will prove that $W_i' \cdot \mathcal{L}' = N_v - 1$. Apply the adjunction formula

$$2g(W_i') - 2 = W_i' \cdot \mathcal{L}' + W_i' \cdot W_i'.$$

This formula holds as $\mathcal{X}_{U_v U'^v}$ is semistable over v. As in the case of $\mathcal{X}_U$, we have $g(W_i') = 0$ and $W_i' \cdot W_i' = -(N_v + 1)$. This gives $W_i' \cdot \mathcal{L}' = N_v - 1$.

Now we are ready to establish the equations for A_0 and A_1. By the definition of $V_1 = \sum_i a_i W_i$, we have $V_1 \cdot \hat{\xi} = 0$. This is just $A_0 + A_1 = 0$ by (4).

On the other hand, the definition of V_1 also gives $(\bar{1} + V_1) \cdot C = \hat{\xi} \cdot C$ for any vertical divisor C of $\mathcal{X}_{U, O_{F_v^{\mathrm{ur}}}}$. Take $C = \sum_{j \in S_1} W_j$. It does not intersect the Zariski closure of 1. Furthermore, by (2) and (3), $W_i \cdot C = N_v + 1$ for $i \in S_0$ and $W_i \cdot C = -(N_v + 1)$ for $i \in S_1$. By (1) and (4), we have $\hat{\xi} \cdot C = 1/2$. Then the identity $(\bar{1} + V_1) \cdot C = \hat{\xi} \cdot C$ becomes

$$\sum_{i \in S_0} a_i (N_v + 1) - \sum_{i \in S_1} a_i (N_v + 1) = \frac{1}{2}.$$

This gives our second equation. The proof is complete. □

4.5 Hecke action on arithmetic Hodge classes

In last subsection, we have the decomposition

$$Z(g, (t_1, t_2)) = \langle Z_*(g, \phi) t_1, t_2 \rangle - \langle Z_*(g, \phi) \xi_{t_1}, t_2 \rangle + \langle Z_*(g, \phi) \xi_{t_1}, \xi_{t_2} \rangle - \langle Z_*(g, \phi) t_1, \xi_{t_2} \rangle.$$

We have also considered a decomposition of the first term on the right-hand side. In this subsection we consider the remaining three terms. The treatment here is an enhanced version of [YZZ, §7.3].

Two easy terms

Recall that κ_U° is the degree of L_U on a connected component of $X_{U, \overline{F}}$.

Proposition 4.7.

$$\langle Z_*(g,\phi)\xi_{t_1}, t_2\rangle = -\frac{1}{2}\kappa_U^\circ\ E_*(0,g,r(t_1,t_2)\phi)_U \cdot \langle \xi_{t_2}, t_2\rangle,$$

$$\langle Z_*(g,\phi)\xi_{t_1}, \xi_{t_2}\rangle = -\frac{1}{2}\kappa_U^\circ\ E_*(0,g,r(t_1,t_2)\phi)_U \cdot \langle \xi_{t_2}, \xi_{t_2}\rangle.$$

Proof. We first compute $\langle Z_*(g,\phi)\xi_{t_1}, t_2\rangle$. By definition, $Z_*(g,\phi)\xi_{t_1}$ is a linear combination of $Z(x)\xi_{t_1}$. By construction, the correspondence $Z(x)$ keeps the canonical bundle up to a multiple under pull-back and push-forward. More precisely, one has

$$Z(x)\xi_{t_1} = (\deg Z(x))\xi_{t_1 x}, \quad \forall x \in \mathbb{B}_f^\times.$$

Note that $\langle Z(x)\xi_{t_1}, t_2\rangle$ is nonzero only if $\xi_{t_1 x}$ and t_2 lie in the same geometrically connected component of X_U. It follows that

$$\langle Z_*(g,\phi)\xi_{t_1}, t_2\rangle = \langle Z_*(g,\phi)_{U,q(t_1^{-1}t_2)}\xi_{t_1}, t_2\rangle = \deg Z_*(g,\phi)_{U,q(t_1^{-1}t_2)} \cdot \langle \xi_{t_2}, t_2\rangle.$$

Here $Z_*(g,\phi)_{U,q(t_1^{-1}t_2)}$ consists of the $q(t_1^{-1}t_2)$-component of $Z_*(g,\phi)_U$ as introduced in [YZZ, §4.2.4]. By [YZZ, Proposition 4.2],

$$\deg Z(g,\phi)_{U,q(t_1^{-1}t_2)} = -\frac{1}{2}\kappa_U^\circ\ E(0,g,r(t_1,t_2)\phi)_U.$$

This gives the formula for $\langle Z_*(g,\phi)\xi_{t_1}, t_2\rangle$. The same method also proves the formula for $\langle Z_*(g,\phi)\xi_{t_1}, \xi_{t_2}\rangle$. □

Almost eigenvector

It remains to consider $\langle Z_*(g)t_1, \xi_{t_2}\rangle$. We follow the treatment of [YZZ, §7.3.2] with some modification to fit the current setting.

For any $x \in \mathbb{B}_f^\times$, let $\mathcal{Z}(x)$ be the Zariski closure of $Z(x)$ in $\mathcal{X}_U \times_{O_F} \mathcal{X}_U$. Note that U is maximal by assumption. The following are true:

(1) $\mathcal{Z}(x_1)$ commutes with $\mathcal{Z}(x_2)$ for any $x_1, x_2 \in \mathbb{B}_f^\times$;

(2) $\mathcal{Z}(x) = \prod_{v\nmid\infty} \mathcal{Z}(x_v)$ for any $x \in \mathbb{B}_f^\times$;

(3) for any $x \in \mathbb{B}_f^\times$, both structure projections from $\mathcal{Z}(x)$ to $\mathcal{X}_U$ are finite.

In the proof of Proposition 4.7, we already see that

$$Z(x)\xi = (\deg Z(x))\ \xi.$$

In other words, ξ is an eigenvector of $Z(x)$ over X_U. For the arithmetic version, we will see that $\hat{\xi}$ generally fails to be an eigenvector of $\mathcal{Z}(x)$, but the failure is explicitly computable.

Define an arithmetic class $D(x)$ on $\mathcal{X}_U$ by

$$D(x) := \mathcal{Z}(x)\hat{\xi} - (\deg Z(x))\ \hat{\xi}.$$

Then $D(x)$ is a vertical arithmetic $\mathbb{Q}$-divisor since it is zero on the generic fiber.

If $x \in \mathbb{B}_v^\times$ for some non-archimedean place v nonsplit in $\mathbb{B}$, then we have $\deg Z(x) = 1$ and $D(x) = 0$. In fact, since $U_v = O_{\mathbb{B}_v}^\times$, the double coset $U_v x U_v = xU_v$ is a single coset depending only on $v(q(x))$. As a consequence, $Z(x)$ is just an automorphism of X_U, and thus $\mathcal{Z}(x)$ is an automorphism of $\mathcal{X}_U$. For any subgroup $U'^v \subset U^v$, we have a similar automorphism on $\mathcal{X}_{U_v U'^v}$ determined by x, and this automorphism does not change the relative dualizing sheaf of $\mathcal{X}_{U_v U'^v}$. By the norm map, we see that $\mathcal{Z}(x)$ fixes the arithmetic class $\hat{\xi}$.

If $x \in \mathbb{B}^\Sigma$, then $D(x)$ is a *constant* $\mathbb{Q}$*-divisor*, i.e., the pull-back of an arithmetic $\mathbb{Q}$-divisor from $\mathrm{Spec}(O_{F'})$, where F' is the algebraic closure of F in the functor field of X_U. Note that F' is the abelian extension of F with Galois group $\pi_0(X_{U,\bar{F}}) = F_+^\times \backslash \mathbb{A}_f^\times / q(U)$ via the class field theory. See the reason for the constancy of $D(x)$ in [YZZ, §7.3.2].

Hence, for all $x \in \mathbb{B}_f^\times$, $D(x)$ is a constant $\mathbb{Q}$-divisor, i.e., the pull back of an arithmetic $\mathbb{Q}$-divisor from $\mathrm{Spec}(O_{F'})$. By abuse of notation, we also denote by $D(x)$ the arithmetic degree of the arithmetic $\mathbb{Q}$-divisor on $\mathrm{Spec}(O_{F'})$. Hence we get a number $D(x) \in \mathbb{R}$. It is more convenient to introduce

$$D_0(x) := \frac{1}{\deg Z(x)} D(x).$$

By definition, $D_0(x)$ is *additive* in that

$$D_0(x) = \sum_{v \nmid \infty} D_0(x_v).$$

The sum has only finitely many nonzero terms.

Now we have the following basic result.

Lemma 4.8. *For any* $t \in C_U$*,*

$$\langle Z(x)t, \xi \rangle = \deg Z(x) \langle t, \xi \rangle - \deg Z(x) \sum_{v \nmid \infty} D_0(x_v).$$

Proof. This is a direct consequence of [YZZ, Lemma 7.7], which asserts

$$\langle Z(x)D, \xi \rangle = \deg Z(x) \langle D, \xi \rangle - \deg(D) D(x), \quad D \in \mathrm{Div}(X_{U,\overline{F}}),\ x \in \mathbb{B}_v^\times.$$

There is a gap in the proof of the loc. cit. due to the extra term caused by the weak admissibility, but the conclusion still holds. In fact, the loc. cit. proves that

$$\langle \overline{Z(x)D}, \hat{\xi} \rangle = \deg Z(x) \langle \overline{D}, \hat{\xi} \rangle - \deg(D) D(x).$$

On the other hand, by Lemma 4.1,

$$\langle D, \xi \rangle = \langle \overline{D}, \hat{\xi} \rangle + (\kappa_U^\circ)^{-1} \deg(D),$$

and

$$\langle Z(x)D, \xi \rangle = \langle \overline{Z(x)D}, \hat{\xi} \rangle + (\kappa_U^\circ)^{-1} \deg Z(x) \deg(D).$$

This implies the original statement. □

The last term

Now we are ready to compute $\langle Z_*(g,\phi)t_1, \xi_{t_2}\rangle$.

Proposition 4.9.

$$\langle Z_*(g,\phi)t_1, \xi_{t_2}\rangle = -\frac{1}{2}\kappa_U^\circ\ E_*(0,g,r(t_1,t_2)\phi)_U\ \langle[1],\xi\rangle + \frac{1}{2}\sum_{v\notin\Sigma}\mathcal{F}_\phi^{(v)}(g,(t_1,t_2)),$$

where

$$\mathcal{F}_\phi^{(v)}(g,(t_1,t_2)) = \sum_{u\in\mu_U^2\backslash F^\times}\sum_{a\in F^\times} W_a^v(0,g,u,r(t_1,t_2)\phi)\, f_{\phi_v,a}(g,(t_1,t_2),u)$$

with

$$f_{\phi_v,a}(g,(t_1,t_2),u) = (1-N_v^{-2})|d_v|^{\frac{3}{2}}|au^{-1}|_v\,\kappa_U^\circ \sum_{y\in\mathbb{B}_v(au^{-1})/U_v^1} r(g,(t_1,t_2))\phi_v(y,u)D_0(t_{1,v}^{-1}yt_{2,v}).$$

Here $\mathbb{B}_v(a) = \{x\in\mathbb{B}_v : q(x)=a\}$.

Proof. Denote $t = t_1t_2^{-1}$. We have

$$\langle Z_*(g,\phi)t_1,\xi_{t_2}\rangle = \langle Z_*(g,\phi)_{q(1/t)}t_1,\xi_{t_2}\rangle = \langle Z_*(g,\phi)_{q(1/t)}t_1,\xi\rangle = \langle Z_*(g,\phi)_{q(1/t)}[1],\xi\rangle.$$

Here the first equality holds as in the proof of Proposition 4.7, the second equality holds by a similar reason of geometrically connected components, and the third equality holds by the Galois action associated to t_1.

Recall that from [YZZ, §4.2.4] we have

$$Z_*(g)_{q(1/t)} \quad = \quad w_U \sum_{u\in\mu_U'\backslash F^\times}\sum_{a\in F_+^\times}\sum_{y\in K^t\backslash\mathbb{B}_f(a)} r(g,(t,1))\phi(y,u)Z(t^{-1}y).$$

Here

$$K^t = \mathrm{GSpin}(\mathbb{V}_f)\cap tKt^{-1} = \{(h_1,h_2)\in(tUt^{-1})\times U : q(h_1)=q(h_2)\}$$

acts on

$$\mathbb{B}_f(a) = \{x\in\mathbb{B}_f : q(x)=a\}$$

by $(h_1,h_2): x\mapsto h_1xh_2^{-1}$.

Hence, Lemma 4.8 gives

$$\langle Z_*(g,\phi)_{q(1/t)}[1],\xi\rangle = \deg(Z_*(g,\phi)_{q(1/t)})\langle[1],\xi\rangle + \sum_{v\nmid\infty}\mathcal{F}_\phi^{(v)}(g,(t_1,t_2)),$$

where

$$\mathcal{F}_\phi^{(v)}(g,(t_1,t_2)) = -w_U\sum_{u\in\mu_U'\backslash F^\times}\sum_{a\in F_+^\times}\sum_{y\in K^t\backslash\mathbb{B}_f(a)} r(g,(t,1))\phi(y,u)\cdot\deg Z(t^{-1}y)\cdot D_0(t_v^{-1}y_v).$$

As in the proof of Proposition 4.7, we already have

$$\deg Z(g,\phi)_{U,q(t^{-1})} = -\frac{1}{2}\kappa_U^\circ\ E(0,g,r(t_1,t_2)\phi)_U.$$

It remains to convert the above expression of $\mathcal{F}_\phi^{(v)}(g,(t_1,t_2))$ to the form in the proposition.

Consider the last summation

$$\sum_{y\in K^t\backslash\mathbb{B}_f(a)} r(g,(t,1))\phi(y,u)\deg Z(t^{-1}y)\cdot D_0(t_v^{-1}y_v).$$

By $\deg Z(t^{-1}y) = |Ut^{-1}yU/U|$, the summation is equal to

$$\sum_{y\in K^t\backslash\mathbb{B}_f(a)}\ \sum_{x\in Ut^{-1}yU/U} r(g)\phi(x,q(t)u)D_0(x_v).$$

Note that

$$Ut^{-1}yU/U = Kt^{-1}y/U = t^{-1}(tKt^{-1}y/U) = t^{-1}(K^ty/U^1).$$

The summation becomes

$$\begin{aligned}&\sum_{y\in\mathbb{B}_f(a)/U^1} r(g)\phi(t^{-1}y,q(t)u)D_0(t_v^{-1}y_v)\\ =&\left(\sum_{y\in\mathbb{B}_f^v(a)/(U^v)^1} r(g,(t,1))\phi^v(y,u)\right)\cdot\left(\sum_{y_v\in\mathbb{B}_v(a)/U_v^1} r(g,(t,1))\phi_v(y,u)D_0(t_v^{-1}y_v)\right).\end{aligned}$$

We assume that v is split in $\mathbb{B}$; otherwise $D_0(t_v^{-1}y_v)=0$ identically. It suffices to convert the first summation on the right-hand side in this case. The proof is similar to the proof of [YZZ, Proposition 4.2], except that we do not convert the second summation on the right-hand side.

In fact, by [YZZ, Proposition 2.9],

$$\sum_{y\in\mathbb{B}_f^v(a)/(U^v)^1} r(g,(t,1))\phi^v(y,u) = -\frac{|a|_v}{\mathrm{vol}((U^v)^1)\mathrm{vol}(\mathbb{B}_\infty^1)}W_{au}^v(0,g,u,r(h)\phi).$$

The negative sign comes from the Weil index of $\mathbb{B}^v$, which is -1 since $\mathbb{B}_v$ is a matrix algebra.

Finally, apply equation (4.3.2) in the proof of [YZZ, Proposition 4.2]. Note that $\mathrm{vol}(U_v^1) = (1-N_v^{-2})|d_v|^{\frac{3}{2}}$ by the normalization in [YZZ, §1.6.2]. It remains to check

$$f_{\phi_v,a}(g,(t_1,t_2),u) = f_{\phi_v,a}(g,(t,1),u).$$

This can be obtained by writing the sum over $\mathbb{B}_v(au^{-1})/U_v^1$ as an integral over $\mathbb{B}_v(au^{-1})$. □

For simplicity, write

$$\mathcal{F}_\phi^{(v)}(g) = \mathcal{F}_\phi^{(v)}(g,(1,1)),\quad f_{\phi_v,a}(g,u) = f_{\phi_v,a}(g,(1,1),u),\quad f_{\phi_v,a}(1,u) = f_{\phi_v,a}(1,(1,1),u).$$

Proposition 4.10. *For any non-archimedean place v nonsplit in $\mathbb{B}$, $f_{\phi_v,a}(1,u)\neq 0$ only if $a\in O_{F_v}$ and $u\in O_{F_v}^\times$. In that case,*

$$f_{\phi_v,a}(1,u)=|d_v|^{\frac{3}{2}}\frac{1+N_v^{-1}}{1-N_v^{-1}}\Big((r+2)N_v^{-(r+1)}-rN_v^{-(r+2)}-(r+2)N_v^{-1}+r\Big)\log N_v.$$

Here $r=v(a)$.

Proof. This is essentially computed in [Zh1]. By definition,

$$f_{\phi_v,a}(1,u)=(1-N_v^{-2})|d_v|^{\frac{3}{2}}|au^{-1}|_v\,\kappa_U^\circ\sum_{y\in\mathbb{B}_v(au^{-1})/U_v^1}\phi_v(y,u)D_0(y).$$

It is nonzero only if $u\in O_{F_v}^\times$ and $a\in O_{F_v}$, which we assume in the following. Identify $\mathbb{B}_2=M_2(F_v)$ and $O_{\mathbb{B}_v}=M_2(O_{F_v})$. Note $r=v(a)\geq 0$. Denote

$$M_2(O_{F_v})_r=\{y\in M_2(O_{F_v})_r: v(\det(y))=r\}.$$

Then the summation equals

$$\sum_{y\in(\mathbb{B}_v(au^{-1})\cap O_{\mathbb{B}_v})/U_v^1}D_0(y)=\sum_{y\in M_2(O_{F_v})_r/\mathrm{GL}_2(O_{F_v})}D_0(y)=\sum_{y\in \mathrm{GL}_2(O_{F_v})\backslash M_2(O_{F_v})_r/\mathrm{GL}_2(O_{F_v})}D(y).$$

Note that the double coset in the last summation corresponds exactly to the classical Hecke correspondence $T(p_v^r)$. Hence, the above further equals

$$T(p_v^r)\hat{\xi}-\deg(T(p_v^r))\hat{\xi}.$$

Here $\deg(T(p_v^r))=\sigma_1(p_v^r)=1+N_v+\cdots+N_v^r$.

By [Zh1, Proposition 4.3.2],

$$T(p_v^r)\overline{\mathcal{L}}-\sigma_1(p_v^r)\ \overline{\mathcal{L}}=-2\sum_{i=0}^{r}iN_v^{r-i}\log N_v+\log(N_v^{r\sigma_1(p_v^r)})$$

In fact, the proposition considers a morphism

$$T(p_v^r)\mathcal{L}\longrightarrow\mathcal{L}^{\otimes\deg(T(p_v^r))}$$

and computes the norms of this morphism at non-archimedean places in part 1 and at archimedean places in part 2. Note that the result in part 2 of the proposition should be $N(m)^{\sigma_1(m)}$ instead of $N(m)^{2\sigma_1(m)}$. The sum of the logarithms of these norms gives the formula.

An elementary computation gives

$$T(p_v^r)\overline{\mathcal{L}}-\sigma_1(p_v^r)\ \overline{\mathcal{L}}=\frac{(r+2)N_v^{-(r+1)}-rN_v^{-(r+2)}-(r+2)N_v^{-1}+r}{(1-N_v^{-1})^2}N_v^r\log N_v.$$

The result follows by $\overline{\mathcal{L}}=\kappa_U^\circ\cdot\hat{\xi}$. □

The expression of $f_{\phi_v,a}(1,u)$ in the above lemma happens to be very close to that of $W'_{a,v}(0,1,u)-\frac{1}{2}\log|a|_vW_{a,v}(0,1,u)$ in Lemma 3.4 (1). They will give great cancelation in our matching of the derivative series and the height series.

5 Comparison of the two series

In this section, we will combine results in the last two sections to prove Theorem 1.1. The upshot is to apply Lemma 2.2 to the difference

$$\mathcal{D}(g,\phi) = \mathcal{P}rI'(0,g,\phi)_U - 2Z(g,(1,1))_U.$$

Here we take $t_1 = t_2 = 1$ for the CM points.

We will see that the right-hand side is a sum of finitely many non-singular pseudo-Eisenstein series and non-singular pseudo-theta series. Then Lemma 2.2 will imply that $\mathcal{D}(g,\phi)$ is the sum of the corresponding Eisenstein series and theta series. Since $\mathcal{D}(g,\phi)$ is cuspidal, its constant must be zero. This implies that the sum of the constant terms of the corresponding Eisenstein series and theta series is zero, which gives an equality involving the modular height of X_U. After computing all other terms, we get a formula of the modular height.

To start with, let $(F, E, \mathbb{B}, U, \phi)$ be as in §3.2. By Theorem 3.1,

$$\mathcal{P}rI'(0,g,\phi)_U = \mathcal{P}r'I'(0,g,\phi)_U - \mathcal{P}r'\mathcal{J}'(0,g,\phi)_U,$$

By Theorem 4.2,

$$Z(g,(1,1))_U = \langle Z_*(g,\phi)_U 1, 1\rangle - \langle Z_*(g,\phi)_U 1, \xi_1\rangle - \langle Z_*(g,\phi)_U \xi_1, 1\rangle + \langle Z_*(g,\phi)_U \xi_1, \xi_1\rangle.$$

Then the difference

$$\begin{aligned}\mathcal{D}(g,\phi) \;=\;& \mathcal{P}r'I'(0,g,\phi)_U - 2\langle Z_*(g,\phi)_U 1, 1\rangle\\ &-\mathcal{P}r'\mathcal{J}'(0,g,\phi)_U + 2\langle Z_*(g,\phi)_U 1, \xi_1\rangle\\ &+2\langle Z_*(g,\phi)_U \xi_1, 1\rangle - 2\langle Z_*(g,\phi)_U \xi_1, \xi_1\rangle.\end{aligned}$$

In the following, for each of the three lines on the right-hand side of the above expression of $\mathcal{D}(g,\phi)$, we will describe the computational result, check that it is non-singular in the pseudo sense, and give its contribution in the equality after applying Lemma 2.2.

Third line

Start with the third line, which has the simplest expression. By Proposition 4.7,

$$\langle Z_*(g,\phi)_U \xi_1, 1\rangle = -\frac{1}{2}\kappa_U^\circ\ E_*(0,g,\phi)_U \cdot \langle \xi_1, 1\rangle,$$

$$\langle Z_*(g,\phi)_U \xi_1, \xi_1\rangle = -\frac{1}{2}\kappa_U^\circ\ E_*(0,g,\phi)_U \cdot \langle \xi_1, \xi_1\rangle.$$

Here κ_U° denotes the degree of L_U on a geometrically connected component of X_U.

The contribution of $2\langle Z_*(g,\phi)_U \xi_1, 1\rangle - 2\langle Z_*(g,\phi)_U \xi_1, \xi_1\rangle$ after Lemma 2.2 is

$$\kappa_U^\circ \cdot (\langle \xi_1, \xi_1\rangle - \langle \xi_1, 1\rangle) \cdot E(0,g,\phi)_U. \qquad (5.0.1)$$

Second line

Now we consider the second line. Denote $c_3' = (1 + \log 4)[F : \mathbb{Q}]$. By Proposition 3.2,

$$\mathcal{P}r'\mathcal{J}'(0, g, \phi) = -(c_0 + c_3')E_*(0, g, \phi) - \sum_{v\nmid\infty} C_*(0, g, \phi)(v) + 2\sum_{v\nmid\infty} E'(0, g, \phi)(v).$$

Here we have Eisenstein series

$$\begin{aligned} E(s, g, \phi) &= \sum_{u\in\mu_U^2\backslash F^\times} \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s r(\gamma g)\phi(0, u), \\ C(s, g, \phi)(v) &= \sum_{v\nmid\infty} \sum_{u\in\mu_U^2\backslash F^\times} \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s c_{\phi_v}(\gamma g, 0, u)\ r(\gamma g^v)\phi^v(0, u), \end{aligned}$$

with

$$c_{\phi_v}(g, y, u) = r_E(g)\phi_{1,v}(y, u)W_{0,v}^{\circ}{}'(0, g, u, \phi_{2,v}) + \log\delta(g_v)r(g)\phi_v(y, u);$$

and we have a pseudo-Eisenstein series

$$E'(0, g, \phi)(v) = \sum_{u\in\mu_U^2\backslash F^\times} \sum_{a\in F^\times} W_a^v(0, g, u, \phi^v)\left(W_{a,v}'(0, g, u, \phi_v) - \frac{1}{2}\log|a|_v \cdot W_{a,v}(0, g, u, \phi_v)\right).$$

By Proposition 4.9,

$$\langle Z_*(g, \phi)1, \xi_1\rangle = -\frac{1}{2}\kappa_U^{\circ}\ E_*(0, g, \phi)_U\ \langle 1, \xi\rangle + \frac{1}{2}\sum_{v\notin\Sigma} \mathcal{F}_\phi^{(v)}(g),$$

where the pseudo-Eisenstein series

$$\mathcal{F}_\phi^{(v)}(g) = \sum_{u\in\mu_U^2\backslash F^\times} \sum_{a\in F^\times} W_a^v(0, g, u, \phi)\, f_{\phi_v,a}(g, u)$$

with

$$f_{\phi_v,a}(g, u) = (1 - N_v^{-2})|d_v|^{\frac{3}{2}}|au^{-1}|_v\, \kappa_U^{\circ} \sum_{y\in\mathbb{B}_v(au^{-1})/U_v^1} r(g)\phi_v(y, u)D_0(t_{1,v}^{-1}yt_{2,v}).$$

The difference gives

$$\begin{aligned} & -\mathcal{P}r'\mathcal{J}'(0, g, \phi)_U + 2\langle Z_*(g, \phi)1, \xi_1\rangle \\ = \ & (c_0 + c_3' - \kappa_U^{\circ}\langle 1, \xi\rangle)E_*(0, g, \phi) + \sum_{v\nmid\infty} C_*(0, g, \phi)(v) \\ & -2\sum_{v\in\Sigma_f} E'(0, g, \phi)(v) - 2\sum_{v\notin\Sigma}(E'(0, g, \phi)(v) - \frac{1}{2}\mathcal{F}_\phi^{(v)}(g)) \end{aligned}$$

This is a finite sum of Eisenstein series and pseudo-Eisenstein series, by the following considerations using the explicit local results.

(1) For any $v\in\Sigma_f$, the explicit result of Lemma 3.4(2) implies that

$$W'_{a,v}(0,1,u,\phi_v)-\frac{1}{2}\log|a|_v\cdot W_{a,v}(0,1,u,\phi_v),$$

as a function of $(a,u)\in F_v^\times\times F_v^\times$, satisfies the condition of Lemma 2.3(2). Therefore, $E'(0,g,\phi)(v)$ is a non-singular pseudo-Eisenstein series in this case. Denote by

$$E(0,g,\phi_v^+\otimes\phi^v)+E(0,g,\phi_v^-\otimes\phi^v)$$

the associated Eisenstein series. Note that Lemma 2.3(1) and Lemma 3.4(2) further give

$$r(w)\phi_v^+(0,u)+r(w)\phi_v^-(0,u)=0,\quad \forall u\in F_v^\times.$$

(2) For any $v\notin\Sigma$, the pseudo-Eisenstein series

$$E'(0,g,\phi)(v)-\frac{1}{2}\mathcal{F}_\phi^{(v)}(g)=\sum_{u\in\mu_U^2\backslash F^\times}\sum_{a\in F^\times}W_a^v(0,g,u,\phi^v)\tilde{f}_{\phi_v,a}(g,u),$$

where

$$\tilde{f}_{\phi_v,a}(g,u)=\left(W'_{a,v}(0,g,u,\phi_v)-\frac{1}{2}\log|a|_v\cdot W_{a,v}(0,g,u,\phi_v)\right)-\frac{1}{2}f_{\phi_v,a}(g,u)$$

for any $a,u\in F_v^\times$, $g\in\mathrm{GL}_2(F_v)$. By the explicit results of Lemma 3.4(1) and Proposition 4.10, $\tilde{f}_{\phi_v,a}(1,u)\neq0$ only if $u\in O_{F_v}^\times$ and $v(a)\geq -v(d_v)$. Moreover, for $u\in O_{F_v}^\times$ and $a\in O_{F_v}$,

$$\tilde{f}_{\phi_v,a}(1,u)=(-\zeta_v'(2)/\zeta_v(2)+\log|d_v|)\,W_{a,v}(0,1,u)+|d_v|^{\frac{3}{2}}\frac{1-|d_v|}{N_v-1}\log N_v.$$

By Lemma 2.3(2), we see that $E'(0,g,\phi)(v)-\frac{1}{2}\mathcal{F}_\phi^{(v)}(g)$ is a non-singular pseudo-Eisenstein series in this case. The associated Eisenstein series is of the form

$$(-\zeta_v'(2)/\zeta_v(2)+\log|d_v|)\,E(0,g,\phi)+E(0,g,\phi_v^+\otimes\phi^v)+E(0,g,\phi_v^-\otimes\phi^v).$$

The last two terms are 0 for almost all $v\notin\Sigma$. Moreover, Lemma 2.3(2) also gives for all $v\notin\Sigma$,

$$\phi_v^+(0,u)+\phi_v^-(0,u)=0,\quad\forall u\in F_v^\times.$$

Therefore, the contribution of $-\mathcal{P}r'\mathcal{J}'(0,g,\phi)_U+2\langle Z_*(g,\phi)1,\xi_1\rangle$ after Lemma 2.2 is

$$\begin{aligned}&(c_0+c_3'-\kappa_U^\circ\langle1,\xi\rangle+2\sum_{v\notin\Sigma}(\zeta_v'(2)/\zeta_v(2)-\log|d_v|))E(0,g,\phi)\\&+\sum_{v\nmid\infty}C(0,g,\phi)(v)-2\sum_{v\nmid\infty}\left(E(0,g,\phi_v^+\otimes\phi^v)+E(0,g,\phi_v^-\otimes\phi^v)\right).\end{aligned}\qquad(5.0.2)$$

First line

It remains to consider

$$\mathcal{P}r'I'(0,g,\phi)_U - 2\langle Z_*(g,\phi)_U 1,1\rangle.$$

By Theorem 3.1, the current $\mathcal{P}r'I'(0,g,\phi)_U$ has the same expression as the old $\mathcal{P}rI'(0,g,\phi)_U$ in [YZ, Theorem 7.2]. By Theorem 4.2, the current $\langle Z_*(g,\phi)_U 1,1\rangle$ has the expression of the old $Z(g,(1,1),\phi))_U$ in [YZ, Theorem 8.6] with the extra term $-\frac{1}{2}[F:\mathbb{Q}]E_*(0,g,\phi)$. Consequently, the current difference $\mathcal{P}r'I'(0,g,\phi)_U - 2\langle Z_*(g,\phi)_U 1,1\rangle$ has the expression of the old $\mathcal{D}(g,\phi)$ in [YZ, §9.1] with an extra term $[F:\mathbb{Q}]E_*(0,g,\phi)$.

Note that the choice of ϕ in [YZ] is slightly different from what we have here. In [YZ, §7.2], it has an extra set S_2 of two non-archimedean places v of F split in E with certain degenerate ϕ_v. This assumption is made to kill the terms close to $E(s,g,\phi)$. However, the computations for $\mathcal{D}(g,\phi)$ holds for our $\mathcal{P}r'I'(0,g,\phi)_U - 2\langle Z_*(g,\phi)_U 1,1\rangle$ by pretending $S_2=\emptyset$.

Hence, the translation of the computational results of [YZ, §9.1] to the current setting gives

$$\begin{aligned}
&\mathcal{P}r'I'(0,g,\phi)_U - 2\langle Z_*(g,\phi)_U 1,1\rangle\\
= & -2\sum_{v\nmid\infty \text{ nonsplit}} \fint_{C_U} (\mathcal{K}_\phi^{(v)}(g,(t,t)) - \mathcal{M}_\phi^{(v)}(g,(t,t))\log N_v)dt\\
&+\sum_{v\in\Sigma_f}(2\log N_v)\fint_{C_U} j_{\bar v}(Z_*(g,\phi)_U t,t)dt\\
&+\sum_{v\nmid\infty}\sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in E^\times} d_{\phi_v}(g,y,u)\ r(g)\phi^v(y,u)\\
&+\Big(\frac{2i_0(1,1)}{[O_E^\times:O_F^\times]}-c_1\Big)\sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in E^\times} r(g)\phi(y,u)\\
&+[F:\mathbb{Q}]E_*(0,g,\phi)\\
&-[F:\mathbb{Q}](\gamma+\log(4\pi)-1)E_*(0,g,\phi).
\end{aligned}$$

Here the last line comes from and is equal to

$$-2\sum_{v|\infty}\fint_{C_U}(\overline{\mathcal{K}}_\phi^{(v)}(g,(t,t)) - \mathcal{M}_\phi^{(v)}(g,(t,t)))dt,$$

which follows from Proposition 4.4 and was missed in [YZ, §9.1].

The last expression is a sum of finitely many non-singular pseudo-theta series. Note that neither the fourth line or the fifth line contributes to the result after Lemma 2.2 because they are *degenerate* pseudo theta series. So we only recall the other lines. Recall that

$$\begin{aligned}
&\mathcal{K}_\phi^{(v)}(g,(t_1,t_2)) - \mathcal{M}_\phi^{(v)}(g,(t_1,t_2))(\log N_v)\\
=&\sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in B(v)-E} r(g,(t_1,t_2))\phi^v(y,u)\bar k_{r(t_1,t_2)\phi_v}(g,y,u)
\end{aligned}$$

where

$$\bar{k}_{\phi_v}(g,y,u) = k_{\phi_v}(g,y,u) - m_{r(g)\phi_v}(y,u)\log N_v.$$

In particular.

$$\bar{k}_{\phi_v}(y,u) = k_{\phi_v}(1,y,u) - m_{\phi_v}(y,u)\log N_v$$

extends to a Schwartz function in $\overline{\mathcal{S}}(B(v)_v \times F_v^\times)$. Similarly,

$$j_{\bar{v}}(Z_*(g,\phi)_U t_1, t_2) = \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in B(v)^\times} r(g,(t_1,t_2))\phi^v(y,u)r(t_1,t_2)l_{r(g)\phi_v}(y,u)$$

where $l_{\phi_v}(y,u)$ extends to a Schwartz function in $\overline{\mathcal{S}}(B(v)_v \times F_v^\times)$.

The contribution of $\mathcal{P}r'I'(0,g,\phi)_U - 2\langle Z_*(g,\phi)_U 1,1\rangle$ after Lemma 2.2 is

$$\begin{aligned}
&-2\sum_{v\nmid\infty \text{ nonsplit}} \fint_{C_U} \theta(g,(t,t),\bar{k}_{\phi_v}\otimes\phi^v)dt \\
&+\sum_{v\in\Sigma_f}(2\log N_v)\fint_{C_U}\theta(g,(t,t),l_{\phi_v}\otimes\phi^v)dt \\
&-[F:\mathbb{Q}](\gamma+\log(4\pi)-2)E_*(0,g,\phi).
\end{aligned} \tag{5.0.3}$$

The sum

As a conclusion, the difference $\mathcal{D}(g,\phi)$ is the sum of finitely many non-singular pseudo-Eisenstein series and finitely many non-singular pseudo-theta series. We are finally ready to apply Lemma 2.2. For the conclusion, the sum of (5.0.1), (5.0.2) and (5.0.3) gives

$$\begin{aligned}
\mathcal{D}(g,\phi) = &-2\sum_{v\nmid\infty \text{ nonsplit}} \fint_{C_U} \theta(g,(t,t),\bar{k}_{\phi_v}\otimes\phi^v)dt + \sum_{v\in\Sigma_f}(2\log N_v)\fint_{C_U}\theta(g,(t,t),l_{\phi_v}\otimes\phi^v)dt \\
&+\sum_{v\nmid\infty} C(0,g,\phi)(v) - 2\sum_{v\nmid\infty}\left(E(0,g,\phi_v^+\otimes\phi^v)+E(0,g,\phi_v^-\otimes\phi^v)\right) \\
&+c_4\cdot E(0,g,\phi)_U,
\end{aligned} \tag{5.0.4}$$

where

$$c_4 = c_0 + c_3' - 2\kappa_U^\circ\langle 1,\xi\rangle + \kappa_U^\circ\langle\xi_1,\xi_1\rangle + 2\sum_{v\notin\Sigma}(\zeta_v'(2)/\zeta_v(2) - \log|d_v|) - [F:\mathbb{Q}](\gamma+\log(4\pi)-2).$$

Here we have used the identity $\langle 1,\xi\rangle = \langle 1,\xi_1\rangle$, which holds by considering geometrically connected components. Moreover, we have the following result.

Lemma 5.1.

$$\kappa_U^\circ\langle\xi_1,\xi_1\rangle = -2\,h_{\overline{\mathcal{L}}_U}(X_U), \qquad \kappa_U^\circ\langle 1,\xi\rangle = -h_{\overline{\mathcal{L}}_U}(P_U) + [F:\mathbb{Q}].$$

Proof. Denote $|\pi_0|$ the number of geometrically connected component of X_U. The first result goes as follows:

$$\kappa_U^\circ\langle\xi_1,\xi_1\rangle = \frac{\kappa_U^\circ}{|\pi_0|}\langle\xi,\xi\rangle = \frac{\kappa_U^\circ}{|\pi_0|}\frac{1}{(\kappa_U^\circ)^2}\langle\overline{\mathcal{L}}_U,\overline{\mathcal{L}}_U\rangle = \frac{1}{\deg(L_U)}\langle\overline{\mathcal{L}}_U,\overline{\mathcal{L}}_U\rangle = -\frac{1}{\deg(L_U)}\widehat{\deg}(\hat{c}_1(\overline{\mathcal{L}}_U)^2).$$

The first equality holds by considering geometrically connected components of X_U, and the other equalities holds by definition. Note that the negative sign of the last equality is due to different normalizations of the intersection numbers, which is originally from the negative sign in the arithmetic Hodge index theorem.

The second result goes as follows:

$$\kappa_U^\circ\langle 1,\xi\rangle = \langle 1,\overline{\mathcal{L}}_U\rangle = \langle \bar{P}_U,\overline{\mathcal{L}}_U\rangle + [F:\mathbb{Q}] = -h_{\overline{\mathcal{L}}_U}(P_U) + [F:\mathbb{Q}].$$

Here the second equality follows from Lemma 4.1. □

By the lemma, c_4 contains the height $h_{\overline{\mathcal{L}}_U}(X_U)$ which we need to compute. For $h_{\overline{\mathcal{L}}_U}(P_U)$, by Theorem 1.2, the main result of Part II of [YZ],

$$h_{\overline{\mathcal{L}}_U}(P_U) = -\frac{L_f'(0,\eta)}{L_f(0,\eta)} + \frac{1}{2}\log\frac{d_{\mathbb{B}}}{d_{E/F}}.$$

It cancels the major part of

$$c_0 = 2\frac{L'(0,\eta)}{L(0,\eta)} + \log|d_E/d_F| = 2\frac{L_f'(0,\eta)}{L_f(0,\eta)} + \log|d_E/d_F| - [F:\mathbb{Q}](\gamma+\log 4\pi).$$

More precisely,

$$c_0 + 2\,h_{\overline{\mathcal{L}}_U}(P_U) = \log|d_{\mathbb{B}}d_F| - [F:\mathbb{Q}](\gamma+\log 4\pi)$$

By $c_3' = (1+\log 4)[F:\mathbb{Q}]$, we further have

$$c_0 + c_3' + 2\,h_{\overline{\mathcal{L}}_U}(P_U) = \log|d_{\mathbb{B}}d_F| - [F:\mathbb{Q}](\gamma+\log\pi-1).$$

Hence, we can simplify c_4 to get

$$c_4 = -2\,h_{\overline{\mathcal{L}}_U}(X_U) + 2\sum_{v\notin\Sigma}(\zeta_v'(2)/\zeta_v(2) - \log|d_v|) + \log|d_{\mathbb{B}}d_F| - [F:\mathbb{Q}](2\gamma+2\log(2\pi)-1).$$

The constant terms

Note that $\mathcal{D}(g,\phi)$ is a cusp form, so its constant term must be 0. Then the constant terms of the right-hand side of (5.0.4) should be 0. This will give the result we need.

In the following, we first treat the case $|\Sigma|>1$ and then mention the difference for the easier case $|\Sigma|=1$. While it is straightforward to write down the constant terms of the theta series, it takes a little extra effort to treat those for the Eisenstein series. We claim that the constant terms of the Eisenstein series

$$E(0,g,\phi)_U,\qquad C(0,g,\phi)(v),\qquad E(0,g,\phi_v^+\otimes\phi^v) + E(0,g,\phi_v^-\otimes\phi^v),$$

are respectively equal to

$$\sum_{u\in\mu_U^2\backslash F^\times} r(g)\phi(0,u),$$
$$\sum_{u\in\mu_U^2\backslash F^\times} c_{\phi_v}(g,0,u)\ r(g^v)\phi^v(0,u),$$
$$\sum_{u\in\mu_U^2\backslash F^\times} (r(g_v)\phi_v^+(0,u)+r(g_v)\phi_v^-(0,u))\ r(g^v)\phi^v(0,u).$$

In other words, the contribution from the intertwining part at $s=0$ is 0. This will be a consequence of our assumption that $|\Sigma|>1$. In fact, the result for $E(0,g,\phi)_U$ is immediately a consequence of [YZZ, Proposition 2.9(3)]. To treat the other two Eisenstein series, take $C(0,g,\phi)(v)$ for example. As in the proof of [YZZ, Proposition 2.9(3)], at $s=0$, the intertwining part of $C(s,g,\phi)(v)$ is a product of $\zeta_F(s+1)$ with the normalized local components. Each local component at a place in $\Sigma\smallsetminus\{v\}$ contributes a zero at $s=0$, and $\zeta_F(s+1)$ contributes a pole at $s=0$. The product gives a zero at $s=0$ of order at least $|\Sigma\smallsetminus\{v\}|-1\geq 1$. This proves the claim.

Taking the constant terms of (5.0.4), we end up with

$$\begin{aligned}
0 \;=\; & -2\sum_{v\nmid\infty \text{ nonsplit}}\ \sum_{u\in\mu_U^2\backslash F^\times} r(g)(\bar{k}_{\phi_v}\otimes\phi^v)(0,u)\\
&+\sum_{v\in\Sigma_f}(2\log N_v)\sum_{u\in\mu_U^2\backslash F^\times} r(g)(l_{\phi_v}\otimes\phi^v)(0,u)\\
&+\sum_{v\nmid\infty}\ \sum_{u\in\mu_U^2\backslash F^\times} c_{\phi_v}(g,0,u)\ r(g^v)\phi^v(0,u)\\
&-2\sum_{v\nmid\infty}\ \sum_{u\in\mu_U^2\backslash F^\times}(r(g_v)\phi_v^+(0,u)+r(g_v)\phi_v^-(0,u))\ r(g^v)\phi^v(0,u)\\
&+c_4\sum_{u\in\mu_U^2\backslash F^\times} r(g)\phi(0,u).
\end{aligned}$$

The goal is to get a formula of c_4 from the expression. Then it suffices to take a specific $g\in\mathrm{GL}_2(\mathbb{A})$ such that

$$\sum_{u\in\mu_U^2\backslash F^\times} r(g)\phi(0,u)\neq 0.$$

Note that $g=1$ does not work since $\phi_v(0,u)=0$ for any $v\in\Sigma_f$. Define $g=(g_v)_v\in\mathrm{GL}_2(\mathbb{A})$ by

$$g_v=\begin{cases} w=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}, & v\in\Sigma_f,\\ 1, & v\notin\Sigma_f.\end{cases}$$

Now we simplify the above equality for this g.

By the above discussion, we already have

$$r(g_v)\phi_v^+(0,u)+r(g_v)\phi_v^-(0,u)=0$$

for any $v \nmid \infty$. It follows the fourth line of the right-hand side of the equality is 0. The equation becomes

$$\begin{aligned} 0 &= -2 \sum_{v \nmid \infty \text{ nonsplit}} \sum_{u \in \mu_U^2 \backslash F^\times} r(g)(\bar{k}_{\phi_v} \otimes \phi^v)(0,u) \\ &\quad + \sum_{v \in \Sigma_f} (2 \log N_v) \sum_{u \in \mu_U^2 \backslash F^\times} r(g)(l_{\phi_v} \otimes \phi^v)(0,u) \\ &\quad + \sum_{v \nmid \infty} \sum_{u \in \mu_U^2 \backslash F^\times} c_{\phi_v}(g,0,u)\ r(g^v)\phi^v(0,u) \\ &\quad + c_4 \sum_{u \in \mu_U^2 \backslash F^\times} r(g)\phi(0,u). \end{aligned} \tag{5.0.5}$$

Here we recall that

$$\bar{k}_{\phi_v}(y,u) = k_{\phi_v}(1,y,u) - m_{\phi_v}(y,u) \log N_v$$

extends to a Schwartz function in $\overline{\mathcal{S}}(B(v)_v \times F_v^\times)$.

Note that each of the first three lines of the right-hand side of (5.0.5) has a sum over places certain places v of F. In the following, for each non-archimedean place v, we consider the contribution of this fixed v from our these three lines.

(1) If v is split in E, then only the third line has contribution from v. In this case, by [YZ, Lemma 7.6],

$$c_{\phi_v}(1,0,u) = \log |d_v|\ \phi_v(0,u).$$

(2) If v is nonsplit in E but split in $\mathbb{B}$, then both the first line and the the third line has contribution from v. In this case, by [YZ, Lemma 7.4, Lemma 7.6, Lemma 8.7],

$$-2\bar{k}_{\phi_v}(0,u) + c_{\phi_v}(1,0,u) = \log |d_v|\ \phi_v(0,u).$$

See also [YZ, Proposition 9.2].

(3) If v is nonsplit in $\mathbb{B}$, by Lemma 3.5, Lemma 3.6, and Lemma 4.6,

$$-2r(w)\bar{k}_{\phi_v}(0,u) + 2r(w)l_{\phi_v}(0,u)\log N_v + c_{\phi_v}(w,0,u) = (-\log|d_v| + \alpha_v \log N_v) r(w)\phi_v(0,u),$$

where

$$\alpha_v = 1 - \frac{N_v - 1}{2(N_v+1)}.$$

Note that the expressions in Lemma 3.5 and Lemma 3.6 depend on the parity of $v(d_v)$, but their combined expression for α_v happens to be uniform for all $v(d_v)$. This can be explained by the fact that Lemma 3.5 treats $W'_{0,v}(0,g,u,\phi_{2,v})$, while Lemma 3.6 treats $W'_{a,v}(0,g,u,\phi_{2,v})$.

Taking all these into consideration, the equation becomes

$$0 = \left(\sum_{v \notin \Sigma} \log |d_v| + \sum_{v \in \Sigma_f} (-\log |d_v| + \alpha_v \log N_v) + c_4 \right) \sum_{u \in \mu_U^2 \backslash F^\times} r(g)\phi(0,u).$$

Note that
$$\sum_{u\in\mu_U^2\backslash F^\times} r(g)\phi(0,u)>0$$
by our choice of g. We get an equation
$$\sum_{v\notin\Sigma}\log|d_v| + \sum_{v\in\Sigma_f}(-\log|d_v| + \alpha_v\log N_v) + c_4 = 0. \tag{5.0.6}$$
This is obtained for the case $|\Sigma|>1$.

If $|\Sigma|=1$, we claim that (5.0.6) also holds. In this case, the constant terms for $E(0,g,\phi)_U$ and other similar series might contain nonzero intertwining parts by [YZZ, Proposition 2.9(3)]. We may figure our the effect of this by extra argument. Alternatively, (5.0.4) simply implies
$$\mathcal{D}(g,\phi) = c_4\cdot E(0,g,\phi)_U,$$
since the other terms are zero by the computational results. Comparing the constant terms, we easily have $c_4=0$, since $\mathcal{D}(g,\phi)$ is cupidal. This agrees with (5.0.6).

Logarithmic derivative

Recall that
$$c_4 = -2\,h_{\overline{\mathcal{L}}_U}(X_U) + 2\sum_{v\notin\Sigma}(\zeta_v'(2)/\zeta_v(2) - \log|d_v|) + \log|d_{\mathbb{B}}d_F| - [F:\mathbb{Q}](2\gamma+2\log(2\pi)-1)$$
Then (5.0.6) becomes
$$-2\,h_{\overline{\mathcal{L}}_U}(X_U) + 2\sum_{v\notin\Sigma}\zeta_v'(2)/\zeta_v(2) + \log|d_{\mathbb{B}}d_F^2| - [F:\mathbb{Q}](2\gamma+2\log(2\pi)-1) + \sum_{v\in\Sigma_f}\alpha_v\log N_v = 0.$$
Note that
$$\frac{\zeta_F'(2)}{\zeta_F(2)} = \sum_{v\nmid\infty}\frac{\zeta_v'(2)}{\zeta_v(2)}, \qquad \frac{\zeta_v'(2)}{\zeta_v(2)} = -\frac{N_v^{-2}}{1-N_v^{-2}}\log N_v.$$
The first equality holds because the Euler product of $\zeta_F(s)$ is absolutely convergent for $\mathrm{Re}(s)>1$. Hence, we finally end up with
$$-2\,h_{\overline{\mathcal{L}}_U}(X_U) + 2\frac{\zeta_F'(2)}{\zeta_F(2)} + \sum_{v\in\Sigma_f}\left(\alpha_v + \frac{2N_v^{-2}}{1-N_v^{-2}} + 1\right)\log N_v + \log|d_F^2| - [F:\mathbb{Q}](2\gamma+2\log(2\pi)-1) = 0.$$
Here the local term
$$\alpha_v + \frac{2N_v^{-2}}{1-N_v^{-2}} + 1 = 1 - \frac{N_v-1}{2(N_v+1)} + \frac{2N_v^{-2}}{1-N_v^{-2}} + 1 = \frac{3}{2} + \frac{1}{N_v-1} = \frac{3N_v-1}{2(N_v-1)}.$$
Therefore,
$$h_{\overline{\mathcal{L}}_U}(X_U) = \frac{\zeta_F'(2)}{\zeta_F(2)} + \sum_{v\in\Sigma_f}\frac{3N_v-1}{4(N_v-1)}\log N_v + \log|d_F| - (\gamma+\log(2\pi)-\frac{1}{2})[F:\mathbb{Q}].$$

Functional equation

We can convert the logarithmic derivative at 2 to that at -1 by the functional equation. In fact, the completed Dedekind zeta function

$$\tilde{\zeta}_F(s) = \tilde{\zeta}_{F,\infty}(s)\zeta_F(s)$$

with the gamma factor

$$\tilde{\zeta}_{F,\infty}(s) = (\pi^{-s/2}\Gamma(s/2))^{[F:\mathbb{Q}]}$$

has functional equation

$$\tilde{\zeta}_F(1-s) = |d_K|^{s-\frac{1}{2}}\tilde{\zeta}_F(s).$$

Note that

$$\frac{\tilde{\zeta}'_{F,\infty}(2)}{\tilde{\zeta}_{F,\infty}(2)} = -\frac{1}{2}(\gamma + \log\pi)[F:\mathbb{Q}],$$
$$\frac{\tilde{\zeta}'_{F,\infty}(-1)}{\tilde{\zeta}_{F,\infty}(-1)} = -\frac{1}{2}(\gamma + \log(4\pi))[F:\mathbb{Q}] + [F:\mathbb{Q}].$$

It follows that

$$\begin{aligned} h_{\overline{\mathcal{L}}_U}(X_U) &= \frac{\tilde{\zeta}'_F(2)}{\tilde{\zeta}_F(2)} + \log|d_F| - \frac{1}{2}[F:\mathbb{Q}](\gamma + \log(4\pi) - 1) + \sum_{v\in\Sigma_f}\frac{3N_v - 1}{4(N_v - 1)}\log N_v \\ &= -\frac{\tilde{\zeta}'_F(-1)}{\tilde{\zeta}_F(-1)} - \frac{1}{2}[F:\mathbb{Q}](\gamma + \log(4\pi) - 1) + \sum_{v\in\Sigma_f}\frac{3N_v - 1}{4(N_v - 1)}\log N_v \\ &= -\frac{\zeta'_F(-1)}{\zeta_F(-1)} - \frac{1}{2}[F:\mathbb{Q}] + \sum_{v\in\Sigma_f}\frac{3N_v - 1}{4(N_v - 1)}\log N_v. \end{aligned}$$

This prove Theorem 1.1.

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