Triple product L-series and Gross–Schoen cycles I: split case (Draft version)

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1 Introduction to the main results

1.1 Local linear forms and pairing

Let F be a local field and E a cubic semisimple algebra over F. More precisely, E is one of the three forms: $F \oplus F \oplus F$, $F \oplus K$ for a quadratic extension K of F, and a cubic extension E of F. Let B be a quaternion algebra over F. Thus either $B = M_2(F)$ or $B = D$ a division algebra. Let π be a admissible representation of $B_E^\times$. Assume that the central character ω of π is trivial when restricted to $F^\times$

$$\omega|_{F^\times} = 1.$$

Consider the space of

$$\mathscr{L}(\pi) := \mathrm{Hom}_{B^\times}(\pi, \mathbb{C}).$$

Then it have been shown that this space has dimension ≤ 1. More precisely, if B' is the complement of B in $\{M_2(F), D\}$ and let π' denote the Jacquet-Langlands correspondence of π on B' (zero if π is not discrete). Then the work of Prasad (non-archimedean) and Loke (archimedean) shows that

$$\dim \mathscr{L}(\pi) + \dim \mathscr{L}(\pi') = 1.$$

Let $\epsilon(\pi)$ be the local root number

$$\epsilon(\pi) := \epsilon(\frac{1}{2}, \pi, \psi \circ \mathrm{tr}_{E/F}) \in \{\pm 1\}.$$

The definition here does not depend on the choice of non-trivial character ψ of F. Then the non-vanishing of these spaces can determined by $\epsilon(\pi)$:

$$\epsilon(\pi) = 1 \Longleftrightarrow B = M_2(F)$$

$$\epsilon(v) = -1 \Longleftrightarrow B = D.$$

Assume that π is tempered, then the following integration of matrix coefficients with respect to a Haar measure on $F^\times \backslash B^\times$ is convergent:

$$\int_{F^\times \backslash B^\times} (\pi(b)f, \widetilde{f}) db^\times, \qquad f \otimes \widetilde{f} \in \pi \otimes \widetilde{\pi}$$

This integration defines a linear form on $\pi \otimes \widetilde{\pi}$ which is invariant under $B^\times \times B^\times$, i.e., an element in

$$\mathscr{L}(\pi) \otimes \mathscr{L}(\widetilde{\pi}) = \mathrm{Hom}_{B^\times \times B^\times}(\pi \otimes \widetilde{\pi}, \mathbb{C}).$$

One can show that this linear form is nonzero if and only if $\mathscr{L}(\pi) \neq 0$. In order to obtain a linear form on $\pi \otimes \widetilde{\pi}$, we need to evaluate the integral in the following spherical case:

1. E/F and π are unramified, f and $\widetilde{f}$ are spherical vector such that $(f, \widetilde{f}) = 1$;
2. dg takes value 1 on the maximal compact subgroup of $B^\times$.

In this case, one can show that the integration is given by

$$\frac{\zeta_E(2)}{\zeta_F(2)} \frac{L(1/2, \sigma)}{L(1, \sigma, ad)}.$$

See Ichino [14], Lemma 2.2. Thus we can define a normalized linear form

$$m \in \mathscr{L}(\pi) \otimes \mathscr{L}(\widetilde{\pi})$$

$$m(f, \widetilde{f}) := \frac{\zeta_F(2)}{\zeta_E(2)} \frac{L(1, \sigma, ad)}{L(1/2, \sigma)} \int_{F^\times \backslash B^\times} (\pi(b)f, \widetilde{f}) db^\times.$$

Define a bilinear form on $\mathscr{L}(\pi) \otimes \mathscr{L}(\widetilde{\pi})$ by

$$\langle \ell, \widetilde{\ell} \rangle = \frac{\ell \otimes \widetilde{\ell}}{m} \in \mathbb{C}.$$

If π is tempered and unitary then this pairing induces a positive hermitian form on $\mathscr{L}(\pi)$.

1.2 Global linear forms and Ichino's formula

Let F be a number field with ring of adeles $\mathbb{A}$ and E a cubic semisimple algebra over F. More precisely, E is one of the three forms: $F \oplus F \oplus F$, $F \oplus K$ for a quadratic extension K of F, and a cubic extension E of F. Let σ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$. In [30], Piatetski-Shapiro and Rallis defined an eight dimensional representation r_8 of the L-group of the algebraic group $Res_F^E GL_2$. Thus we have a Langlands L-series $L(s, \sigma, r_8)$. When $E = F \oplus F \oplus F$ and $\sigma = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$, this L-function is the Rankin type triple product L-function. When E is a field, the L-function $L(s, \sigma, r_8)$ is the Asai L-function of σ for the cubic extension E/F. Without assuming of confusion, we will simply denote the L-function by $L(s, \sigma)$.

Assume that the central character ω of σ is trivial when restricted to $\mathbb{A}^\times$

$$\omega|_{\mathbb{A}^\times} = 1.$$

Then the σ is self-dual and we have a functional equation for the Rankin L-series $L(s, \sigma)$

$$L(s, \sigma) = \epsilon(s, \sigma) L(1 - s, \sigma).$$

And the global root number $\epsilon(1/2, \sigma) = \pm 1$. For a fixed non-trivial additive character ψ of $F \backslash \mathbb{A}$, we have a decomposition

$$\epsilon(s, \sigma, \psi) = \prod \epsilon(s, \sigma_v, \psi_v).$$

The local root number $\epsilon(1/2, \sigma_v, \psi_v) = \pm 1$ does not depend on the choice of ψ_v. Thus we have a well-defined set of places of F:

$$\Sigma = \{v : \qquad \epsilon(1/2, \sigma_v, \psi_v) = -1.\}$$

Let $\mathbb{B}$ be a quaternion algebra over $\mathbb{A}$ which is obtained from $M_2(\mathbb{A})$ with $M_2(F_v)$ replaced by D_v if $\epsilon(1/2, \sigma_v, \psi_v) = -1$, and let π be the admissible representation of $\mathbb{B}_E^\times$ which is obtained from σ with σ_v replaced by σ'_v if $v \in \Sigma$. Define

$$\mathscr{L}(\pi) := \mathrm{Hom}_{\mathbb{B}^\times}(\pi, \mathbb{C}).$$

Then we have

$$\dim \mathscr{L}(\pi) = \dim \otimes_v \mathscr{L}(\pi_v).$$

Fix a Haar measure $db = \otimes db_v$ on $\mathbb{A}^\times \backslash \mathbb{B}^\times$ then local pairing with respect to dg_v:

$$\langle \cdot, \cdot \rangle_v : \qquad \mathscr{L}(\pi_v) \times \mathscr{L}(\widetilde{\pi}_v) \longrightarrow \mathbb{C}$$

defines a global pairing

$$\langle \cdot, \cdot \rangle = \otimes_v \langle \cdot, \cdot \rangle_v : \qquad \mathscr{L}(\pi) \otimes \mathscr{L}(\widetilde{\pi}) \longrightarrow \mathbb{C}.$$

The pairing here depends on the choice of decomposition $db = \otimes db_v$.

Assume that the global root number is 1. Then $|\Sigma|$ is even. In this case, $\mathbb{B}$ is the base change $B_{\mathbb{A}}$ of a quaternion algebra B over F, and π is an irreducible cuspidal automorphic representation of $\mathbb{B}_E^\times$. Thus we may view elements in π and $\widetilde{\pi}$ as functions on $B_E^\times \backslash \mathbb{B}_E^\times$ with duality given by Tamagawa measures. As the central characters of π (resp. $\widetilde{\pi}$) is trivial when restricted to $\mathbb{A}^\times$, we can define an element $\ell_\pi \in \mathscr{L}(\pi)$ by periods integral:

$$\ell_\pi(f) := \int_{Z(\mathbb{A}) B^\times \backslash \mathbb{B}^\times} f(b) db.$$

Here the Haar measure is normalized as Tamagawa measure. Jacquet's conjecture says that $\ell_\pi \neq 0$ if and only if $L(1/2, \sigma) \neq 0$. this conjecture has been proved by Harris and Kudla [13]. A refinement of Jacquet's conjecture is the following formula due to Ichino:

Theorem 1.2.1 (Ichino [14]).

$$\langle \ell_\pi, \ell_{\widetilde{\pi}} \rangle = \frac{1}{2^c} \frac{\zeta_E(2)}{\zeta_F(2)} \frac{L(1/2, \sigma)}{L(1, \sigma, ad)}.$$

Here the constant c is $3, 2,$ and 1 respectively if $E = F \oplus F \oplus F$, $E = F \oplus K$ for a quadratic K, and a cubic field extension E of F respectively.

1.3 Derivative formula in term of Gross–Schoen cycles

Now we assume that the global root number

$$\epsilon(1/2, \sigma) = -1.$$

Then Σ is odd, the central value $L(\frac{1}{2}, \sigma) = 0$ as forced by the functional equation, and we are led to consider the first derivative $L'(\frac{1}{2}, \sigma)$. In this case π is no longer an automorphic representation and the construction in the previous subsection is not available. Instead, heights of certain cohomologically trivial cycles will provide an invariant linear form as now we describe.

We will need to impose certain constraints as follows:

1. F is a totally real field.

2. $E = F \oplus F \oplus F$ is split. We may thus write $\sigma = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$ where each σ_i is a cuspidal automorphic representation of $GL_2(\mathbb{A})$. In this case, the condition on the central character of σ can be rewritten as

$$\omega_1 \cdot \omega_2 \cdot \omega_3 = 1.$$

3. For $i = 1, 2, 3$ and $v|\infty$, all $\sigma_{i,v}$ are discrete of weight 2. It follows that the odd set Σ must contain all archimedean places.

For any open compact subgroup U of $\mathbb{B}_f^\times$, we have a Shimura curve Y_U defined over F such that for any archimedean place τ, we have the usual uniformization as follows. Let $B = B(\tau)$ be a quaternion algebra over F with ramification set $\Sigma(\tau) := \Sigma \setminus \{\tau\}$ which acts on Poincaré double half plane $\mathscr{H}^\pm = \mathbb{C} \setminus \mathbb{R}$ by fixing an isomorphism $B \otimes_\tau \mathbb{R} = M_2(\mathbb{R})$. Then we have the following identification of analytic space at τ:

$$Y_{U,\tau}^{\mathrm{an}} = B^\times \backslash \mathscr{H}^\pm \times \mathbb{B}_f^\times / U.$$

We also have a similar unformization as a rigid space at a finite place in Σ using Drinfeld's upper half plane.

When U various, the curves Y_U form a projective system with an action of $\mathbb{B}^\times$. We write Y for the projective limit, and let $\mathbb{B}^\times$ act on Y so that $\mathbb{B}_\infty^\times$ trivially on Y. Then it is easy to see that the subgroup of $\mathbb{B}^\times$ acting trivially on Y is $D := \mathbb{B}_\infty^\times \cdot \overline{F^\times}$ where $\overline{F^\times}$ is the topological closure of $F^\times$ in $\mathbb{A}^\times/F_\infty^\times$. Notice that Y still has a scheme structure over F but it is not of finite type. Define the space of homologically trivial cycles on $X := Y^3$ by direct limit:

$$\mathrm{Ch}^2(X)^{00} := \varinjlim_{U_E=U^3} \mathrm{Ch}^2(X_{U_E})^{00}.$$

The height pairing on each level X_{U_E} defines a pairing on their limit which is positive definite by the index conjecture. Define the Mordell–Weil group of π by

$$\mathrm{MW}(\pi) := \mathrm{Hom}_{\mathbb{B}^\times}(\pi, \mathrm{Ch}^2(X)^{00}).$$

We have an induced height pairing

$$\langle \cdot, \cdot \rangle : \quad \mathrm{MW}(\pi) \otimes \mathrm{MW}(\widetilde{\pi}) \longrightarrow \mathbb{C}.$$

This pairing is conjecturally positive definite.

A refinement of the Birch and Swinneron-Dyer or Beilison–Bloch conjectures says that $\mathrm{MW}(\pi)$ is finite dimensional and

$$\dim \mathrm{MW}(\pi) = \mathrm{ord}_{s=1/2} L(s, \pi).$$

Theorem 1.3.1. *For each π as above, there is an explicitly constructed element*

$$y_\pi \in \mathscr{L}(\pi) \otimes \mathrm{MW}(\widetilde{\pi})$$

such that

$$\langle y_\pi, y_{\widetilde{\pi}} \rangle = \frac{\zeta_F(2)^2 L'(1/2, \sigma)}{2L(1, \sigma, ad)}$$

Here the paring on the left hand side is the product of pairings on $\mathscr{L}(\pi)$ and $\mathrm{MW}(\pi)$.

There are some consequences:

1. $L'(1/2, \sigma) = 0$ if and only if it is zero for all conjugates of σ;

2. assume that σ is unitary, then we take $\widetilde{f} = \bar{f}$. The Hodge index conjecture implies $L'(1/2, \sigma) \geq 0$. This is an consequence of the Riemann Hypothesis.

In the following we give a construction of y_π. For each U, let $Y_{U,\xi}$ be the Gross–Schoen cycle on Y_U^3 which is obtained form the diagonal cycle by some modification with respect to the Hodge class ξ_U on Y_U as constructed in [12] and [36]. Such a class form a projective system and thus defined a class in

$$Y_\xi \in \mathrm{Ch}_1(X)^{00} := \varprojlim_{U} \mathrm{Ch}^2(X_{U_E})^{00}$$

where the supscript 00 means cycles with trivial projections $Y_U \times Y_U$.

Let $\phi \in C_0^\infty(\mathbb{B}_E^\times/D_E)$ where $D_E = D^3$, then we can define correspondence

$$Z(\phi) \in \mathrm{Ch}^3(X \times X) := \varinjlim_{U_E} \mathrm{Ch}^3(X_{U_E} \times X_{U_E}).$$

Thus we can define a co-cycle

$$Z(\phi)Y_\xi \in \mathrm{Ch}^2(X)^{00} := \varinjlim_{U_E} \mathrm{Ch}^2(X_{U_E})^{00}.$$

In this way, we obtain homomorphism

$$y \in \mathrm{Hom}_{\mathbb{B}^\times \times \mathbb{B}_E^\times}(C_0^\infty(\mathbb{B}_E^\times/D_E), \quad \mathrm{Ch}^2(X)^{00}).$$

Here $\mathbb{B}^\times \times \mathbb{B}_E^\times$ acts on $C_0^\infty(\mathbb{B}_E^\times/D_E)$ by left and right translation, on $\mathrm{Ch}^2(X)^{00}$ by projection to the natural action of second factor $\mathbb{B}_E^\times$.

It has been proved by Gross–Schoen that this homomorphism factors though the action on weight 2 form i.e., the quotient:

$$C_0^\infty(\mathbb{B}_E^\times/D_E) \longrightarrow \oplus \pi \otimes \widetilde{\pi}$$

where π runs through the set of Jacquet–Langlands correspondences of cusp forms of parallel weight 2. Here the projection is induced by usual action $\rho(f)$ on σ with respect to a Haar measure $\mathbb{B}_E^\times/D_E = (\mathbb{B}^\times/D)^3$ normalized so that for an open compact subgroup U_E of $\mathbb{B}_E^\times/D_E$, $-\mathrm{vol}(U)^{-1}$ is the Euler characteristic of X_{U_E} when U_E is sufficiently small. In this way, we have constructed for each σ a pairing

$$y_\pi \in \mathrm{Hom}_{\mathbb{B}^\times \times \mathbb{B}_E^\times}(\pi \times \widetilde{\pi}, \mathrm{Ch}^2(X)^{00}) = \mathscr{L}(\pi) \otimes \mathrm{MW}(\widetilde{\pi}).$$

1.4 Symmetric and exterior products

Let $\mathscr{S}_3$ be the permutation group of three letters. Then $\mathscr{S}_3$ acts on X and induces an action on $\mathrm{Ch}^2(X)^{00}$ and on $\mathbb{B}_E^\times = (\mathbb{B}^\times)^3$ as usual. For a representation π of $\mathbb{B}_E^\times$, and $s \in \mathscr{S}_3$, we have a representation π^s and a linear isomorphism $\pi \longrightarrow \pi^s$ which we still denote a

$$\mathrm{MW}(\pi) \longrightarrow \mathrm{MW}(\pi^s): \qquad \ell^s(v) = \ell(v^{s^{-1}})^s.$$

Assume that $\pi_1 = \pi_2$. Then for s a permutation (12) we have $\pi^s = \pi$. Thus s gives an involution $W(\pi)$ and decompose them as a direct sum of $\pm$ eigen spaces. The Beilonson–Bloch conjecture in this case gives

$$\dim \mathrm{MW}(\pi)^- = \mathrm{ord}_{s=1/2} L(s, \mathrm{Sym}^2\sigma_1 \otimes \sigma_3),$$

and

$$\dim \mathrm{MW}(\pi)^+ = \mathrm{ord}_{s=1/2} L(s, \sigma_3 \otimes \omega_1)$$

where ω_1 is the central character of σ_1. The second equality is compatible with the usual BSD

$$\dim \mathrm{MW}(\pi_3 \otimes \omega_1) = \mathrm{ord}_{s=1/2} L(s, \sigma_3 \otimes \omega_1)$$

via an isomorphism

$$\mathrm{MW}(\pi)^+ \simeq \mathrm{MW}(\pi_3 \otimes \omega_1)$$

Here $\mathrm{MW}(\pi_3 \otimes \omega_1)$ is the Mordell–Weil group for σ_3 which is define as

$$\mathrm{MW}(\pi_3 \otimes \omega_1) = \mathrm{Hom}_{\mathbb{B}^\times}(\pi_3 \otimes \omega_1, \mathrm{Ch}^1(Y)^0_{\mathbb{C}}).$$

The composition $\mathrm{MW}(\pi) \longrightarrow \mathrm{MW}(\pi)^+ \simeq \mathrm{MW}(\pi_3 \otimes \omega_1)$ is given as follows: define a correspondence between Y^3 and Y by

$$\alpha : Y^2 \longrightarrow Y^3 \times Y, \qquad (x, y) \mapsto (x, x, y) \times (y).$$

Then Composing with α_* defines a map

$$\mathrm{MW}(\pi) = \mathrm{Hom}_{\mathbb{B}_E^\times}(\pi, W_{\mathbb{C}}) \longrightarrow \mathrm{Hom}_{\mathbb{B}^\times}(\pi, \mathrm{Ch}^1(Y)^0_{\mathbb{C}}).$$

One can show the elements in the image factor through the canonical map

$$\pi_1 \otimes \pi_1 \longrightarrow \omega_1$$

and thus are in $\mathrm{MW}(\pi_3 \otimes \omega_1)$.

Since $\mathscr{L}(\widetilde{\pi}) := \mathrm{Hom}_{\mathbb{B}^\times}(\widetilde{\pi}, \mathbb{C})$ is one dimensional, it is given by a sign $\epsilon(s) = \pm 1$. By work of Prasad [28],

$$\epsilon(s) = \epsilon(\mathrm{Sym}^2 \sigma_1 \otimes \sigma_3).$$

Since $y_{\widetilde{\pi}}$ is invariant under s, we thus have

Corollary 1.4.1. *Assume that $\pi_1 = \pi_2$.*

1. *If $\epsilon(\mathrm{Sym}^2 \sigma_1 \otimes \sigma_3) = 1$, then $\epsilon(\omega_1 \otimes \sigma_3) = -1$ and*

$$y_\pi \in \mathscr{L}(\pi) \otimes W(\pi_3 \otimes \omega_1),$$

and

$$\langle y_\pi, y_{\widetilde{\pi}} \rangle = \frac{\zeta_F(2)^2 L(1/2, \mathrm{Sym}^2 \sigma_1 \otimes \sigma_3)}{2L(1, \sigma, ad)} L'(1/2, \omega_1 \otimes \sigma_3).$$

2. *If $\epsilon(\mathrm{Sym}^2 \sigma_1 \otimes \sigma_3) = -1$, then $\epsilon(\omega_1 \otimes \sigma_3) = 1$ and*

$$y_\pi \in \mathscr{L}(\pi) \otimes W(\widetilde{\pi})^-,$$

and

$$\langle y_\pi, y_{\widetilde{\pi}} \rangle = \frac{\zeta_F(2)^2 L(1/2, \omega_1 \otimes \sigma_3)}{2L(1, \sigma, ad)} L'(1/2, \mathrm{Sym}^2 \sigma_1 \otimes \sigma_3).$$

Finally we assume that $\pi_1 = \pi_2 = \pi_3$ with trivial central characters. Then $\mathscr{S}_3$ acts on π and then $\mathrm{MW}(\pi)$ and decompose it into subspaces according three irreducible representations of $\mathscr{S}_3$

$$\mathrm{MW}(\pi) = \mathrm{MW}(\pi)^+ \oplus \mathrm{MW}^- \oplus \mathrm{MW}(\pi)^0$$

where $\mathrm{MW}(\pi)^+$ is the space of invariants under $\mathscr{S}_3$, and $\mathrm{MW}(\pi)^-$ is the space where $\mathscr{S}_3$ acts as sign function, and $\mathrm{MW}(\pi)^0$ is the space where $\mathscr{S}_3$ is acts as a direct sum of the unique 2 dimensional representation. Then the Beilinson–Bloch conjecture gives

$$\dim \mathrm{MW}(\pi)^+ = 0$$

$$\dim \mathrm{MW}(\pi)^- = \mathrm{ord}_{s=1/2} L(s, \mathrm{Sym}^3 \sigma_1)$$

$$\dim \mathrm{MW}(\pi)^0 = 2\mathrm{ord}_{s=1/2} L(s, \sigma_1).$$

The action of $\mathscr{S}_3$ on $\mathscr{L}(\pi)$ is either trivial or given by sign function. By Prasad's theorem we have:

Corollary 1.4.2. *Assume that $\pi_1 = \pi_2 = \pi_3$.*

1. *If $\epsilon(\mathrm{Sym}^3 \sigma_1) = 1$ or $\epsilon(\sigma_1) = -1$, then*

$$y_\pi = 0$$

2. *If $\epsilon(\mathrm{Sym}^3) = -1$ and $\epsilon(\sigma_1) = 1$, then*

$$y_\pi \in \mathscr{L}(\pi)^- \otimes W(\widetilde{\pi})^-,$$

and

$$\langle y_\pi, y_{\widetilde{\pi}} \rangle = \frac{\zeta_F(2)^2 L(1/2, \sigma_1)^2}{2L(1, \sigma, ad)} L'(1/2, \mathrm{Sym}^3 \sigma_1).$$

1.5 Strategy of proof

The strategy of proof of the height formula will be analogous in spirit to the proof of Gross-Zagier formula. Basically it contains the analytic and geometric sides and the comparison between them.

Assume that Σ is odd. First we will describe the analytic side, i.e., the analytic kernal function for the central derivative. Though we want to focus on our case under the assumption in the lase section, the same argument for the analytic kernel also works in all cases (i.e., Σ is arbitrary, F is any number field and E is any cubic semisimple F-algebra.)

By the work of Garrett and Piatetski-Shapiro-Rallis, we have an integral representation of $L(s, \sigma)$ using pull-back of Siegel-Eisenstein series. More precisely, let $\mathbb{B}$ be the unique (incoherent) quaternion algebra over $\mathbb{A}$ ramified at exactly at Σ. One can associate to $\phi \in \mathscr{S}(\mathbb{B}_E)$ the Siegel-Eisenstein series $E(s, g, \phi)$. Due to the incoherence, $E(s, g, \phi)$ vanishes at $s = 0$. We obtain an integral representation

$$\int_{[\mathbb{G}]} E'(g, 0, \Phi)\varphi(g)dg = \frac{L'(\frac{1}{2}, \sigma, r_8)}{\zeta_F^2(2)} \prod_v \alpha(\Phi_v, W_{\varphi_v}). \tag{1.5.1}$$

where $\alpha(\Phi_v, W_{\varphi_v})$ is the normalized local zeta integral

$$\alpha(\Phi_v, W_{\varphi_v}) = \frac{\zeta_{F,v}^2(2)}{L(\frac{1}{2}, \sigma_v, r_8)} \int_{F^\times N_0 \backslash \mathbb{G}} \Phi(\eta g) W_{\varphi_v}(g) dg. \tag{1.5.2}$$

By the local theta lifting we have a map

$$\theta_v : \sigma_v \otimes r_v \to \pi_v \otimes \widetilde{\pi}_v \tag{1.5.3}$$

which can be normalized such that if $\theta_v(\phi_v, \varphi_v) = f_v \otimes \widetilde{f}_v$, we have

$$\int_{U(F_v)\backslash \mathrm{SL}_2(F_v)} r(g)\phi_v(1) W_{\varphi_v}(g) dg = (f_v, \widetilde{f}_v).$$

Under this normalization (and certain choice of measures), we have

$$\alpha(\Phi_v, W_{\varphi_v}) = (-1)^{sign(B_v)} \frac{\zeta_{F_v}(2)}{L(\frac{1}{2}, \sigma)} \int_{F_v^\times \backslash B_v^\times} (\pi_v(b) f_v, \widetilde{f}_v) db^\times.$$

In this method we obtain $E'(g, 0, \Phi)$ as a kernel function. This kind of Siegel-Eisenstein series has been studied extensively. In particular, its first derivative was firstly studied by Kudla in [19]. It is natural to consider its Fourier expansion:

$$E'(g, 0, \Phi) = \sum_{T \in \mathrm{Sym}_3(F)} E'_T(g, 0, \Phi).$$

For nonsingular $T \in \mathrm{Sym}_3(F)$, we have an Euler expansion as a product of local Whittaker functions (for $Re(s) \gg 0$)

$$E_T(g, s, \Phi) = \prod_v W_{T,v}(g, s, \phi).$$

It is known that the Whittaker functional $W_{T,v}(g, s, \phi)$ can be extended to an entire function on the complex plane for the s-variable and that $W_{T,v}(g, 0, \phi)$ vanishes if T cannot be represented as moment matrix of three vectors in the quadratic space B_v. This motivates the following definition. For $T \in \mathrm{Sym}_3(F)_{reg}$ (here "reg" meaning that T is regular), let $\Sigma(T)$ be the set of places over which T is anisotropic. Then $\Sigma(T)$ has even cardinality and the vanishing order of $E_T(g, s, \phi)$ at s is at least

$$|\{v : T \text{ is not representable in } B_v\}| = |\Sigma \cup \Sigma(T)| - |\Sigma \cap \Sigma(T)|.$$

Since $|\Sigma|$ is odd, $E_T(g, s, \phi)$ always vanishes at $s = 0$. And its derivative is non-vanishing only if Σ and $\Sigma(T)$ is nearby: they differ by precisely one place v, i.e., only if $\Sigma(T) = \Sigma(v)$ with

$$\Sigma(v) = \begin{cases} \Sigma \setminus \{v\} & \text{if } v \in \Sigma \\ \Sigma \cup \{v\} & \text{otherwise} \end{cases}$$

Moreover when $\Sigma(T) = \Sigma(v)$, the derivative is given by

$$E'_T(g, 0, \phi) = \prod_{w \neq v} W_{T,w}(g_w, 0, \phi_w) \cdot W'_{T,v}(g_v, 0, \phi_v).$$

We thus obtain a decomposition of $E'(g, 0, \phi)$ according to the difference of Σ_T and Σ:

$$E'(g, 0, \phi) = \sum_v E'_v(g, 0, \phi) + E'_{sing}(g, 0, \phi) \tag{1.5.4}$$

where

$$E'_v(g, 0, \phi) = \sum_{\Sigma_T = \Sigma(v)} E'_T(g, 0, \phi) \tag{1.5.5}$$

and

$$E'_{sing}(g, 0, \phi) = \sum_{T, det(T)=0} E'_T(g, 0, \phi).$$

Moreover, the local Whittaker functional $W'_{T,v}(g, 0, \phi_v)$ is closely related to the evaluation of local density. In the spherical case (i.e., $B_v = M_2(F_V)$ is split, ψ_v is unramified, ϕ_v is the characteristic function of the maximal lattice $M_2(\mathscr{O}_v)^3$), $W'_{T,v}(g, 0, \phi_v)$ has essentially been calculated by Katsurada ([17]).

Now two difficulties arise:

1. The vanishing of singular Fourier coefficients (parameterized by singular $T \in \mathrm{Sym}_3(F)$) are not implied by local reason. Hence it is hard to evaluate the first derivative E'_T for singular T.

2. The explicit calculation of $W'_{T,v}(e, 0, \phi_v)$ for a general ϕ_v seems to be extremely complicated.

The solution is to utilize the uniqueness of linear form (note that we have a lot of freedom to choose appropriate ϕ) and to focus on certain very special ϕ_v. More precisely, define the open subset $B^3_{v,reg}$ of B^3_v to be all $x \in B^3_v$ such that the components of x generates a non-degenerate subspace of B_v of dimension 3. Then we can prove

1. If ϕ_v is supported on $B^3_{v,reg}$ for $v \in S$ where S contains at least two finite places, then for singular T and $g \in \mathbb{G}(\mathbb{A}^S)$, we have

$$E'_T(g, v, \phi) = 0.$$

2. If the test function ϕ_v is "regular at a sufficiently higher order" (see Definition 2.7.1), we have for all non-singular T with $\Sigma_T = \Sigma(v)$ and $g \in \mathbb{G}(\mathbb{A}^v)$:

$$E'_T(g, 0, \phi) = 0.$$

To conclude the discussion of analytic kernel function, we choose ϕ_v to be a test function "regular at a sufficiently higher order" for $v \in S$ where S is a set of finite places with at least two elements such that any finite place outside S is spherical. And we always choose the standard Gaussian at all archimedean places. Then for $g \in \mathbb{G}(\mathbb{A}^S)$, we have

$$E'(g,0,\phi) = \sum_v E'_v(g,0,\phi) \tag{1.5.6}$$

where the sum runs over v outside S and

$$E'_v(g,0,\phi) = \sum_{T, \Sigma(T)=\Sigma(v)} E'_T(g,0,\phi) \tag{1.5.7}$$

where the sum runs over nonsingular T.

Moreover, we can have a decomposition of its holomorphic projection, denoted by $E'(g,0,\phi)_{hol}$. And it has a decomposition

$$E'(g,0,\phi)_{hol} = \sum_v \sum_{T, \Sigma(T)=\Sigma(v)} E'_T(g,0,\phi)_{hol} \tag{1.5.8}$$

where we only change $E'_T(g,0,\phi)$ to $E'_T(g,0,\phi)_{hol}$ when $\Sigma(T) = \Sigma(v)$ for v an archimedean place. So similarly we may define $E'_v(g,0,\phi)_{hol}$.

This yields an analytic kernel function of the central derivative $L'(\frac{1}{2},\sigma)$ for all three possibilities of the cubic algebra E.

Now we describe the geometric kernel function under the further assumptions appeared in the beginning of the last subsection. The construction of geometric kernel function is similar to that in the proof of Gross-Zagier formula. More precisely, for $\phi \in \mathscr{S}(\mathbb{B}_f)$ we can define a generating function of Hecke operators, denoted by $Z(\phi)$ (see Section 3). Such generating functions have appeared in Gross-Zagier's paper. Works of Kudla-Millson and Borcherds first relate it to the Weil representation. A little extension of our result ([33]) shows that $Z(\phi)$ is a modular form on $GL_2(\mathbb{A})$. Thus it is natural to consider the generating function for a triple $\phi = \otimes_i \phi_i \in \mathscr{S}(\mathbb{B}_f^3)$ fixed by U^6 for a compact open $U \subset B_f^\times$:

$$\deg Z(g,\phi) := \langle \Delta_{U,\xi}, Z(g,\phi)\Delta_{U,\xi}\rangle, \quad g \in \mathrm{GL}_2^3(\mathbb{A}).$$

Now the main ingredient of our proof is the following weak form of an arithmetic Siegel–Weil formula:

$$E'(g,0,\phi) \equiv -2\deg Z(g,\phi), \quad g \in \mathbb{G}(\mathbb{A})$$

where "$\equiv$" means modulo all forms on $\mathbb{G}(\mathbb{A})$ that is perpendicular to σ. Note that this is parallel to the classical Siegel–Weil formula in the coherent case

$$E(g,0,\phi) = 2I(g,\phi).$$

The replacement of "$=$" by "$\equiv$" should be necessary due to representation theory reason.

It follows that we have a decomposition to a sum of local heights:

$$\deg Z(g,\phi) \equiv -\sum_v (\overline{\Delta_{U,\xi}} \cdot \overline{Z(g,\phi)\Delta_{U,\xi}})_v$$

where the intersection takes place on certain "good" model of $Y_U = X_U^3$ as described in [12] and $\overline{\Delta}_{U,\xi}$ is certain extension to the good model. Here we may write the extension as a sum of the Zariski closure and a vertical cycle

$$\overline{\Delta}_{U,\xi} = \widetilde{\Delta}_{U,\xi} + V.$$

The Shimura curve X_U has a natural integral model $\mathscr{X}_U$ over $\mathscr{O}$ and when U_v is maximal, this model $\mathscr{X}_U$ has good reduction at v if $v \notin \Sigma$. In later case, one may take the good model of $Y_{U,v}$ to be the product of $\mathscr{X}_{U,v}^3$ and simply take the Zariski closure as the extension $\overline{\Delta}_{U,\xi}$.

Among the summand in $\overline{\Delta}_{U,\xi}$, one has the Zariski closure of the main diagonal $\Delta_{\{1,2,3\}}$. We simply denote it by $\widetilde{\Delta}$. At v a finite good place, the work of Gross-Keating ([10]) essentially implies that for $g \in \mathbb{G}(\mathbb{A}^S)$:

$$(\overline{\Delta} \cdot Z(g,\phi)\overline{\Delta}) = E'_v(g,0,\phi).$$

And when $v|\infty$, using the complex uniformization we may construct the Green current. And we prove that the contribution from the main diagonal to the archimedean height in the intersection is equal to $E'_v(g,0,\phi)_{hol}$ (1.5.8).

Finally, we need to deal with the partial diagonal and the contribution from the vertical cycle.

1.6 Notations

Some groups: $H = \mathrm{GSpin}(V)$, $\mathbb{G} = \mathrm{GL}^{\circ}_{2,E}$... We will use measures normalized as follows. We first fix a non-trivial additive character $\psi = \otimes_v \psi_v$ of $F\backslash\mathbb{A}$. Then we will take the self-dual measure dx_v on F_v with respect to ψ_v and take the product measure on $\mathbb{A}$. We will take the product measure on $F_v^\times$ as $d^\times x_v = \zeta_{F_v}(1)|x_v|^{-1}dx_v$. Similarly, the measure on B_v and $B_v^\times$ are the self-dual measure dx_v with respect to the character $\psi_v(tr(xy^\iota))$ and $d^\times x_v = \zeta_{F_v}(1)\zeta_{F_v}(2)|\nu(x_v)|^{-2}dx_v$.

2 Analytic kernel functions

In this section, we will review Weil representation and apply it to triple product L-series. We will follow work of Garret, Piateski-Shapiro–Rallis, Waldspurger, Harris–Kudla, Prasad, and Ichino etc. The first main result is Theorem 2.3.1 about integral representation of the triple product L-series using Eisenstein series from the Weil representation on an adelic quaternion algebra.

When the sign of the functional equation is $+1$, then the adelic quaternion algebra is coherent in the sense that it comes form a quaternion algebra over number field, then our main result is the special value formula Theorem 2.4.3.

When the sign is -1, then the quaternion algebra is *incoherent*, and the derivative of the Eisenstein series is the kernel function for the derivative of L-series, see formula (2.3.7). We study the derivative of Eisenstein series for Schwartz function $\phi \in \mathscr{S}(\mathbb{B}^3)$ on an incoherent (adelic) quaternion algebra $\mathbb{B}$ over adeles $\mathbb{A}$ of a number field F. We will first study the non-singular Fourier coefficients T. We show that these coefficients are non-vanishing only if T is represented by elements in $\mathbb{B}$ if we remove one factor at a place v, see formula (3.1.2). In this case, the fourier coefficient can be computed by taking derivative at the local Whittacker functions at v, see Proposition 3.2.2.

Our last result is that the singular Fourier coefficients vanish if the Schwarts function are supported on regular sets for two places of F.

2.1 Weil representation and theta liftings

In this subsection, we will review the Weil representation as its its extension to similitudes by Harris and Kudla, and normalized Shimuzu lifting by Waldspurger.

Extending Weil representation to similitudes

Let F be a local filed. Let n be a positive integer and let Sp_{2n} be the symplectic group with the standard alternating form $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ on F^{2n}. With the standard polarization $F^{2n} = F^n \oplus F^n$, we have two subgroups of Sp_{2n}:

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \middle| a \in GL_n(F) \right\}$$

and

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in Sym_n(F) \right\}.$$

Note that M, N and J generate the symplectic group Sp_{2n}.

Let $(V, (\cdot,\cdot))$ be a non-degenerate quadratic space of even dimension m. Associated to V there is a character χ_V of $F^\times/F^{\times,2}$ defined by

$$\chi_V(a) = (a, (-1)^{m/2} det(V))_F$$

where $(\cdot,\cdot)_F$ is the Hilbert symbol of F and $\det(V) \in F^\times/F^{\times,2}$ is the determinant of the moment matrix $Q(\{x_i\}) = ((x_i, x_j))$ of any basis $x_1, ..., x_m$ of V. Let $\mathrm{O}(V)$ be the orthogonal group.

Let $\mathscr{S}(V^n)$ be the space of Bruhat-Schwartz functions on $V^n = V \otimes F^n$ (for archimedean F, functions corresponding to polynomials in the Fock model). Then the Weil representation $r = r_\psi$ of $Sp_{2n} \times O(V)$ can be realized on $\mathscr{S}(V^n)$ by the following formulae:

$$r(m(a))\phi(x) = \chi_V(\det(a))|det(a)|_F^{\frac{m}{2}}\phi(xa),$$

$$r(n(b))\phi(x) = \psi(tr(bQ(x)))\phi(x)$$

and

$$r(J)\phi(x) = \gamma\widehat{\phi}(x)$$

where γ is an eighth root of unity and $\widehat{\phi}$ is the Fourier transformation of ϕ:

$$\widehat{\phi}(x) = \int_{F^n} \phi(y)\psi(\sum_i x_i y_i)dy$$

for $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$.

Now we want to extend r to representations of groups of similitudes. Let GSp_{2n} and $\mathrm{GO}(V)$ be groups of similitudes with similitude homomorphism ν (to save notations, ν will be used for both groups). Consider a subgroup $R = G(\mathrm{Sp}_{2n} \times \mathrm{O}(V))$ of $\mathrm{GSp}_{2n} \times \mathrm{GO}(V)$

$$R = \{(g, h) \in \mathrm{GSp}_{2n} \times \mathrm{GO}(V) | \nu(g) = \nu(h)\}.$$

Then we can identify $\mathrm{GO}(V)$ (resp., Sp_{2n}) as a subgroup of R consisting of $(d(\nu(h)), h)$ where

$$d(\nu) = \begin{pmatrix} 1_n & 0 \\ 0 & \nu \cdot 1_n \end{pmatrix}$$

(resp. $(g, 1)$). We then have isomorphisms

$$R/\mathrm{Sp}_{2n} \simeq \mathrm{GO}(V), \quad R/O(V) \simeq \mathrm{GSp}_{2n}^+$$

where GSp_{2n}^+ is the subgroup of GSp_{2n} with similitudes in $\nu(\mathrm{GO}(V))$.

We then extend r to a representation of R as follows: for $(g, h) \in R$ and $\phi \in \mathscr{S}(V^n)$,

$$r((g, h))\phi = L(h)r(d(\nu(g)^{-1})g)\phi$$

where

$$L(h)\phi(x) = |\nu(h)|_F^{-n}\phi(h^{-1}x).$$

For F a number field, we patch every local representation to obtain representations of adelic groups.

Local theta lifting

In this subsection, we consider the case when $n = 1$ and V is the quadratic space attached to a quaternion algebra B with its reduced norm. Note that $\mathrm{Sp}_2 = \mathrm{SL}_2$ and $\mathrm{GSp}_2 = \mathrm{GL}_2$. And $\mathrm{GL}_2^+(F) = \mathrm{GL}_2(F)$ unless $F = \mathbb{R}$ and B is the Hamilton quaternion in which case $\mathrm{GL}_2^+(\mathbb{R})$ is the subgroup of $\mathrm{GL}_2(\mathbb{R})$ with positive determinants.

For an infinite-dimensional representation σ of $\mathrm{GL}_2(F)$, let π be the representation of $B^\times$ associated by Jacquet-Langlands correspondence and let $\widetilde{\pi}$ be the contragredient of π. Note that we set $\pi = \sigma$ when $B = M_{2\times 2}$.

We have natural isomorphisms between various groups:

$$1 \to \mathbb{G}_m \to B^\times \times B^\times \to \mathrm{GSO}(V) \to 1$$

where $(b_1, b_2) \in B^\times \times B^\times$ acts on B via $(b_1, b_2)x = b_1 x b_2^{-1}$,

$$\mathrm{GO}(V) = \mathrm{GSO}(V) \rtimes \{1, c\}$$

where c acts on B via the canonical involution $c(x) = x^\iota$ and acts on $\mathrm{GSO}(V)$ via $c(b_1, b_2) = (b_2^\iota, b_1^\iota)^{-1}$, and

$$R' = \{(h, g)) \in \mathrm{GSO}(V) \times \mathrm{GL}_2 | \nu(g) = \nu(h)\}.$$

Proposition 2.1.1 (Shimizu liftings)**.** *There exists an* $\mathrm{GSO}(V) \simeq R'/\mathrm{SL}_2$*-equivariant isomorphism*

$$(\sigma \otimes r)_{\mathrm{SL}_2} \simeq \pi \otimes \widetilde{\pi}. \tag{2.1.1}$$

Proof. Note that this is stronger than the usual Howe's duality in the present setting. The result essentially follows from results on Jacquet-Langlands correspondence. Here we explain why we can replace $\mathrm{GO}(V)$ by $\mathrm{GSO}(V)$. In fact, there are exactly two ways to extend an irreducible representation of $\mathrm{GSO}(V)$ to $\mathrm{GO}(V)$. But only one of them can participate the theta correspondence due to essentially the fact that the *sign* of $\mathrm{GO}(V)$ does not occur in the theta correspondence unless $\dim V \leq 2$. □

We thus denote by θ the R'-equivariant map

$$\theta : \sigma \otimes r \to \pi \otimes \widetilde{\pi}. \tag{2.1.2}$$

Let $\mathscr{W}_\sigma = \mathscr{W}_\sigma^\psi$ be the ψ-Whittaker model of σ and let W_φ be a Whittaker function corresponding to φ. Define

$$\begin{aligned} &\mathscr{S}(V) \otimes \mathscr{W}_\sigma \to \mathbb{C} \\ &\qquad (\phi, W) \mapsto S(\phi, W) = \frac{\zeta(2)}{L(1, \sigma, ad)} \int_{N(F)\backslash SL_2(F)} r(g)\phi(1)W(g)dg. \end{aligned}$$

The integral is absolutely convergent by Lemma 5 of [32] and defines an element in

$$\mathrm{Hom}_{SL_2 \times B^\times}(r \otimes \sigma, \mathbb{C})$$

where $B^\times$ is diagonally embedded into $B^\times \times B^\times$. The factor before the integral is chosen so that $S(\phi, W) = 1$ when everything is unramified. Since

$$\mathrm{Hom}_{\mathrm{SL}_2\times B^\times}(r\otimes\sigma, \mathbb{C}) \simeq \mathrm{Hom}_{B^\times}((r\otimes\sigma)_{\mathrm{SL}_2}, \mathbb{C}) \simeq \mathrm{Hom}_{B^\times}(\pi\otimes\widetilde{\pi}, \mathbb{C})$$

and the last space is of one dimensional spanned by the canonical $B^\times$-invariant pairing between π and its (smooth) dual space $\widetilde{\pi}$, we can normalize the map θ such that

$$S(\phi, W) = (f_1, f_2)$$

where $f_1 \otimes f_2 = \theta(\phi\otimes W)$.

In the global situation where B is a quaternion algebra defined over a number field, then we can define the global theta lifting by

$$\theta(\phi\otimes\varphi)(h) = \frac{\zeta(2)}{2L(1,\sigma,ad)}\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})}\varphi(g_1g)\theta(g_1g,h,\phi)dg_1, \qquad (h,g)\in R'(\mathbb{A}).$$

Proposition 2.1.1. *With definition as above, we have a decomposition $\theta = \otimes\theta_v$ in*

$$\mathrm{Hom}_{R'(\mathbb{A})}(r\otimes\sigma, \pi\otimes\widetilde{\pi}).$$

Proof. It suffices to prove the identity after composing with the pairing on $\pi\otimes\widetilde{\pi}$. More precisely, let $f_1, f_2 \in \pi$ be an element in a cuspidal representation of $B_{\mathbb{A}}^\times$, and $\phi\in\mathscr{S}(V_{\mathbb{A}})$ and $\varphi\in\sigma$ so that

$$f_1\otimes f_2 = \theta(\phi\otimes\varphi).$$

Assume everything is decomposable, we want to compute the product (f_1, f_2) in terms of local terms in

$$\phi\otimes\varphi = \otimes\phi_v\otimes\varphi_v \in r\otimes\sigma = \otimes(r_v\otimes\sigma_v).$$

By definition,

$$(f_1, f_2) = \int_{[B^\times]} f_1(h)f_2(h)dh = \int_{[B^\times]}\theta(\phi\otimes\varphi)(h)dh.$$

Notice that the diagonal embedding

$$B^\times \longrightarrow B^\times\times B^\times/\Delta(F^\times) \longrightarrow \mathrm{GO}(V)$$

is given by the conjugation of $B^\times$ on $V = B$. Let $V = V_1\oplus V_0$ with V_1 consisting of scale elements, and V_0 consisting of trace free elements. Then $B^\times/F^\times$ can be identified with $\mathrm{SO}(V_0)$. Assume that $\phi = \phi_1\otimes\phi_0$ with $\phi_i\in\mathscr{S}(V_i(\mathbb{A}))$. Then

$$\theta(g,h,\phi) = \theta(g,1,\phi_1)\theta(\widetilde{g},h,\phi_0), \qquad (g,h)\in\mathrm{SL}_2(\mathbb{A})\times\mathrm{SO}(V_0)(\mathbb{A}).$$

where $\widetilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$ lifts g. It follows that

$$\begin{aligned}(f_1, f_2) =& \frac{\zeta(2)}{2L(1, \sigma, ad)} \int_{[\mathrm{SO}(V_0)]} dh \int_{[\mathrm{SL}_2(F)]} \varphi(h)\theta(\widetilde{g}, 1, \phi_1)\theta(\widetilde{g}, h, \phi_0) dg \\ =& \frac{\zeta(2)}{2L(1, \sigma, ad)} \int_{[\mathrm{SL}_2]} \varphi(g)\theta(\widetilde{g}, 1, \phi_1) \left(\int_{[\mathrm{SO}(V_0)]} \theta(\widetilde{g}, h, \phi_0) dh \right) dg.\end{aligned}$$

By Siegel-Weil the last integration defines an Eisenstein series:

$$\int_{[\mathrm{SO}(V_0)]} \theta(\widetilde{g}, h, \phi_0) dh = 2 \sum_{\gamma \in P(F)\backslash \mathrm{SL}_2(F)} \omega(\gamma \widetilde{g})\phi_0(0)$$

where the measure on $[\mathrm{SO}(V_0)] = [B^\times]$ is taken as Tamagawas measure which has volume 2. Notice that the the sum on the right hand may not be convergent. To regularize this integral, we may insert $\delta(\gamma g)^s$ for $\mathrm{Re}(s) >> 0$ then take limit $s \longrightarrow 0$. We ignore this process and take formal computation. Bring this to our inner product to obtain:

$$\begin{aligned}(f_1, f_2) =& \frac{\zeta(2)}{2L(1, \sigma, ad)} \int_{[\mathrm{SL}_2]} \varphi(g)\theta(\widetilde{g}, 1, \phi_1) \sum_{\gamma \in P(F)\backslash \mathrm{SL}_2(F)} \omega(\gamma \widetilde{g})\phi_0(0) dg \\ =& \frac{\zeta(2)}{2L(1, \sigma, ad)} \int_{P(F)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(g)\theta(\widetilde{g}, \phi_1)\omega(\widetilde{g})\phi_0(0) dg \\ =& \frac{\zeta(2)}{2L(1, \sigma, ad)} \int_{P(F)N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \left(\int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\theta(n\widetilde{g}, \phi_1) dn \right) \omega(\widetilde{g})\phi_0(0) dg.\end{aligned}$$

Compute the inside integral, we need to compute $\theta_{\phi_1}(ng)$ for $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$:

$$\theta_{\phi_1}(ng) = \sum_{\xi \in F} \omega(n\widetilde{g})\phi_1(\xi) = \sum_{\xi} \psi(\mathrm{tr}(x\xi^2))\omega(\widetilde{g})\phi_1(\xi).$$

Thus the inner integral can be written as

$$\sum_{\xi \in F} \omega(\widetilde{g})\phi_1(\xi) \int_{F\backslash \mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(\mathrm{tr}(x\xi^2)) dx.$$

Since φ is a cusp form, the integral here vanishes if $x = 0$. If $x \neq 0$, it is given by $W\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ for W the whittacker function for φ for the character ψ. On the other hand for $\xi \neq 0$, $\omega(\widetilde{g})\phi(\xi) = \omega(\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}\widetilde{g})\phi_1(1)$. It follows that

$$\begin{aligned}(f_1, f_2) =& \frac{\zeta(2)}{L(1, \sigma, ad)} \int_{P(F)N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \sum_{\xi \in F^\times} \omega(\widetilde{g})\phi_1(\xi) W\left(\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} g \right) \omega(\widetilde{g})\phi_2(0) dg \\ =& \frac{\zeta(2)}{L(1, \sigma, ad)} \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} W(g)\omega(\widetilde{g})\phi_1(1)\omega(\widetilde{g})\phi_0(0) dg.\end{aligned}$$

$\square$

2.2 Trilinear form

In this subsection, we review tri-linear form following Garret, Piatetski-Shapiro and Rallis, Prasad and Loke, and Ichino.

Consider the symplectic form on the six-dimensional space E^2:

$$E^2 \otimes E^2 \xrightarrow{\wedge} E \xrightarrow{\mathrm{tr}} F$$

$$(x, y) \otimes (x', y') \mapsto \mathrm{tr}_{E/F}(xy' - yx').$$

where the first map is by taking wedge product and the second one is the trace map from E to F. Let GSp_6 be the group of similitudes relative to this symplectic form. In this way, we see that elements in $\mathrm{GL}_2(E)$ with determinants in $F^\times$ belong to GSp_6. So we define

$$\mathbb{G} = \left\{ g \in \mathrm{GL}_2(E) | \det(g) \in F^\times \right\}.$$

and identify it with a subgroup of GSp_6.

Let $I(s) = \mathrm{Ind}_P^{\mathrm{GSp}_6} \lambda_s$ be the degenerate principle series of GSp_6. Here, P is the Siegel parabolic subgroup:

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & \nu^t a^{-1} \end{pmatrix} \in \mathrm{GSp}_6 | a \in GL_F(E), \nu \in F^\times \right\}$$

and for $s \in \mathbb{C}$, λ_s is the character of P defined by

$$\lambda_s \left(\begin{pmatrix} a & * \\ 0 & \nu^t a^{-1} \end{pmatrix} \right) = |det(a)|_F^{2s} |\nu|_F^{-3s}.$$

For an irreducible admissible representation σ of $\mathbb{G}$, let $W_\sigma = W_\sigma^\psi$ be the ψ-Whittaker module of σ.

There is a $G(Sp_6 \times O(B_F))$-intertwining map

$$i: \quad \mathscr{S}(B_E) \to I(0) \tag{2.2.1}$$

$$\phi \mapsto \Phi(\cdot, 0)$$

where for $g \in GSp_6$,

$$\Phi(g, 0) = |\nu(g)|^{-3} r(d(\nu(g))^{-1} g) \phi(0).$$

Let $\Pi(B)$ be the image of the map (2.2.1). Similarly, for B', we can define $\Pi(B')$.

Lemma 2.2.1. *For nonarchimedean F,*

$$I(0) = \Pi(B) \oplus \Pi(B'). \tag{2.2.2}$$

Proof. See [13], section. 4, (4.4)-(4.7). Also cf. [18], II.1. □

Now we treat the case when F is archimedean.

If $F = \mathbb{C}$, then one has only one quaternion algebra B over F. In this case we have

$$I(0) = \Pi(B). \tag{2.2.3}$$

This is proved in Lemma A.1 of Appendix of [13].

If $F = \mathbb{R}$, then one has two quaternion algebras, $B = M_{2\times 2}$ and B' the Hamilton quaternion. The replacement of Lemma 2.2.1 is the following isomorphism ([13], (4.8))

$$I(0) = \Pi(B) \oplus \Pi(B') \tag{2.2.4}$$

where $\Pi(B') = \Pi(4,0) \oplus \Pi(0,4)$ where the two spaces are associated to the two quadratic spaces obtained by changing signs of the reduced norm on the Hamilton quaternion.

Local zeta integral of triple product

The local zeta integral of Garrett ([7]) and Piatetski-Shapiro and Rallis ([30]) is a (family of) linear functional on $I(s) \times W_\sigma$ defined as

$$Z(s, \Phi, W) = \int_{F^\times N_0 \backslash \mathbb{G}} \Phi_s(\eta g) W(g) dg, \quad (\Phi, W) \in I(s) \times W_\sigma.$$

Here, N_0 is a subgroup of $\mathbb{G}$ defined as

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} | b \in E, \mathrm{tr}_{E/F}(b) = 0 \right\},$$

and $\eta \in \mathrm{GSp}_6$ is a representative of the unique open orbit of $\mathbb{G}$ acting on $P\backslash GSp_6$. The integral is absolutely convergent for $Re(s) \gg 0$. And the integral $Z(0, \Phi, W)$ is absolutely convergent when the exponent $\Lambda(\sigma) < \frac{1}{2}$. This condition holds if σ is a local component of a cuspidal automorphic representation by the work of Kim-Shahidi ([24]).

Proposition 2.2.2. *For σ with $\Lambda(\sigma) < \frac{1}{2}$, $Z(0, \Phi, W)$ defines a non-vanishing linear functional on $I(0) \times W_\sigma$.*

Proof. See Proposition 3.3 [30] and [16], page 227 (cf. Proposition 2.1 in [13] and Lemma 2.1 in [29]). But we will reprove this later in the proof of Theorem 2.7.4. □

Local tri-linear forms

Let B, B' be the two inequivalent quaternion algebras over F. Define the *sign*(B) to be 0 (resp. 1) if B is split (resp. ramified). For a generic representation σ of $\mathrm{GL}_2(E)$ with central character $\omega_\sigma|_{F^\times} = 1$, let π_B (resp. π'_B) be the representation of $B_E^\times$ (resp. $B_E'^\times$) associated by Jacquet–Langlands correspondence. Here we set $\pi_B = 0$ (resp. $\pi'_B = 0$) if the Jacquet–Langlands correspondence does not exist. Define $m(\pi_B)$ to be the number

$$m(\pi_B) := \dim \mathrm{Hom}_{B^\times}(\pi_B, \mathbb{C}).$$

Theorem 2.2.3 (Prasad-Loke)**.** *We have the dichotomy*

$$m(\pi_B) + m(\pi_{B'}) = 1.$$

And the dichotomy is controlled by the root number

$$m(\pi_B) = 1 \Leftrightarrow \epsilon(\sigma, \frac{1}{2}) = (-1)^{sign(B)}.$$

Proof. Prasad proves this for non-archimedean place, and Loke for archimedean place. $\square$

Define the integral of matrix coefficient:

$$I(f_1, f_2) = \int_{F^\times \backslash B^\times} (\pi_B(b) f_1, f_2) db, \qquad f_1 \in \pi_B,\ f_2 \in \widetilde{\pi}_B. \tag{2.2.5}$$

Proposition 2.2.4 (Ichino, [14])**.** *Under the normalization of θ as in* **??***, we have*

$$Z(\phi, W_\varphi) = (-1)^{sign(B)} \zeta_F(2)^{-1} I(f_1, f_2)$$

where $f_1 \otimes f_2 = \theta(\phi \otimes \varphi)$.

Proof. This is Proposition 5.1 of [14]. $\square$

Proposition 2.2.5. *Assume that π is unitary. Write $I(f) = I(f, \bar{f})$ for $f \in \pi_B$.*

1. *One has the following positivity $I(f) \geq 0$.*
2. *Moreover, the following are equivalent:*
   (a) *$m(\pi_B) = 1$.*
   (b) *Z does not vanish on $\sigma \otimes \pi(B)$.*
   (c) *I does not vanish on π_B.*

Proof. The first one follows essentially from a theorem of He (cf. [15]). We need to prove the second one. Obviously, $c) \Rightarrow a)$. The previous proposition implies that $b) \Leftrightarrow c)$. We are left to prove $a) \Rightarrow b)$. Let B' be the (unique) quaternion algebra non-isomorphic to B. By the dichotomy, $m(\pi_{B'}) = 0$, and thus $Z = I = 0$ identically for B'. First we assume that F is non-archimedean. Then by the direct sum decomposition $I(0) = \pi(B) \oplus \pi(B)$ and the non-vanishing of Z on $I(0) \otimes \sigma$, we conclude that Z does not vanish on $\pi(B) \otimes \sigma$. If F is archimedean, we only need to consider F is real. The assertion is trivial if B is the Hamilton quaternion since then $B^\times/F^\times \simeq \mathrm{SO}(3)$ is compact. We assume that $B = M_{2\times 2,\mathbb{R}}$. Then one can use the same argument as above. $\square$

2.3 Integral representation of L-series

In this subsection, we review integral representation of triple product L-series of Garret, Piatetski-Shapiro and Rallis, and various improvements of Harris–Kudla. Let F be a number field with adeles $\mathbb{A}$, $\mathbb{B}$ a quaternion algebra over $\mathbb{A}$, E a cubic semisimple algebra. We write $\mathbb{B}_E := \mathbb{B} \otimes_F E$ the base changed quaternion algebra over $\mathbb{A}_E := \mathbb{A} \otimes_F E$.

Eisenstein series

For $\phi \in \mathscr{S}(\mathbb{B}_E)$, we define

$$\Phi(g, s) = r(g)\phi(0)\lambda_s(g)$$

where the character λ_s of P defined as

$$\lambda_s(d(\nu)n(b)m(a)) = |\nu|^{-3s}|det(a)|^{2s}.$$

and it extends to a function on GSp_6 via Iwasawa decomposition $GSp_6 = PK$ such that $\lambda_s(g)$ is trivial on K. It satisfies

$$\Phi(d(\nu)n(b)m(a)g, s) = |\nu|^{-3s-3}|det(a)|^{2s+2}\Phi(g, s).$$

It thus defines an section of $Ind_P^{GSp_6}(\lambda_s)$. Then Siegel's Eisenstein series is defined to be

$$E(\Phi, g, s) = \sum_{\gamma \in P(F)\backslash GSp_6(F)} \Phi(\gamma g, s) = \sum_{\gamma \in P(F)\backslash GSp_6(F)} r(g)\phi(0)\lambda_s(g).$$

This is absolutely convergent when $Re(s) > 2$.

Rankin triple L-function

Let σ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$. Let π be the associated to Jacquet-Langlands correspondence of σ on $\mathbb{B}_E^\times$. Let ω_σ be the central character of σ. We assume that

$$\omega_\sigma|_{\mathbb{A}_F^\times} = 1. \tag{2.3.1}$$

Define a finite set of places of F

$$\Sigma(\sigma) = \left\{ v | \epsilon(\sigma_v, \frac{1}{2}) = -1 \right\}. \tag{2.3.2}$$

Define the zeta integral as

$$Z(\Phi, \varphi, s) = \int_{[\mathbb{G}]} E(\Phi, g, s)\varphi(g)dg \tag{2.3.3}$$

where $[\mathbb{G}] = \mathbb{A}^\times\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})$.

Theorem 2.3.1 ([30])**.** *For a cusp form $\phi \in \sigma$ and $Re(s) \gg 0$ we have an Euler product*

$$Z(\Phi, \varphi, s) = \prod_v Z(\Phi_v, \varphi_v, s) = \frac{L(s + \frac{1}{2}, \sigma, r_8)}{\zeta_F(2s+2)\zeta_F(4s+2)} \prod_v \alpha(s, \Phi_v, W_{\varphi_v}) \tag{2.3.4}$$

where

$$\alpha(s, \Phi_v, W_{\varphi_v}) = \frac{\zeta_{F_v}(2s+2)\zeta_{F_v}(4s+2)}{L(s + \frac{1}{2}, \sigma_v, r_8)} Z(\Phi_v, W_{\varphi_v}, s)$$

which equals one for almost all v.

At $s = 0$, the local zeta integral has already appeared earlier in this paper:

$$Z(\Phi_v, W_{\varphi_v}) = \int_{F^\times N_0 \backslash \mathbb{G}} \Phi_{V,s}(\eta g) W_{\varphi_v}(g) dg.$$

This constant is non-zero only if $\epsilon(\mathbb{B}_v) = \epsilon(\sigma_v, \frac{1}{2})$. We also normalize local constants

$$\alpha(\Phi_v, W_{\varphi_v}) = \frac{\zeta_{F,v}^2(2)}{L(\frac{1}{2}, \sigma, r_8)} Z(\Phi_v, W_{\varphi_v}, 0) \tag{2.3.5}$$

For σ_v a local component of an irreducible cuspidal automorphic representation, by Kim-Shahidi's work we have $\lambda(\sigma_v) < 1/2$. Hence the local zeta integral is absolutely convergent for all v at $s = 0$.

Thus the global $Z(\Phi, \varphi, 1/2) \neq 0$ only if $\Sigma(\mathbb{B}) = \Sigma(\sigma)$ and both of them have even cardinality. In this case we have an identity:

$$Z(\Phi, \varphi) = \prod_v Z(\Phi_v, \varphi_v, 0) = \frac{L(\frac{1}{2}, \sigma, r_8)}{\zeta_F(2)^2} \prod_v \alpha(\Phi_v, W_{\varphi_v}) \tag{2.3.6}$$

Assume that $\Sigma = \Sigma(\sigma)$ is odd and that $\mathbb{B}$ is the incoherent quaternion algebra with ramification set Σ. Then for $\phi \in \mathscr{S}(B_{\mathbb{A}_E})$ we have

$$\frac{d}{ds} Z(\Phi, \varphi, s)|_{s=0} = \int_{[\mathbb{G}]} E'(g, 0, \Phi) \varphi(g) dg \tag{2.3.7}$$

$$= \frac{L'(\frac{1}{2}, \sigma, r_8)}{\zeta_F^2(2)} \prod_v \alpha(\Phi_v, W_{\varphi_v}) \tag{2.3.8}$$

2.4 Special value formula

In this subsection, we review a special value formula of Ichino (also independently proved by the senior author of the present paper). We assume that Σ is even. Let B be a quaternion algebra with ramification set Σ. We write V for the orthogonal space (B, q).

Siegel-Weil for similitudes

The theta kernel is defined to be, for $(g, h) \in R(\mathbb{A})$,

$$\theta(g, h, \phi) = \sum_{x \in B_E} r(g, h) \phi(x).$$

It is $R(F)$-invariant. The theta integral is the theta lifting of the trivial automorphic form, for $g \in GSp_6^+(\mathbb{A})$,

$$I(g, \phi) = \int_{[O(B_E)]} \theta(g, h_1 h, \phi) dh$$

where h_1 is any element in $GO(B_E)$ such that $\nu(h_1) = \nu(g)$. It dees not depend on the choice of h_1. When $B = M_{2\times 2}$ the integral needs to be regularized.

$I(g, \phi)$ is left invariant under $GSp_6^+(\mathbb{A}) \cap GSp_6(F)$ and trivial under the center $Z_{GSp_6}(\mathbb{A})$ of GSp_6.

Theorem 2.4.1 (Siegel-Weil). *$E(\Phi, g, 0)$ is holomorphic at $s = 0$ and*

$$E(\Phi, g, 0) = 2I(g, \phi). \tag{2.4.1}$$

Proof. [13]. □

Remark 1. Similar argument shows that $E(g, 0, \phi) \equiv 0$ if $\phi \in \mathscr{S}(B_{\mathbb{A}_E})$ for an incoherent $B_{\mathbb{A}_E}$.

Corollary 2.4.2 (Fourier coefficients for similitudes). *For $g \in GSp^+(\mathbb{A})$, we have*

$$W_T(\Phi, g, 0) = 2|\nu(g)|_{\mathbb{A}}^{-3} \int_{\Omega_T} r(g_1)\Phi(h^{-1}x) d\mu_T(x)$$

Proof. Take $h \in GO(V_{\mathbb{A}})$ with the same similitude as g. We obtain

$$\begin{aligned}
W_T(\Phi, g, 0) &= \int_{\mathrm{Sym}_3(F)} \psi(-Tb) \int_{[O(V)]} \sum_{x \in V(F)} r(h_1 h, g)\Phi(x) dh_1 db \\
&= \int_{\mathrm{Sym}_3(F)} \psi(-Tb) \int_{[O(V)]} \sum_{x \in V(F)} |\nu(g)|_{\mathbb{A}}^{-3} r(g_1)\Phi(h^{-1}h_1^{-1}x) dh_1 db \\
&= |\nu(g)|_{\mathbb{A}}^{-3} \int_{\mathrm{Sym}_3(F)} \psi(-Tb) \int_{[O(V)]} \sum_{x \in V(F)} r(g_1)\Phi(h^{-1}h_1^{-1}x) dh_1 db \\
&= 2|\nu(g)|_{\mathbb{A}}^{-3} \int_{\Omega_T} r(g_1)\Phi(h^{-1}x) d\mu_T(x)
\end{aligned}$$

by Siegel-Weil for $\Phi'(x) = r(g_1)\Phi(h^{-1}x)$. □

Special value formula

Theorem 2.4.3 (Inchino cf. [14], Zhang [35]). *Let $dg = \prod_v dg_v$ be the Tamagawa measure on $B_F^\times \setminus B_{\mathbb{A}}^\times$. For $f = \otimes_v f_v \in \pi$, $\widetilde{f} = \otimes_v \widetilde{f}_v \in \widetilde{\pi}$, we have*

$$\beta(f, \widetilde{f}) = \frac{1}{2^c} \frac{\zeta_E(2)}{\zeta_F(2)} \frac{L(\frac{1}{2}, \sigma)}{L(1, \sigma, ad)} \alpha(f, \widetilde{f}).$$

Proof. Suppose that $f \otimes \widetilde{f} = \theta(\phi \otimes \varphi)$. Then by the seesaw identity and Siegel-Weil formula we have

$$\beta(f, \widetilde{f}) = \frac{1}{2} \int_{[\mathbb{G}]} E(g, 0, \Phi)\varphi(g) dg.$$

This implies that

$$\beta(f, \widetilde{f}) = \frac{L(\frac{1}{2}, \sigma)}{\zeta_F(2)^2} \prod_v \alpha(\Phi_v, W_{\varphi_v}). \tag{2.4.2}$$

Then if θ_v is normalized as in **??**, by Prop. 2.2.4 we have for $\theta_v(\phi_v \otimes \varphi_v) = f_v \otimes \widetilde{f}_v \in \pi_v \otimes \widetilde{\pi}_v$:

$$\alpha(\Phi_v, W_{\varphi_v}) = (-1)^{sign(B_v)} \frac{\zeta_{F,v}(2)}{L(\frac{1}{2}, \sigma_v, r_8)} \int_{B_v^\times} (\pi(b) f_v, \widetilde{f}_v) db. \tag{2.4.3}$$

Now we need to deal with the normalization of theta map and the measure in Prop 2.2.4. Firstly the product of measures dg_v in Prop 2.2.4 differs from the Tamagawa measure on $B_F^\times \setminus B_{\mathbb{A}}^\times$ by a multiple $\zeta_F(2)$.

In the normalization **??**, the spherical case yields

$$(f_v, \widetilde{f}_v) = \zeta_{E_v}(2)^{-1} L(1, \sigma_v, ad).$$

And in this normalization, by the work of Waldspurger we have the comparison

$$(f, \widetilde{f})_{Pet} = 2^c \zeta_E(2)^{-1} L(1, \sigma, ad) \prod_v \frac{\zeta_{E_v}(2)}{L(1, \sigma_v, ad)} (f_v, \widetilde{f}_v).$$

Therefore, to be compatible with the normalization in the theorem, we just need to rescale the formula 2.4.2 by multiplying $2^c \zeta_E(2)^{-1} L(1, \sigma, ad)$ on the left hand side and dividing by $\zeta_{E_v}(2)^{-1} L(1, \sigma_v, ad)$ in the right hand side of 2.4.3.

Finally, since Σ is even, the constant is $\prod_v (-1)^{B_v} = (-1)^{|\Sigma|} = 1$. □

We have the following consequence on the special values of triple product L-series:

Theorem 2.4.4. *Let F be an number field and E/F be a cubic semisimple algebra. For any cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A}_E)$ with central character $\omega|_{\mathbb{A}^\times} = 1$, we have*

1. *(Positivity)*
$$L(\frac{1}{2}, \sigma) \geq 0$$

2. *(Jacquet's conjecture) $L(\frac{1}{2}, \sigma) \neq 0$ if and only if there exists (uniquely determined) quaternion algebra B over F such that the period*
$$\int_{[B^\times]} f(b) db \neq 0$$
for some $f \in \Pi_{B,E}$, the Jacquet-Langlands correspondence of σ.

Proof. These trivially follow from local results above and the global period formula

$$\frac{|\int_{[B^\times]} f(b)db|^2}{(f,f)_{Pet}} = C \cdot L(\frac{1}{2},\sigma)\prod_v \alpha_v(f_v,f_v)$$

where $C > 0$ is an explicit real number and α is proportional to I_v by a positive multiple such that $\alpha_v = 1$ for almost all v. $\square$

Remark 2. The non-vanishing and positivity of the matrix coefficient integral is conjectured to be true for all pair $(SO(n), SO(n+1))$ in the refinement of Gross-Prasad conjecture by Ichino-Ikeda. One consequence of the non-vanishing and positivity (together with the global period formula) is the positivity of the central value of L-function.

2.5 Weak Intertwining property

In the case where Σ is odd, the formulation $\phi \mapsto E'(g,0,\phi)$ is not equivariant under the action of $\mathrm{Mp}_3(\mathbb{A})$. The following gives a weak intertwining property:

Proposition 2.5.1. *Let $\mathscr{A}(G)_0$ be the image of $\Pi(B_\mathbb{A})$ under the map $\Phi \mapsto E(g,0,\Phi)$ for all quaternion algebra B over F. Fix one place v_1 and fix all Φ_v for $v \neq v_1$ and $h \in G_v$, then we have $E'(gh,0,\Phi) - E'(g,0,r(h)\Phi) \in \mathscr{A}_0$ for $\Phi = \Phi^{(v_1)} \otimes \Phi_{v_1}$ and $\Phi_{v_1} \in I_{v_1}(0)$.*

Proof. Let $\alpha(s,h)(g) = \alpha(s,g,h) = \frac{1}{s}(|\frac{\delta(gh)}{\delta(g)}|^s - 1), s \neq 0$. Then it obviously extends to an entire function of s and it is left $P_\mathbb{A}$-invariant. Now for $Re(s) \gg 0$, we have

$$E(gh,s,\Phi) - E(g,s,r(h)\Phi) = sE(g,s,\alpha(s,h)r(h)\Phi)$$

Now note that the section $g \to \alpha(s,h)r(h)\Phi(g)\delta(g)^s$ is a holomorphic section of $I(s)$. Hence the Eisenstein series $E(g,s,\alpha(s,h)r(h)\Phi)$ is holomorphic at $s = 0$ since any holomorphic section of $I(s)$ is a finite linear combination of standard section with holomorphic coefficients. This implies the desired assertion.

Similarly one can prove the $(\mathscr{G},K)$-intertwining if v_1 is archimedean. We skip this and refer to [21]. $\square$

2.6 Singular coefficients

In this subsection we deal with the singular part $E'_{sing}(g,0,\phi)$ of the Siegel-Eisenstein series.

Definition 2.6.1. *For a place v of F, define the open subset $\mathbb{B}^3_{v,\mathrm{sub}}$ of $\mathbb{B}^3_v$ to be all $x \in \mathbb{B}^3_v$ such that the components of x generates a dimension 3 subspace of $\mathbb{B}_v$. We define the subspace $\mathscr{S}(\mathbb{B}^3_{v,\mathrm{sub}})$ of $\mathscr{S}(\mathbb{B}^3_v)$ to be the set of all Bruhat-Schwartz functions ϕ with* $\mathrm{supp}(\phi) \subset \mathbb{B}^3_{v,\mathrm{sub}}$.

Note that $\mathscr{S}(\mathbb{B}^3_{v,\mathrm{sub}})$ is P_v-stable under the Weil representation.

Our main proposition in this section is the following vanishing result:

Proposition 2.6.2. *For an integer $k \geq 1$, fix non-archimedean places $v_1, v_2, ..., v_k$. Let $\phi = \otimes_v \phi_v \in \mathscr{S}(\mathbb{B}^3)$ with $\mathrm{supp}(\phi_{v_i}) \subset \mathbb{B}^n_{v_i,\mathrm{sub}}$ (i=1,2,...,k) . Then for T singular and $g \in G(\mathbb{A})$ with $g_{v_i} \in P_{v_i}$, $(i = 1, 2, ..., k)$, the vanishing order of the analytic function $ord_{s=0} E_T(g, s, \Phi)$ is at least $k - 1$.*

In particular, for T singular, if $k = 1$, then $E_T(g, 0, \Phi) = 0$; and if $k = 2$, then $E'_T(g, 0, \Phi) = 0$.

For $i = 1, 2, 3$, define

$$w_i = \begin{pmatrix} 1_i & & & \\ & & & 1_{3-i} \\ & & 1_i & \\ & -1_{3-i} & & \end{pmatrix}.$$

Lemma 2.6.3. *For a place v, if a Siegel-Weil section $\Phi_s \in I(s)$ is associated to $\phi \in \mathscr{S}_0(\mathbb{B}^3_{v,sub})$, then Φ_s is supported in the open cell Pw_0P for all s.*

Proof. By the definition $\Phi_s(g) = r(g)\phi(0)\delta(g)^s$. Thus it suffices to prove $supp(\Phi_0) \subset Pw_0P$. Note that by the Bruhat decomposition $G = \coprod_i Pw_iP$, it suffices to prove $r(pw_ip)\phi(0) = 0$ for $i = 1, 2, 3$. Since $\mathscr{S}(\mathbb{B}^3_{v,sub})$ is P_v-stable, it suffices to prove $r(w_i)\phi(0) = 0$ for $i = 1, 2, 3$. Since

$$r(w_i)\phi(0) = \gamma \int_{\mathbb{B}^{n-i}} \phi(0, ...0, x_{i+1}, ..., x_3) dx_{i+1}...dx_3$$

for certain eighth-root of unity γ, we compelete the proof since

$$\phi(0, ..., 0, x_{i+1}, ..., x_3) \equiv 0$$

when $i \geq 1$. □

Remark 3. It is probably true that this characterizes all ϕ such that $\mathrm{supp}(\Phi) \subset Pw_0P$.

Proof of Proposition 2.6.2. We will use some results about Siegel-Weil formula and related representation theory. They should be well-known to experts and are proved mostly in series of papers by Kudla-Rallis. We will sketch proofs of some of them but don't claim any originality and we are not sure if there are more straightforward ways.

Suppose $rank(T) = 3 - r$ with $r > 0$. Note that if $T =^t \gamma T' \gamma$, $T' = \begin{pmatrix} 0 & \\ & \beta \end{pmatrix}$ for some $\beta \in GL_{3-r}$ and $\gamma \in GL_3$, we have

$$E_T(g, s, \Phi) = E_{T'}(m(\gamma)g, s, \Phi).$$

Since $m(\gamma) \in P_{v_i}$, it suffices to prove the assertion for

$$T = \begin{pmatrix} 0 & \\ & \beta \end{pmatrix}$$

with β non-singular.

For $Re(s) \gg 0$, we have

$$E_T(g,0,\Phi) = \int_{[N]} \sum_{P(F)\backslash G(F)} \Phi_s(\gamma n g)\psi_{-T}(n)dn$$
$$= \int_{[N]} \sum_{i=0}^{3} \sum_{\gamma \in P\backslash P w_i P} \Phi_s(\gamma n g)\psi_{-T}(n)dn.$$

By Lemma 2.6.3, $\Phi_v(\gamma n_v g_v, s) \equiv 0$ for $\gamma \in Pw_iP, i > 0$, $v \in \{v_1, ..., v_k\}$ and $g_v \in P_v$. Thus for g as in the statement, only the open cell has nonzero contribution in the coefficients

$$E_T(g,s,\Phi) = \int_{N_{\mathbb{A}}} \Phi(w_0 n g)\psi_{-T}(n)dn.$$

This is exactly the Whittaker functional $W_T(g,s,\Phi) = W_T(e,s,r(g)\Phi)$ where r denotes the right regular action of $G(\mathbb{A})$ on $I(s)$.

Let $i : \mathrm{Sp}(3-r) \to \mathrm{Sp}(3)$ be the standard embedding indicated by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1_r & & & \\ & a & & b \\ & & 1_r & \\ & c & & d \end{pmatrix}.$$

Then this induces a map by restriction: $i^* : I(s) \to I^{3-r}(s+\frac{r}{2})$ to principal series on $\mathrm{Sp}(3-r)$. We will frequently use upper/lower index $n-r$ to indicate the rank of the symplectic group we work on.

Lemma 2.6.4. *Let $E_\beta(g,s,i^*M(s)\Phi)$ denote the β-Fourier coefficient of the Eisensetin series defined by section $i^*M * s)\Phi$. Then*

$$W_T(e,s,\Phi) = E_\beta(e, -s + \frac{r}{2}, i^*M(s)\Phi)$$

Remark 4. Note that in general, besides $W_T(g,s,\Phi)$ in $E_T(g,s,\Phi)$ there are also other terms including $E_\beta(e, s + \frac{r}{2}, i^*r(g)\Phi)$. Thus the result above is also consistent with the functional equation $E(g,s,\Phi) = E(g,-s,M(s)\Phi)$.

Proof. Then we have

$$\begin{aligned}
&W_T(e,s,\Phi)\\
=&\int_N \Phi_s(wng)\psi_{-T}(n)dn\\
=&\int_{N_r}\int_{N_{3-r,3}} \Phi_s(wn_1n_2g)\psi_{-T}(n_1n_2)dn_1dn_2\\
=&\int_{N_r}\left(\int_{N_{3-r,3}} \Phi_s(ww_{n-r}^{-1}w_{n-r}n_1(x,y)w_{n-r}^{-1}w_{n-r}n_2(z)g)dn_1\right)\psi_{-\beta}(n_2)dn_2\\
=&\int_{N_r}\left(\int_{U_{3-r,3}} \Phi_s(w^{(r)}u(x,y)w_{n-r}n_2(z)g)du\right)\psi_{-\beta}(n_2)dn_2\\
=&E_\beta(e,s-\frac{r}{2},i^*U(s)\Phi)
\end{aligned}$$

where

$$u(x,y)=\begin{pmatrix} 1_r & y & x & \\ & 1_{3-r} & & \\ & & 1_r & \\ & & -{}^ty & 1_{3-r}\end{pmatrix}$$

$$n(x,y)=\begin{pmatrix} 1_r & & x & y\\ & 1_{3-r} & {}^ty & \\ & & 1_r & \\ & & & 1_{3-r}\end{pmatrix}$$

$$u(x,y)=w_{3-r}n(x,y)w_{3-r}^{-1}$$

and

$$U_r(s)\Phi=\int_{U_{3-r,3}}\Phi_s(w^{(r)}ug)du,\quad w^{(r)}=\begin{pmatrix} & & 1_r & \\ & 1_{3-r} & & \\ -1_r & & & \\ & & & 1_{3-r}\end{pmatrix}.$$

Apply the functional equation to the Eisenstein series $E(g,s,i^*M(s)\Phi)$,

$$W_T(e,s,\Phi)=E_\beta(e,-s+\frac{r}{2},M(s-\frac{r}{2})\circ i^*U(s)\Phi).$$

And now applying the relation (page 37., [22]),

$$M(s-\frac{r}{2})\circ i^*U(s)=i^*M(s),$$

we obtain

$$W_T(e,s,\Phi)=E_\beta(e,-s+\frac{r}{2},i^*M(s)\Phi).$$

□

Now we have an Euler product when $Re(s) \gg 0$,

$$W_T(e,s,\Phi)=\prod_v W_{\beta,v}(e,-s+\frac{r}{2},i^*M_v(s)\Phi_v).$$

Note that by the standard Gindikin-Karpelevich type argument, for the spherical vector $\Phi_v^0(s)$ at a non-archimedean v and when χ_v is unramified, we have

$$M_v(s)\Phi_v^0(s)=\frac{a_v(s)}{b_v(s)}\Phi_v^0(-s)$$

where

$$a_v(s)=L_v(s+\varrho_3-3,\chi_v)\zeta_v(2s-1)$$

and

$$b_v(s)=L_v(s+\varrho_3,\chi_v)\zeta_v(2s+2).$$

Thus, for a finite set outside which everything is unramified,

$$M(s)\Phi(s)=\frac{a(s)}{b(s)}\left(\bigotimes_{v\in S}\frac{b_v(s)}{a_v(s)}M_v(s)\Phi_v(s))\right)\otimes\Phi_S^0(-s).$$

For a local Siegel-Weil section Φ_v for all v, $\frac{b_v(s)}{a_v(s)}M_v(s)\Phi_v$ is holomorphic at $s=0$ and there is a non-zero constant independent of Φ such that

$$\frac{b_v(s)}{a_v(s)}M_v(s)\Phi_v(s))|_{s=0}=\lambda_v\Phi_v(0).$$

Thus

$$\begin{aligned}
&W_T(e,s,\Phi)\\
=&\prod_v W_{\beta,v}(e,-s+\frac{r}{2},i^*M_v(s)\Phi_v)\\
=&\Lambda_{3-r}(-s+\frac{r}{2})\frac{a(s)}{b(s)}\prod_{v\in S'_\beta}\frac{1}{\Lambda_{3-r,v}(-s+\frac{r}{2})}W_{\beta,v}(e,-s+\frac{r}{2},i^*(\frac{b_v(s)}{a_v(s)}M_v(s)\Phi_v))\\
=&\Lambda_{3-r}(-s+\frac{r}{2})\frac{a(s)}{b(s)}\prod_{v\in S'_\beta}A_{\beta,v}(s,\Phi).
\end{aligned}$$

where S_β is the set of all primes such that outside S_β, Φ_v is the spherical vector, ψ_v is unramifed and $\mathrm{ord}_v(det(\beta))=0$.

Since $ord_{s=0}\Lambda_{3-r,v}(-s+\frac{r}{2})=0$, $\frac{b_v(s)}{a_v(s)}M_v(s)\Phi_v)$ is holomorphic and $W_\beta(e,s,\Phi)$ extends to an entire function, we know that $A_{\beta,v}(s,\Phi)$ is holomorphic at $s=0$. And

$$A_{\beta,v}(0,\Phi)=\frac{\lambda_v}{\Lambda_{3-r,v}(0)}W^{3-r}_{\beta,v}(e,\frac{r}{2},i^*\Phi_v(0)).$$

Lemma 2.6.5. *Define a linear functional*

$$\iota : \mathscr{S}(\mathbb{B}_v^3) \to \mathbb{C}$$
$$\phi_v \mapsto A_{\beta,v}(0, \Phi_v).$$

Then, we have $\iota(r(n(b))\phi_v) = \psi_{v,T}(b)\iota(\phi_v)$, *i.e.,* $\iota \in \mathrm{Hom}_N(\mathscr{S}(\mathbb{B}_v^3), \psi_T)$.

Proof. Let $b = \begin{pmatrix} x & y \\ {}^t y & z \end{pmatrix} \in \mathrm{Sym}_3(F_v)$. Since M_n is Sp(3)-intertwining,

$$\begin{aligned}
&W_{\beta,v}(e, -s + \frac{r}{2}, i^*(M_v(s)r(n(b))\Phi_v)) \\
=&W_{\beta,v}(e, -s + \frac{r}{2}, i^*(r(n(b))M_v(s)\Phi_v)) \\
=&\int_{Sym_{3-r}} (M_v(s)\Phi_v)(w_{3-r}n\left(\begin{pmatrix} 0 & 0 \\ 0 & z' \end{pmatrix}\right) n(b))\psi_{-\beta}(z')dz' \\
=&\int_{Sym_{3-r}} (M_v(s)\Phi_v)(u(x,y)w_{3-r}n\left(\begin{pmatrix} 0 & 0 \\ 0 & z'+z \end{pmatrix}\right))dz' \\
=&\int_{Sym_{3-r}} (M_v(s)\Phi_v)(w_{3-r}n\left(\begin{pmatrix} 0 & 0 \\ 0 & z'+z \end{pmatrix}\right))\psi_{-\beta}(z')dz' \\
=&\psi_\beta(z)W_{\beta,v}(e, -s + \frac{r}{2}, i^*M_v(s)\Phi_v) \\
=&\psi_T(b)W_{\beta,v}(e, -s + \frac{r}{2}, i^*M_v(s)\Phi_v).
\end{aligned}$$

Thus, the linear functional $\Phi_s \mapsto A_{\beta,v}(s, \Phi)$ defines an element in $\mathrm{Hom}_N(I(s), \psi_T)$. In particular, when $s = 0$, the composition ι of $A_{\beta,v}$ with the G-intertwining map $\mathscr{S}(\mathbb{B}_v^3) \to I(0)$ defines a linear functional in $\mathrm{Hom}_N(\mathscr{S}(\mathbb{B}_v^3), \psi_T)$. □

Then the map ι factors through the ψ_T-twisted Jacquet module $\mathscr{S}(\mathbb{B}_v^3)_{N,T}$ (i.e., the maximal quotient of $\mathscr{S}(\mathbb{B}^3)$ on which N acts by character ψ_T). Thus by the following result of Rallis, ι is trivial on $\mathscr{S}(\mathbb{B}_{v,\mathrm{sub}}^3)$ when T is singular:

Lemma 2.6.6. *The map* $\mathscr{S}(\mathbb{B}^3) \to \mathscr{S}(\mathbb{B}^3)_{N,T}$ *can be realized as the restriction* $\mathscr{S}(\mathbb{B}^3) \to \mathscr{S}(\Omega_T)$.

Now since $\mathrm{ord}_{s=0}\frac{a(s)}{b(s)} = 0$, we can now conclude that

$$\mathrm{ord}_{s=0}W_T(e, s, \Phi) \geq k$$

if $\phi_{v_i} \in \mathscr{S}(\mathbb{B}_{v_i,\mathrm{reg}}^3)$ since the restriction to Ω_T is zero.

For a general $g \in G_{\mathbb{A}}$, we have

$$\begin{aligned}
&W_T(g, s, \Phi) \\
=&W_T(e, s, r(g)\Phi) \\
=&\Lambda_{n-r}(-s + \frac{r}{2})\frac{a(s)}{b(s)} \prod_{v \in S'_\beta} A_{\beta,v}(s, r(g_{v_i})\Phi_v)
\end{aligned}$$

where $S_{\beta,g}$ is a finite set of place that depends also on g.

Since $\mathscr{S}(\mathbb{B}_v^3) \to I(0)$ is G-equivariant, $A_{\beta,v}(0, r(g_v)\Phi_v) = \iota(r(g_v)\phi_v)$. Since $g_{v_i} \in P_{v_i}$, we have $r(g_{v_i})\phi_{v_i} \in \mathscr{S}(\mathbb{B}^3_{v_i,sub})$ and by the same argument above $A_{\beta,v_i}(0, r(g_{v_i})\Phi_{v_i}) = 0$. This completes the proof of Proposition 2.6.2. □

Remark 5. The proof would be much shorter if it were true that $W_{T,v}(g, s, \Phi_v)$ extends to $\mathbb{C}$ and holomorphic at $s = 0$ for singular T.

Now it is easy to extend to the similitude group GSp_3. Recall that we have a decomposition of $E'(g, 0, \phi)$ according to the difference of $\Sigma(T)$ and Σ:

$$E'(g,0,\phi) = \sum_v E'_v(g,0,\phi) + E'_{sing}(g,0,\phi) \tag{2.6.1}$$

where

$$E'_v(g,0,\phi) = \sum_{\Sigma(T)=\Sigma(v)} E'_T(g,0,\phi) \tag{2.6.2}$$

and

$$E'_{sing}(g,0,\phi) = \sum_{T,\,\det(T)=0} E'_T(g,0,\phi).$$

Corollary 2.6.7. *The same assumption as in Proposition 2.6.2, then we have for T singular and $g \in \mathrm{GSp}_3(\mathbb{A})$ with $g_{v_i} \in P_{v_i}, (i = 1, 2, ..., k)$, the vanishing order of the analytic function $\mathrm{ord}_{s=0}E_T(g, s, \Phi)$ is at least $k = 1$. In other words, for such g we have*

$$E'_{sing}(g,0,\phi) = 0.$$

Proof. For $g \in GSp_3(\mathbb{A})$, one still have

$$W_T(g,s,\Phi) = \Lambda_{3-r}(-s+\frac{r}{2})\frac{a(s)}{b(s)}\prod_{v\in S'_\beta} A_{\beta,v}(s, r(g_{v_i})\Phi_v)$$

for a finite set of places $S_{\beta,g}$. □

2.7 Test function

Let F be a non-archimedean field. Let B be a quaternion algebra over F. And we have the moment map

$$\mu : B^3 \to \mathrm{Sym}_3(F).$$

Definition 2.7.1. *We call a function $\phi \in \mathscr{S}(B^3_{reg})$ of "regular of order k" if it satisfies the condition that $\mu(\mathrm{supp}(\phi)) + p^{-k}Sym_n(\mathscr{O}) \subseteq \mu(B^3_{\mathrm{reg}})$.*

Even though it looks that such functions are very special, they in fact generate $\mathscr{S}(B^3_{reg})$ under the action of a very small subgroup.

Lemma 2.7.2. *Let k be any fixed integer. Then $\mathscr{S}(B^3_{reg})$ is generated by all ramified function of order k under the action of elements $m(aI_3) \in \mathrm{Sp}_3$ for all $a \in F^\times$.*

Proof. Without loss of generality, we can assume that k is even and that $\phi = 1_U \in \mathscr{S}(B^3_{reg})$ is the characteristic function some open compact set $U \subseteq B^3$. Then $\mu(U)$ is an compact open subset of $\mathrm{Sym}_3(F)_{\mathrm{reg}}$. Let $\mathbb{Z}^3_+ = \{(a_1, a_2, a_3) \in \mathbb{Z}^3 | a_1 \leq a_2 \leq a_3\}$. Then the "elementary divisors" defines a map $\delta : b \in \mathrm{Sym}_3(F) \to (a_1, a_2, a_3) \in \mathbb{Z}^3_+$. One can check that it is locally constant on $\mathrm{Sym}_3(F)_{reg}$. Hence the composition of this map and the moment map μ is also locally constant on B^3_{reg}. In particular, this gives a partition of U into disjoint union of finitely many open subsets. So we can assume that $\delta \circ \mu$ is constant on U, say, $\delta \circ \mu(U) = \{(a_1, a_2, a_3)\}$.

Consider $m(a\phi)$ which is certain multiple of 1_{aU}. Choose $a = p^{-A}$ for some integer $A > 1 + a_1 + (a_2 - a_1) + (a_3 - a_1)$. Then we are left to prove that that such $1_{p^{-A-k/2}U}$ is a highly ramified function of order k. It suffices to prove that, for any $x \in U$ and $t \in \mathrm{Sym}_3(\mathscr{O})$, $\mu(p^{-A-k/2}x) + p^{-k}t$ belongs to $\mu(B^3_{\mathrm{reg}})$. Note that

$$\mu(p^{-A-k/2}x) + p^{-k}t = p^{-k-2A+2[\frac{a_1-1}{2}]}(\mu(p^{-[\frac{a_1-1}{2}]}x) + p^{2A-2[\frac{a_1-1}{2}]}t).$$

Now $\mu(p^{-[\frac{a_1-1}{2}]}x) \in \mathrm{Sym}_3(\mathscr{O})$. It is well-known that for $T \in \mathrm{Sym}_3(\mathscr{O})_{\mathrm{reg}}$, T and $T+p^{2+\det(T)}T'$ for any $T' \in \mathrm{Sym}_3(\mathscr{O})$ defines isomorphic integral quadratic forms of rank n. Equivalently, $T+p^{2+\det(T)}T' =^t \gamma T \gamma$ for some $\gamma \in \mathrm{GL}_3(\mathscr{O})$. Now it is easy to see that $\mu(p^{-A-k/2}x)+p^{-k}t \in \mu(B^3_{\mathrm{reg}})$. ☐

The nice property of a ramified function of high order is exhibited in the vanishing of the Whitatker function.

Proposition 2.7.3. *Suppose that $\phi \in \mathscr{S}(B^3_{reg})$ is regular of sufficiently large order k depending on the conductor of the additive character ψ. Then we have*

$$W_T(\phi, e, s) \equiv 0$$

for regular $T \notin \mu(B^3_{\mathrm{reg}})$ and any $s \in \mathbb{C}$. In particular, $W_T(\phi, e, 0) = W'_T(\phi, e, 0) = 0$.

Proof. When $Re(s) \gg 0$, we have

$$\begin{aligned} &W_T(\phi, e, s) \\ &= \int_{\mathrm{Sym}_3(F)} \psi(-b(T - \mu(x))) \int_{B^3} \phi(x)\delta(wb)^s dx db \\ &= \int_{\mathrm{Sym}_3(F)} \psi(b(T' - T))\delta(wb)^s M\phi(T') db dT' \end{aligned}$$

where $M\phi(T') = \int_{\mu^{-1}(T')} \phi(x) d_T x$ is the orbital integral and defines a function $M\phi \in \mathscr{S}(Sym_3(F)_{\mathrm{reg}})$. Since as a function of $b \in \mathrm{Sym}_3(F)$, $\delta(wb)$ is invariant under the translation of $\mathrm{Sym}_3(\mathscr{O})$, we have

$$\begin{aligned}&\int_{\mathrm{Sym}_3(F)} \psi(bt)\delta(wb)^s db\\ &= \left(\int_{\mathrm{Sym}_3(O)} \psi(xt)dx\right) \sum_{b\in\mathrm{Sym}_3(F)/\mathrm{Sym}_3(\mathscr{O})} \psi(bt)\delta(wb)^s\end{aligned}$$

which is zero unless $t \in p^{-k}\mathrm{Sym}^3(\mathscr{O})$ for some k depending on the conductor of the additive character ψ.

Therefore the nonzero contribution to the integral are from $T' - T \in p^{-k}\mathrm{Sym}^3(\mathscr{O})$ and $M\phi(T') \neq 0$. The assumption in the proposition forces that T' is not in $\mu(\mathrm{supp}(\phi))$. But this in turn implies that $M\phi(T') = 0$!

In conclusion, we proves that, if ϕ is of highly ramified of order at least k and $\mathrm{Re}(s) \gg 0$, we have

$$W_T(\phi, e, s) \equiv 0$$

Hence after analytic continuation, we still have $W_T(\phi, e, s) \equiv 0$ for all $s \in \mathbb{C}$!

□

To conclude the discussion of analytic kernel function, we choose ϕ_v to be a test function "ramified of sufficiently higher order" for $v \in S$ where S is a set of finite places with at least two elements such that any finite place outside S is spherical. And we always choose the standard Gaussian at all archimedean places. Then for $g \in \mathbb{G}(\mathbb{A}^S)$, we have

$$E'(g, 0, \phi) = \sum_v \sum_{\Sigma(T)=\Sigma(v)} E'_T(g, 0, \phi) \tag{2.7.1}$$

where the sum runs over v outside S and nonsingular T.

In this subsection, let F be a non-archimedean local field and let B be a quaternion algebra over F.

Recall that we denote by B^3_{sub} the open subset of B^3 consisting of $x \in B^3$ such that $F(x) = Fx_1 + Fx_2 + Fx_3$ is of dimension 3. Denote by B^3_{reg} the open subset of B^3 consisting of $x \in B^3$ such that $F(x) = Fx_1 + Fx_2 + Fx_3$ is regular (=non-degenerate). Note that $B^3_{reg} \subseteq B^3_{sub}$ and they are equal if V is anisotropic. Moreover B^3_{sub} is exactly the open submanifold of B^3 such that the moment map

$$\mu : B^3 \to \mathrm{Sym}_3(F)$$

is submersive when restricted to B^3_{sub}.

Let $\sigma = \otimes_{i=1}^3 \sigma_i$ be unitary irreducible admissible representation of $\mathbb{G}^\circ$ with each σ_i of infinite dimensional and with $\lambda(\sigma) < 1/2$. Let $\lambda(\sigma_i)$ be zero if it is supercuspidal and $|\lambda|$ if $\sigma = \mathrm{Ind}_B^G(\chi|\cdot|^\lambda|\cdot|^{-\lambda})$ for a unitary χ. Let $\lambda(\sigma)$ be the sum of $\lambda(\sigma_i)$. Note that if σ is local component of global automorphic cuspidal representation, we have $\lambda(\sigma) < 1/2$ by work of Kim-Shahidi (Ramanujam conjecture predicts that $\lambda(\sigma) = 0$).

Theorem 2.7.4. *Assume that* $\mathrm{Hom}_{\mathbb{G}^\circ}(R_3(B)\times\sigma,\mathbb{C})\neq 0$. *Then the local zeta integral* $Z(\Phi,W)$ *is non-zero for some choice of* $W\in\mathscr{W}(\sigma,\psi)$ *and* $\Phi\in R_3(B)$ *attached to* $\phi\in\mathscr{S}(B^3_{reg})$.

Corollary 2.7.5. *For a finite place* v *of* F, *there exists* $\theta(\phi_v,\varphi_v)=f_v\otimes\widetilde{f_v}$ *(**??**), such that the local factor* $\alpha(f_v,\widetilde{f_v})$ *is nonvanishing.*

In the following we want to show that the theorem is true in some special cases including the case when V is anisotropic. More precisely, we want to study the quotient $\widetilde{\mathscr{S}}(V^3)$ of $\mathscr{S}(V^3)$ under $\mathrm{SL}_2(F)^3$ modulo the submodule generated by $\mathscr{S}(V^3)_{\mathrm{reg}}$. Let $\sigma=\sigma_1\otimes\sigma_2\otimes_3$ be irreducible representation of $\mathrm{SL}_2(F)^3$ with every σ_i infinite dimensional. We show that σ can't a quotient of $\widetilde{\mathscr{S}}(V^3)$ if V is anisotropic. The proof is by induction.

Let F be a non-archimedean local field and let (V,q) be a non-degenerate orthogonal space over F of even dimension then we have a Weil representation of $\mathrm{SL}_2(F)$ on $\mathscr{S}(V)$, the space of Bruhat-Schwartz functions on V. Let $\alpha_i:\mathscr{S}(V)\longrightarrow\sigma_i$ $(i=1,\cdots m)$ be some $\mathrm{SL}_2(F)$-surjective morphisms to irreducible and admissible $\mathrm{SL}_2(F)$ representations. Then we have an $\mathrm{SL}_2(F)^m$-equivariant morphism

$$\alpha:\quad \mathscr{S}(V^m)\longrightarrow\sigma:=\sigma_1\otimes\cdots\otimes\sigma_m.$$

The main result of this note is to prove the following:

Theorem 2.7.6. *Let* W *be a non-degenerate subspace of* V *(with respect to the norm* $q|_W$*) such that*

$$\dim W+m\leq\dim V.$$

Assume every σ_i *are nontrivial and one of the following conditions hold for every proper non-degenerate subspace* W' *of* V *perpendicular to* W *and its orthogonal complement* W''*:*

1. σ_i *is not a quotient of* $\mathscr{S}(W'')$*;*
2. $\mathscr{S}(W')$ *does not have* $\mathrm{SL}_2(F)$*-invariant functional.*

There is a function $\phi\in\mathscr{S}(V^m)$ *such that* $\alpha(\phi)\neq 0$ *and that the support* $\mathrm{supp}(\phi)$ *of* ϕ *contains only elements* $x=(x_1,\cdots,x_m)$ *such that*

$$W(x):=W+Fx_1+\cdots Fx_m$$

is non-degenerate of dimension $\dim W+m$.

The following is a statement with only condition on W:

Theorem 2.7.7. *Let* W *be a non-degenerate subspace of* V *(with respect to the norm* $q|_W$*) such that*

$$\dim W+m\leq\dim V.$$

Assume every σ_i *are nontrivial and one of the following conditions hold for the orthogonal complement subspace* W' *of* W*:*

1. *W' is anisotropic;*

2. $\dim V \leq 3$ *and the Weil constant* γ *for hyperplane is not* 1*;*

3. $\dim V = 4$*,* $W = 0$*, and the Weil constant* γ *for hyperplane is not* 1*.*

There is a function $\phi \in \mathscr{S}(V^m)$ *such that* $\alpha(\phi) \neq 0$ *and that the support* $\mathrm{supp}(\phi)$ *of* ϕ *contains only elements* $x = (x_1, \cdots, x_m)$ *such that*

$$W(x) := W + Fx_1 + \cdots Fx_m$$

is non-degenerate of dimension $\dim W + m$*.*

Lets start with the following Proposition which allows us to modify any test function to a function with support at points $x = (x_i) \in V^m$ with components x_i of nonzero norm $q(x_i) \neq 0$.

Proposition 2.7.8. *Let* $\phi \in \mathscr{S}(V^m)$ *be an element with nonzero image in* σ*. Then there is a function* $\widetilde{\phi} \in \mathscr{S}(V^m)$ *supported on*

$$\mathrm{supp}\widetilde{\phi} \subset \mathrm{supp}\phi \cap V^m_{q\neq 0}.$$

The key to prove this proposition is the following lemma:

Lemma 2.7.9. *Let* π *be an irreducible and admissible representation of* $\mathrm{SL}_2(F)$ *with* $\dim > 1$*. Then the group* $N(F)$ *of matrixes* $n(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ *does not have any non-zero invariant on* π*.*

Proof. Assume that v is a nonzero invariant of π under $N(F)$. Then v defines an element in $\mathrm{Hom}_N(\widetilde{\pi}, \mathbb{C})$. It follows that $\widetilde{\pi}$ is not supersingular. So is π. It follows that π can be embedded into induced representation $\mathrm{Ind}_B^{\mathrm{SL}_2(F)}(\chi)$ of $\mathrm{SL}_2(F)$ from a quasi-character χ of the group B of upper triangular matrices. Notice that χ is determined by a character μ of $F^\times$:

$$\chi \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \mu(a).$$

Thus v can be realized as a function f on $\mathrm{SL}_2(F)$ such that

$$f(bgn) = \chi(b)f(g), \qquad b \in B, \quad n \in N.$$

Since $\mathrm{SL}_2(F) = B \cup BwN$ with $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, f is determined by its value at identity matrix e and the element w. Notice that π is admissible, $f(k) = f(e)$ for k in a small open compact subgroup K of $\mathrm{SL}_2(F)$. In particular, for $t \in F^\times$ with small norm,

$$f\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = f(e).$$

On other hand, from the decomposition

$$\begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix} = \begin{pmatrix} t^{-1} & 1 \\ 0 & t \end{pmatrix} w \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix}$$

we have

$$f\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \mu(t)^{-1} f(w).$$

Thus when $|t|$ is sufficiently small,

$$\mu(t)^{-1} f(w) = f(e).$$

It follows that μ and χ must be trivial and $f(w) = f(e)$. It follows that f is a constant function on $\mathrm{SL}_2(F)$ which implies that σ is trivial. Thus we have a contradiction for the existence of nonzero v under N. $\square$

Proof of Proposition. First we reduce to the case where ϕ is a pure tensor $\phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_m$ with $\phi_i \in \mathscr{S}(V)$. Indeed, cover $\mathrm{supp}(\phi)$ by local open subsets U_i such that ϕ is a constant c_i on U_i and that U_i is a product of open subsets of V, then

$$\phi = \sum c_i \alpha(1_{U_i}).$$

Thus one of $\alpha(1_{U_i}) \neq 0$. It is clear that 1_{U_i} is a pure tensor.

Assume now that $\phi = \otimes \phi_i$ is a pure tensor. Then

$$\alpha(\phi) = \otimes \alpha_i(\phi_i).$$

Applying the lemma, we obtain elements $t_i \in F$ such that

$$\sigma_i(n(t_i))\alpha_i(\phi_i) - \alpha_i(\phi_i) \neq 0.$$

The left hand side equals to $\alpha(\widetilde{\phi_i})$ with

$$\widetilde{\phi_i} = n(t_i)\phi_i - \phi_i.$$

By definition

$$\widetilde{\phi_i}(x) = (\psi(t_i q(x)) - 1)\phi_i(x).$$

Thus

$$\mathrm{supp}\widetilde{\phi_i} \subset \mathrm{supp}(\phi_i) \cap V_{q \neq 0}.$$

Write $\widetilde{\phi} = \otimes \widetilde{\phi_i}$, then $\alpha(\widetilde{\phi}) \neq 0$ and the support of $\widetilde{\phi}$ is included in $\mathrm{supp}(\phi) \cap V_{q\neq 0}^m$. $\square$

In the following, we proceed to prove the theorem by induction on m. First we need to give a description of maximal $\widetilde{\mathrm{SL}}_2(F)$-trivial quotient of $\mathscr{S}(V)$. For any non-degenerate orthogonal space (V, q), let $\mathscr{S}(V)_0$ denote the $\mathrm{SL}_2(F)$-submodule of $\mathscr{S}(V)$ generated by $\mathscr{S}(V_{q\neq 0})$, let $\widetilde{\mathscr{S}}(V)$ denote the quotient of $\mathscr{S}(V)$ by $\mathscr{S}(V)_0$.

Lemma 2.7.10. *The* $\mathrm{SL}_2(F)$*-modulo* $\widetilde{\mathscr{S}}(V)$ *is the maximal quotient of* $\mathscr{S}(V)$ *over which* $\mathrm{SL}_2(F)$ *acts trivially.*

Proof. For any $t \in F$ let $n(t)$ denote $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then $\phi \in \mathscr{S}(V)$, the element $n(t)\phi - \phi$ is clearly in $\mathscr{S}(V_q)$. Thus the $n(t)$ acts trivially on $\widetilde{\mathscr{S}}(V)$. In other words, $N(F)$ is included in the normal subgroup of $\mathrm{SL}_2(F)$ of elements acting trivially on $\widetilde{\mathscr{S}}(V)$. Since $\mathrm{SL}_2(F)$ is simple, $\mathrm{SL}_2(F)$ acts trivially on $\widetilde{\mathscr{S}}(F)$. Conversely, if $\mathscr{S}(V) \longrightarrow M$ is a quotient where $\mathrm{SL}_2(F)$ acts trivially, then for any $\phi \in \mathscr{S}(V_{q\neq 0})$, there is an $t \in F$ such that $\psi(tq(x)) \neq 1$ for all $x \in \mathrm{supp}(\phi)$. It follows that $\phi = (n(t) - 1)\widetilde{\phi}$ for $\widetilde{\phi}(x) \in \mathscr{S}(V)$ defined by

$$\widetilde{\phi}(x) = \begin{cases} (\psi(tq(x)) - 1)^{-1}\phi(x), & \text{if } \phi(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

It follows that ϕ has trivial image in M. ☐

Proof of Theorem 1 in case $m = 1$. Since W is non-degenerate, we have an orthogonal decomposition $V = W \oplus W'$, and an identification $\mathscr{S}(V) = \mathscr{S}(W) \otimes \mathscr{S}(W')$. The action of $\mathrm{SL}_2(F)$ is given by actions of the double cover $\widetilde{\mathrm{SL}}_2(F)$ on $\mathscr{S}(W)$ and $\mathscr{S}(W')$ respectively. Consider the restriction α_0 of α on $\mathscr{S}(W) \otimes \mathscr{S}(W')_0$. If α_0 is trivial, then α factors through a morphism

$$\widetilde{\alpha} : \mathscr{S}(W) \otimes \widetilde{\mathscr{S}}(W') \longrightarrow \sigma.$$

It follows that $\widetilde{\mathscr{S}}(W') \neq 0$ and that σ is a quotient of $\mathscr{S}(W)$ as the action of $\mathrm{SL}_2(F)$ on $\mathscr{S}(W')$ is trivial. This contradicts to the assumption in the theorem.

Thus α_0 is non-trivial. Choose any ϕ such that $\alpha(\phi) \neq 0$ and that ϕ is a pure tensor:

$$\phi = f \otimes f', \qquad f \in \mathscr{S}(W), \quad f' \in \mathscr{S}(W')_0.$$

By assumption, f' is generated by $\mathscr{S}(W'_{q\neq 0})$ over $\mathrm{SL}_2(F)$. Thus we have a decomposition

$$f' = \sum g_i f'_i, \qquad f'_i \in \mathscr{S}(W'_{q\neq 0}) \quad g_i \in \widetilde{\mathrm{SL}}_2(F).$$

Then we have decomposition

$$\phi = \sum_i g_i \phi_i. \qquad \phi_i := (g_i^{-1} f) \otimes f_i.$$

One of $\alpha(\phi_i) \neq 0$. Thus we may replace ϕ by this ϕ_i to conclude that the support of ϕ consists of points $x = (w, w')$ with $W(x) = W \oplus Fw'$ non-degenerate. Applying the proposition, we may further assume that ϕ has support on $V_{q\neq 0}$. ☐

Proof of Theorem 1: completion. We want to prove the theorem in case $m > 1$. We use induction on m. Thus we assume that we have a $\phi' \in \mathscr{S}(V^{m-1})$ with nonzero image in $\sigma_1 \otimes \cdots \otimes \sigma_{m-1}$ under

$$\alpha' = \alpha_1 \otimes \cdots \otimes \alpha_{m-1}$$

such that the support of ϕ consists of elements $x' = (x_1, \cdots, x_{m-1})$ with non-degenerate $W(x')$ of dimension $\dim W + m - 1$. For any $x \in \mathrm{supp}(\phi')$ by applying the proved case $m = 1$ to the subspace $W(x')$, we have a $\phi_{x'} \in \mathscr{S}(V)$ such that $\mathrm{supp}(\phi_{x'})$ contains only elements x_m with non-degenerate

$$W(x')(x_m) = W(x), \qquad x = (x_1, \cdots, x_m)$$

of dimension $\dim W + m$. By computing moment matrix of W'', we see that this last condition is open in x'. Thus there is an open subset $U(x')$ of x' such that above non-degenerate for all elements in $U(x')$.

As x' varies in $\mathrm{supp}(\phi')$, $U(x')$ covers $\mathrm{supp}(\phi')$. By the compactness of $\mathrm{supp}(\phi')$, we can find finitely many $U(x_i')$ to cover $\mathrm{supp}(\phi')$. By replacing $U(x_i')$ by sub-coverings of $U(x_i') \cap \mathrm{supp}(\phi')$, we may assume that ϕ_{x_i} takes constants c_i on every $U(x_i')$. Thus we have an decomposition

$$\phi' = \sum_i c_i 1_{U(x_i')}.$$

As $\alpha'(\phi') \neq 0$, for one of x_i, say y, $\alpha'(1_{U(y)}) \neq 0$. Now we define

$$\phi = 1_{U(y)} \otimes \phi_y \in \mathscr{S}(V^m)$$

Then ϕ satisfies conditions in Theorem. $\square$

Proposition 2.7.11. *The module $\widetilde{\mathscr{S}}(V) = 0$ in following cases:*

1. *if (V, q) is anisotropic;*
2. *if (V, q) is a hyperbolic plane and the Weil constant $\gamma(q) \neq 1$.*

Proof. The case of anisotropic (V, q) is easy as $\mathscr{S}(V_{q\neq 0})$ is a codimension 1 subspace in $\mathscr{S}(V)$. The quotient is given by evaluation at 0. Thus

$$\mathscr{S}(V) = \mathscr{S}(V_{q\neq 0}) + w\mathscr{S}(W_{q\neq 0})$$

as w acts as the Fourier transform up to a scale multiple.

It remains to deal the case of hyperbolic plane:

$$V = F^2, \qquad q(x, y) = xy.$$

Then $V_{q\neq 0}$ is defined by equation $xy \neq 0$, the complement of the union of coordinates axes. Let $\widetilde{\mathscr{S}}(V)$ be the quotient of $\mathscr{S}(V)$ by the $\mathrm{SL}_2(F)$-module generated by $\mathscr{S}(V_{q\neq 0})$. We need to show that $\widetilde{\mathscr{S}}(V) = 0$. Notice that the Weil representation induces an representation of $O(V) \times \mathrm{SL}_2(F)$ on $\widetilde{\mathscr{S}}(V)$.

The quotient $\mathscr{S}(V)/\mathscr{S}(V_{q\neq 0})$ is identified with $\mathscr{S}(V_{q=0})$, the locally constant functions with compact support. We want to study the quotient

$$\mathscr{S}(V)/(\mathscr{S}(V_{q\neq 0}) + w\mathscr{S}(V_{q\neq 0}))$$

by studying the Fourier transform of functions in $\mathscr{S}(V_{q\neq 0})$ and their restriction to the coordinate axes. Write $\phi \in \mathscr{S}(V_{q\neq 0})$ as a combination of pure tenors:

$$\phi = \sum u_i \otimes v_i, \qquad u_i, v_i \in \mathscr{S}(F).$$

Then the Fourier transform is given by

$$\widehat{\phi} = \sum_i \widehat{v}_i \otimes \widehat{u}_i.$$

The restriction of $\widehat{\phi}$ on coordinate axes is given by

$$f_1(x) = \widehat{\phi}(x,0) = \sum \widehat{v}_i(x)\widehat{u}_i(0), \qquad f_2(y) := \widehat{\phi}(0,y) = \sum \widehat{v}_i(0)\widehat{u}_i(y).$$

Recall that ϕ satisfies the conditions

$$0 = \phi(x,0) = \sum u_i(x)v_i(0), \qquad 0 = \phi(0,y) = \sum u_i(0)v_i(y)$$

which implies

$$0 = \sum \widetilde{u}_i(0)v_i(0), \quad 0 = \sum u_i(0)\widehat{v}_i(0).$$

In other words, we have $\widehat{f}_1(0) = \widehat{f}_2(0) = 0$. It is easy to show that the quotient

$$\mathscr{S}(V)/(\mathscr{S}(V_{q\neq 0}) + w\mathscr{S}(V_{q\neq 0})) \simeq \mathbb{C}^2$$

by evaluation

$$\phi \mapsto L(\phi) := \left(\int_F \phi(x,0)dx, \int_F \phi(0,y)dy\right).$$

It follows that $\widetilde{\mathscr{S}}(V)$ is a quotient of $\mathbb{C}^2$. The Weil representation of $O(V) \simeq F^\times \rtimes \{1,\tau\}$ induces an irreducible representation on $\mathbb{C}^2$: the subgroup $F^\times$ acts on $\mathbb{C}^2$ given by characters $t \longrightarrow (|t|, |t|^{-1})$, the factor-switch operator τ switches two factors of $\mathbb{C}^2$. It follows that $\widetilde{\mathscr{S}}(V)$ as a quotient of $\mathbb{C}^2$ as an $O(V)$-module is either 0 or isomorphic $\mathbb{C}^2$. In either case, the action of $\mathrm{SL}_2(F)$ is trivial as it is simple and commutes with action of $O(V)$. If $\widetilde{\mathscr{S}}(V) \simeq \mathbb{C}^2$, then $w \in \mathrm{SL}_2(F)$ has the following formula acting on $\mathbb{C}^2$: write $\phi = \phi_1 \otimes \phi_2$ then we have

$$L(\phi) = \left(\phi_2(0)\widehat{\phi}_1(0), \phi_1(0)\widehat{\phi}_2(0)\right)$$

$$L(w\phi) = \gamma(q)\left(\phi_2(0)\widehat{\phi}_1(0), \phi_1(0)\widehat{\phi}_2(0)\right).$$

It follows that w acts on $\mathbb{C}^2$ with formula

$$(x,y) \mapsto \gamma(x,y).$$

Thus w is non-trivial on $\mathbb{C}^2$ if $\gamma \neq 1$, a contradiction. $\square$

Proof of Theorem 2. We will use the same strategy as in the proof of Theorem 1.

The proof under the first two conditions is easy. When $m = 1$, we may use Proposition 6. For large m, in either of these two case, the orthogonal complement of $W(x')$ satisfies condition 1. So the induction step in the proof of Theorem 1 holds by Proposition 6.

It remains to deal with condition 3: $\dim V = 4$, $W = 0$. We may assume that $m \geq 3$ by repeating some α_i. First we want to show the following claim: *There is $\phi \in \mathscr{S}(V^3)$ with supported on elements $(x_1, x_2, x_3) \in V^3$ which are linearly independent and have nonzero norms.*

We want to construct such ϕ step by step. The first step is to use Proposition 4 to find a $\phi_1 \in \mathscr{S}(V_{q\neq 0})$ with nonzero image in σ_1 . The second step is to use the fact that $\mathscr{S}(V)$ is the sum of $\mathscr{S}(V \setminus 0)$ and $w\mathscr{S}(V \setminus 0)$. The proof of Theorem 1 in case $m = 1$ will implies the existence of $\phi_2 \in \mathscr{S}(V^2)$ with nonzero image in $\sigma_1 \otimes \sigma_2$ with support on points $(x_1, x_2) \in V^2$ which are linearly independent with nonzero norm. If $Fx_1 + Fx_2$ is non-degenerate, then the complement has dimension 2 thus we can use Proposition 6 to finish the proof. Otherwise, we can embedded $Fx_1 + Fx_2$ into a non-degenerate hyperplane W. The proof of Theorem 1 then gives a $\phi \in \mathscr{S}(V^3)$ supported on points (x_1, x_2, x_3) which are linearly independent with nonzero norms. This proves the claim.

If $M = Fx_1 + Fx_2 + Fx_3$ is non-degenerate then we are done by proof of Theorem 1. Otherwise, the one dimensional kernel N of the pairing on M is included in M. Thus two x_i's will generate the module M/N, say x_1, x_2 after reordering. Now the orthogonal complement of non-degenerate $W := Fx_1 + Fx_2$ is two dimensional. The proof Theorem 1 and Proposition 6 will finish the rest of proof. $\square$

2.8 First Decomposition

Fix an incoherent quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ with ramification set Σ. We assume that $\mathbb{B}$ has totally positive component $\mathbb{B}_v$ at archimedean places. We consider the Eisenstein series $E(g, s, \phi)$ for $\phi \in \mathscr{S}(\mathbb{B}^3)$. We always take ϕ_∞ to be standard Gaussian. In this case this Eisenstein series vanishes at $s = 0$.

For $T \in \mathrm{Sym}_3(F)_{reg}$, let $\Sigma(T)$ be the set of places over which T is anisotropic. Then $\Sigma(T)$ has even cardinality and the vanishing order of $E_T(g, s, \phi)$ at $s =$ is at least

$$|\Sigma \cup \Sigma(T)| - |\Sigma \cap \Sigma(T)|.$$

Since $|\Sigma|$ is odd, $E_T(g, s, \phi)$ always vanishes at $s = 0$. And its derivative is non-vanishing only if Σ and $\Sigma(T)$ is nearby: they differ by precisely one place v. Thus we define

$$\Sigma(v) = \begin{cases} \Sigma \setminus \{v\} & \text{if } v \in \Sigma \\ \Sigma \cup \{v\} & \text{otherwise} \end{cases}$$

When $\Sigma(T) = \Sigma(v)$, the derivative is given by

$$E'_T(g, 0, \phi) = \prod_{w \neq v} W_{T,w}(g_w, 0, \phi_w) \cdot W'_{T,v}(g_v, 0, \phi_v).$$

We thus obtain a decomposition of $E'(g,0,\phi)$ according to the difference of $\Sigma(T)$ and Σ:

$$E'(g,0,\phi) = \sum_v E'_v(g,0,\phi) + E'_{sing}(g,0,\phi) \tag{2.8.1}$$

where

$$E'_v(g,0,\phi) = \sum_{\Sigma(T)=\Sigma(v)} E'_T(g,0,\phi) \tag{2.8.2}$$

and

$$E'_{sing}(g,0,\phi) = \sum_{T,\,\det(T)=0} E'_T(g,0,\phi).$$

2.9 Holomorphic projection

In this section, we want to study holomorphic projection of $E'(g,0,\phi)$.

Firstly let us try to study holomorphic projection for a cusp form φ on $\mathrm{GL}_2(\mathbb{A})$. Fix a non-trivial additive character ψ of $F\backslash\mathbb{A}$, say $\psi = \psi_0 \circ \mathrm{tr}_{F/\mathbb{Q}}$ with ψ_0 the standard additive character on $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$, and let W be the corresponding Whittacker function:

$$W_\varphi(g) = \int_{F\backslash\mathbb{A}} \varphi(n(b)g)\psi(-b)db.$$

Then φ has a Fourier expansion

$$\varphi(g) = \sum_{a\in F^\times} W_\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\right).$$

We say that φ is *holomorphic of weight* 2 , if $W_\phi = W_\infty \cdot W_f$ has a decomposition with W_∞ satisfying the following properties:

$$W_\infty(g) = \begin{cases} ye^{2\pi i(x+iy)}e^{2i\theta} & \text{if } y>0 \\ 0 & \text{otherwise}\end{cases}$$

for the decomposition of $g \in \mathrm{GL}_2(\mathbb{R})$:

$$g = z\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}.$$

For any Whittacker function W of $\mathrm{GL}_2(\mathbb{A})$ which is holomorphic of weight 2 as above with $W_f(g_f)$ compactly supported modulo $Z(\mathbb{A}_f)N(\mathbb{A}_f)$, the Poinaré series is define as follows:

$$\varphi_W(g) := \lim_{t\to 0+} \sum_{\gamma\in Z(F)N(F)\backslash G(F)} W(\gamma g)\delta(\gamma g)^t,$$

where

$$\delta(g) = |a_\infty/d_\infty|, \qquad g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k, \qquad k \in K$$

where K is the standard maximal compact subgroup of $\mathrm{GL}_2(\mathbb{A})$. Let φ be a cusp form and assume that both W and φ have the same central character. Then we can compute their inner product as follows:

$$\begin{aligned}
(\varphi, \varphi_W) &= \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g)\bar{\varphi}_W(g)dg \\
&= \lim_{t \longrightarrow 0} \int_{Z(\mathbb{A})N(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g)\bar{W}(g)\delta(g)^t dg \\
&= \lim_{t \longrightarrow 0} \int_{Z(\mathbb{A})N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} W_\varphi(g)\bar{W}(g)\delta(g)^t dg. \qquad (2.9.1)
\end{aligned}$$

Let φ_0 be the holomorphic projection of φ in the space of holomorphic forms of weight 2. Then we may write

$$W_{\phi_0}(g) = W_\infty(g_\infty)W_{\varphi_0}(g_f)$$

with W_∞ as above. Then (11.1) is a product of integrals over finite places and integrals at infinite places:

$$\int_{Z(\mathbb{R})N(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})} |W_\infty(g_\infty)|^2 dg = \int_0^\infty y^2 e^{-4\pi y} dy/y^2 = (4\pi)^{-1}.$$

In other words, we have

$$(\varphi, \varphi_W) = (4\pi)^{-g} \int_{Z(\mathbb{A}_f)N(\mathbb{A}_f)\backslash \mathrm{GL}_2(\mathbb{A}_f)} W_{\varphi_0}(g_f)\bar{W}(g_f)dg_f. \qquad (2.9.2)$$

As $\bar{W}$ can be any Whittacker function with compact support modulo $Z(\mathbb{A}_f)N(\mathbb{A}_f)$, the combination of (10.1) and (10.2) gives

Lemma 2.9.1. *Let φ be a cusp form with trivial central character at each infinite place. Then the holomorphic projection φ_0 of ϕ has Wittacher function $W_\infty(g_\infty)W_{\varphi_0}(g_f)$ with $W_{\varphi_0}(g_f)$ given as follows:*

$$W_{\varphi_0}(g_f) = (4\pi)^g \lim_{t \longrightarrow 0+} \int_{Z(F_\infty)N(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} W_\varphi(g_\infty g_f)\bar{W}_\infty(g_\infty)\delta(g_\infty)^t dg_\infty.$$

3 Local Whittacker integrals

In this section, we study the derivative of Eisenstein series for Schwartz function $\phi \in \mathscr{S}(\mathbb{B}^3)$ on an incoherent (adelic) quaternion algebra $\mathbb{B}$ over adeles $\mathbb{A}$ of a number field F. We will first study the non-singular Fourier coefficients T. We show that these coefficients are non-vanishing only if T is represented by elements in $\mathbb{B}$ if we remove one factor at a place v, see formula (3.1.2). In this case, the fourier coefficient can be computed by taking derivative at the local Whittacker functions at v, see Proposition 3.2.2. Our second main result is that the singular Fourier coefficients vanish if the Schwarts function are supported on regular sets for two places of F.

3.1 Nonarchimedeanl local Whittaker integral

Let F be a nonarchimedean local field with integer ring $\mathscr{O}$ whose residue field has odd characteristic p. And let ϖ be a uniformizer and $q = |\mathscr{O}/(\varpi)|$ be the cardinality of the residue field. Assume further that the additive character ψ is unramified.

Now we recall some relevant results about Whittaker integral and local density. Let $B = M_2(F)$ and ϕ_0 the characteristic function of $M_2(\mathscr{O}_F)$. Let $T \in \mathrm{Sym}_3(\mathscr{O})$. It is a fact that $W_T(e, s, \phi_0)$ is a polynomial of $X = q^{-s}$. Moreover, there are two polynomial $\gamma_v(T, X)$ and $F_v(T, X)$ such that

$$W_T(e, s, \phi_0) = \gamma_v(T, q^{-2s}) F_v(T, q^{-2s})$$

where

$$\gamma_v(T, X) = (1 - q^{-2}X)(1 - q^{-2}X^2).$$

To describe $F_v(T, X)$ we need several invariants of $T \in \mathrm{Sym}_3(\mathscr{O}_v)$. Suppose that $T \sim \mathrm{diag}[u_i \varpi^{a_i}]$ with $a_1 < a_2 < a_3$. Then we define the following invariants:

- $GK(T) = (a_1, a_2, a_3)$;
- $\eta(T)$ is defined to be 1 (resp. -1) depending on T is isotropic (resp. anisotropic).
- $\xi(T)$ is defined to be $\left(\frac{-u_1 u_2}{\varpi}\right) = (-u_1 u_2, \varpi)$ if $a_1 \equiv a_2 \,(mod\, 2)$ and $a_2 < a_3$, otherwise zero.
- $\sigma(T) = 2$ if $a_1 \equiv a_2 \,(mod\, 2)$, otherwise 1.

Theorem 3.1.1 ([17]). *Assume that $p > 2$.*

$$
\begin{aligned}
F_v(T,X) = &\sum_{i=0}^{a_1} \sum_{j=0}^{(a_1+a_2-\sigma)/2-i} q^{i+j} X^{i+2j} \\
&+ \eta \sum_{i=0}^{a_1} \sum_{j=0}^{(a_1+a_2-\sigma)/2-i} q^{(a_1+a_2-\sigma)/2-j} X^{a_3+\sigma+i+2j} \\
&+ \xi^2 q^{(a_1+a_2-\sigma)/2+1} \sum_{i=0}^{a_1} \sum_{j=0}^{a_3-a_2+2\sigma-4} \xi^j X^{a_2-\sigma+2+i+j}.
\end{aligned}
$$

In [17], this is proved for $F = \mathbb{Q}_p$. But the method of course extends to general F.

Two corollaries now follow immediately. Firstly, we have a formula for the central value of Whittaker integral $W_{T,v}(e, 0, \phi_0)$.

Corollary 3.1.2. *The Whittaker function at $s = 0$ is given by*

$$
W_{T,v}(e, 0, \phi_0) = (1 - q^{-2})^2 \beta_v(T)
$$

where

1. *When T is anisotropic, we have*
$$\beta_v = 0.$$

2. *When T is isotropic, we have three cases*

   (a) *If $a_1 \neq a_2 \bmod 2$, we have*
$$
\beta_v(T) = 2\Big(\sum_{i=0}^{a_1} (1+i) q^i + \sum_{i=a_1+1}^{(a_1+a_2-1)/2} (a_1+1) q^i\Big).
$$

   (b) *If $a_1 \equiv a_2 \bmod 2$ and $\xi = 1$, we have*
$$
\begin{aligned}
\beta_v(T) =& 2\Big(\sum_{i=0}^{a_1} (i+1) q^i + \sum_{i=a_1+1}^{(a_1+a_2-2)/2} (a_1+1) q^i\Big) \\
&+ (a_1+1)(a_3-a_2+1) q^{(a_1+a_2)/2}.
\end{aligned}
$$

   (c) *If $a_1 \equiv a_2 \bmod 2$ and $\xi = -1$, we have*
$$
\begin{aligned}
\beta_v(T) =& 2\Big(\sum_{i=0}^{a_1} (i+1) q^i + \sum_{i=a_1+1}^{(a_1+a_2-2)/2} (a_1+1) q^i\Big) \\
&+ (a_1+1) q^{(a_1+a_2)/2}.
\end{aligned}
$$

Proof. In fact, we obtain that where

$$\beta_v(T) = (1+\eta)(\sum_{i=0}^{a_1}(i+1)q^i + \sum_{i=a_1+1}^{(a_1+a_2-\sigma)/2}(a_1+1)p^i) \\ + p^{(a_1+a_2-\sigma+2)/2}(a_1+1)R_\xi$$

where

$$R_\xi = \begin{cases} 0, & \text{if } \xi = 0, \\ & \text{or } \xi = -1 \text{ and } a_3 \neq a_2 \mod 2 \\ a_3 - a_2 + 2\sigma - 3, & \text{if } \xi = 1 \\ 1, & \text{if } \xi = -1 \text{ and } a_3 \equiv a_2 \mod 2 \end{cases}$$

□

The second consequence is a formula of the central derivative $W'_{T,v}(e,0,\phi_0)$. Since the central value $W_{T,v}(e,0,\phi_0) = 0$, we have

$$W'_{T,v}(e,0,\phi_0) = \gamma(T,1)\frac{\partial}{\partial X}F_v(T,X)|_{X=1}.$$

Note that $\gamma(T,1) = (1-p^{-2})^2$. It can be verified that

$$\nu(T) := \frac{\partial}{\partial X}F_v(T,X)|_{X=1}$$

is given as follows: let $T \sim \mathrm{diag}[t_i]$ with $a_i = \mathrm{ord}(t_i)$ in the order $a_1 \leq a_2 \leq a_3$, then

1. If $a_1 \neq a_2 \, mod \, 2$, we have

$$\nu(T) = \sum_{i=0}^{a_1}(1+i)(3i - a_1 - a_2 - a_3)q^i + \sum_{i=a_1+1}^{(a_1+a_2-1)/2}(a_1+1)(4i - 2a - 1 - a_2 - a_3)q^i.$$

2. If $a_1 \equiv a_2 \, mod \, 2$, we have $a_2 \neq a_3 \, mod \, 2$ hence

$$\nu(T) = \sum_{i=0}^{a_1}(i+1)(3i - a_1 - a - 2 - a_3)q^i \\ + \sum_{i=a_1+1}^{(a_1+a_2-2)/2}(a_1+1)(4i - 2a - 1 - a_2 - a_3)q^i \\ - \frac{a_1+1}{2}(a_3 - a_2 + 1)q^{(a_1+a_2)/2}.$$

Theorem 3.1.3. *Let ϕ_0 be the characteristic function of the maximal order of $B_v = M_2(\mathbb{Q}_p)$. Let ϕ_0' be the characteristic function of maximal order of the (ramified) quaternion algebra B_v'. Suppose that T is anisotropic. Then we have (cf. Theorem 5.3.5)*

$$\nu_v(T)\log q = \frac{W_T'(e,0,\phi_0)}{W_T(e,0,\phi_0')}.$$

This follows from formulae above and the following lemma

Lemma 3.1.4. *Let T be anisotropic. We have*

$$W_T(e,0,\phi_0') = 2(1+q)(1+q^{-1}).$$

3.2 Archimedean Whittaker integral

We want to compute the Whittaker integral $W_T(g,\phi,g,s)$ when $F=\mathbb{R}$, $B=\mathbb{H}$, and

$$\phi(x) = e^{-2\pi tr(Q(x))}, \quad x \in B^3 = \mathbb{H}^3.$$

It is K_∞-invariant. Recall that the additive character

$$\psi(x) = e^{2\pi i x}, \quad x \in \mathbb{R}.$$

Lemma 3.2.1. *Let $g = n(b)m(a)k$ be the Iwasawa decomposition. Then*

$$W_T(g,s,\phi_\infty) = \psi(Tb)\lambda_s(m({}^ta^{-1}))|det(a)|^4 W_{{}^taTa}(e,s,\phi_\infty).$$

Proof. By definition the Whittaker integral is

$$\begin{aligned}
&W_T(g,s,\phi_\infty)\\
=&\int_{Sym_3(\mathbb{R})}\psi(-Tu)r(wn(u)g)\phi_\infty(0)\lambda_s(wn(u)g)du\\
=&\psi(Tb)\int_{Sym_3(\mathbb{R})}\psi(-Tu)r(wn(u)m(a))\phi_\infty(0)\lambda_s(wn(u)m(a))du
\end{aligned}$$

Since $wn(u)m(a) = wm(a)n(a^{-1}u{}^ta^{-1}) = m({}^ta^{-1})wn(a^{-1}u{}^ta^{-1})$, we obtain

$$\begin{aligned}
&W_T(g,s,\phi_\infty)\\
=&\psi(Tb)\lambda_s(m({}^ta^{-1}))\int_{Sym_3(\mathbb{R})}\psi(-Tu)r(wn(a^{-1}u{}^ta^{-1}))\phi_\infty(0)\lambda_s(wn(a^{-1}u{}^ta^{-1}))du\\
=&\psi(Tb)\lambda_s(m({}^ta^{-1}))|det(a)|^4\int_{Sym_3(\mathbb{R})}\psi(-Tau{}^ta)r(wn(u))\phi_\infty(0)\lambda_s(wn(u))du\\
=&\psi(Tb)\lambda_s(m({}^ta^{-1}))|det(a)|^4 W_{{}^taTa}(e,s,\phi_\infty).
\end{aligned}$$

∎

Thus it suffices to consider only $g = e$ the identity element of $Sp_6(\mathbb{R})$:

$$W_T(e,s,\phi) = \int_{Sym_3(\mathbb{R})} \psi(-Tu)det(1+u^2)^{-s} r(wu)\phi(0)du.$$

Lemma 3.2.2. *When $Re(s) \gg 0$, we have*

$$W_T(e,s,\phi) = \frac{\pi^6}{2^6}\int_{Sym_3(\mathbb{R})} \psi(-Tu)det(1+iu)^{-s}det(1-iu)^{-s-2}du.$$

Proof. By definition we have

$$r(wu)\phi(0) = \int_{\mathbb{H}^3} \psi(uQ(x))\phi(x)dx.$$

Let $u = k \cdot diag(u_1,u_2,u_3)^t k$ for $k \in SO(3)$ be the Cartan decomposition. We have

$$r(wu)\phi(0) = \int_{\mathbb{H}^3} \psi(uQ(x))\phi(x)dx = \int_{\mathbb{H}^3} \psi((u_i)\cdot^t kQ(x)k)e^{-2\pi tr(Q(x))}dx.$$

Substitute $x \mapsto xk$ and note that ${}^tkQ(x)k = Q(xk)$,

$$r(wu)\phi(0) = \prod_{j=1}^{3}\int_{\mathbb{H}} e^{\pi(iu_j-1)q(x_j)}dx_j = (\frac{\pi^2}{4})^3\prod_{j}^{3}\frac{1}{(1-iu_j)^2}.$$

Equivalently,

$$r(wu)\phi(0) = \frac{\pi^6}{64}det(1-iu)^{-2}.$$

Since $det(1+u^2) = det(1-iu)det(1+iu)$, the lemma now follows. □

Now we introduce a function ([31, pp.274]) for $g, h \in Sym_n(\mathbb{R})$ and $\alpha, \beta \in \mathbb{C}$

$$\eta(g,h;\alpha,\beta) = \int_{x>\pm h} e^{-gx}det(x+h)^{\alpha-2}det(x-h)^{\beta-2}dx$$

which is absolutely convergent when $g > 0$ and $Re(\alpha), Re(\beta) > \frac{n}{2}$. Here we point out that dx is the Euclidean measure viewing $Sym_n(\mathbb{R})$ as $\mathbb{R}^{n(n+1)/2}$ naturally. This measure is not self-dual but only up to a constant $2^{n(n-1)/4}$. In the following we always use the Euclidean measure as [31] does. For two elements $h_1, h_2 \in Sym_n(\mathbb{R})$, by $h_1 \sim h_2$ we mean that $h_1 = kh_2k^{-1}$ for some $k \in O(n)$, the real orthogonal group for the standard positive definite quadratic space.

Lemma 3.2.3. *When $Re(s) > 1$, we have*

$$W_T(e,s,\phi) = \kappa(s)\Gamma_3(s+2)^{-1}\Gamma_3(s)^{-1}\eta(2\pi, T; s+2, s)$$

where

$$\kappa(s) = \frac{\pi^{6s+12}}{2^3}$$

Proof. Before we proceed let us recall some well-known results. Let $z \in Sym_n(\mathbb{C})$ with $Re(z) > 0$, then we have for $s \in \mathbb{C}$ with $Re(s) > \frac{n-1}{2}$,

$$\int_{Sym_n(\mathbb{R})_+} e^{-tr(zx)} det(x)^{s-\frac{n+1}{2}} dx = \Gamma_n(s) det(z)^{-s} \tag{3.2.1}$$

where

$$\Gamma_n(s) = \pi^{\frac{n(n-1)}{4}} \Gamma(s)\Gamma(s-\frac{1}{2})...\Gamma(s-\frac{n-1}{2}).$$

For instance, when $n = 1$, we have when $Re(z) > 0$ and $Re(s) > 0$

$$\int_{\mathbb{R}_+} e^{-zx} x^{s-1} dx = \Gamma(s) z^{-s}.$$

Applying 3.2.1 to $z = v + 2\pi iu$ for $u, v \in \mathbb{R}$, we obtain when $Re(s) > \frac{n-1}{2}$,

$$\widehat{f}(u) = \Gamma_n(s) det(v + 2\pi iu)^{-s}$$

where

$$f(x) = \begin{cases} e^{-vx} det(x)^{s-\frac{n+1}{2}} & x > 0 \\ 0 & otherwise \end{cases}.$$

Take inverse Fourier transformation, we obtain

$$\int_{Sym_n(\mathbb{R})} e^{2\pi iux} det(v + 2\pi iu)^{-s} du = \begin{cases} \frac{1}{2^{n(n-1)/2}\Gamma_n(s)} e^{-vx} det(x)^{s-\frac{n+1}{2}} & x > 0 \\ 0 & otherwise. \end{cases} \tag{3.2.2}$$

By 3.2.1 above for $n = 3$, we have

$$W_T(e, s, \phi) = \frac{\pi^6}{2^6} \frac{\pi^{3s+6}}{\Gamma_3(s+2)} \int_{Sym_3(\mathbb{R})} e^{-2\pi i Tu} det(1 + iu)^{-s} \int_{Sym_3(\mathbb{R})_+} e^{-\pi(1-iu)x} det(x)^s dx 2^{3/2} du.$$

Here du is changed to the Euclidean measure and the constant multiple $2^{3/2}$ comes from the ratio between the self-dual measure and the Euclidean one. Interchange the order of the two integrals

$$\frac{2^{3/2}\pi^6}{2^6} \frac{\pi^{3s+6}}{\Gamma_3(s+2)} \int_{Sym_3(\mathbb{R})_+} e^{-\pi x} det(x)^s \left(\int_{Sym_3(\mathbb{R})} e^{2\pi iu(\frac{1}{2}x-T)} det(1+iu)^{-s} du \right) dx.$$

By 3.2.2 again for $n = 3$, we obtain

$$\frac{2^{3/2}\pi^6}{2^6} \frac{\pi^{3s+6}}{\Gamma_3(s+2)} \int_{x>0, x>2T} e^{-\pi x} det(x)^s \frac{(2\pi)^6}{2^3\Gamma_3(s)} e^{-2\pi(\frac{x}{2}-T)} det(2\pi(\frac{x}{2} - T))^{s-2} dx.$$

which yields:

$$\frac{\pi^{6s+12}}{2^{3/2}\Gamma_3(s+2)\Gamma_3(s)} \int_{x>0, x>2T} e^{-2\pi(x-T)} det(x)^s det(x - 2T)^{s-2} dx.$$

Finally we may substitute $x \to T + x$ to complete the proof. ∎

We define ([31, pp.280,(3.2)]) for $z \in Sym_n(\mathbb{C})_+$ (z with $Re(z) > 0$)

$$\zeta_n(z, \alpha, \beta) = \int_{Sym_n(\mathbb{R})_+} e^{-zx} det(x+1)^{\alpha - \frac{n+1}{2}} det(x)^{\beta - \frac{n+1}{2}} dx. \tag{3.2.3}$$

Lemma 3.2.4 (Shimura)**.** *For $z \in Sym_n(\mathbb{C})$ with $Re(z) > 0$, the integral of $\zeta_n(z; \alpha, \beta)$ is absolutely convergent for $\alpha \in \mathbb{C}$ and $Re(\beta) > \frac{n-1}{2}$. And $\omega(z, \alpha, \beta) := \Gamma_n(\beta)^{-1} det(z)^\beta \zeta_n(z, \alpha, \beta)$ can be extended to a holomorphic function to $(\alpha, \beta) \in \mathbb{C}^2$.*

Proof. See [31, Thm. 3.1]. □

The following proposition gives an inductive way of computing the Whittaker integral $W_T(e, s, \phi)$, or equivalently $\eta(2\pi, T; s+2, s)$.

Proposition 3.2.5. *Assume that $sign(T) = (p, q)$ with $p + q = 3$ so that we have $4\pi T \sim diag(a, -b)$ for $a \in \mathbb{R}^p_+, b \in \mathbb{R}^q_+$. Let $t = diag(a, b)$. Then we have*

$$\eta(2\pi, T; s+2, s) = 2^{6s} e^{-t/2} |det(T)|^{2s} \xi(T, s)$$

where

$$\xi(T, s) = \int_M e^{-(aW + bW')} det(1+W)^{2s} \zeta_p(ZaZ, s+2, s - \frac{3-p}{2})$$
$$\times \zeta_q(Z'bZ', s, s + \frac{q+1}{2}) dw.$$

where $M = \mathbb{R}^p_q$, $W = w \cdot^t w$, $W' =^t ww$, $Z = (1+W)^{1/2}$ and $Z' = (1+W')^{1/2}$.

Proof. We may assume that $4\pi T = kt'k^{-1}$ where $k \in O(3)$ and $t' = diag(a, -b)$. Then it is easy to see that

$$\eta(2\pi, T; s+2, s) = \eta(2\pi, t'/(4\pi); s+2, s) = |det(T)|^{2s} \eta(t/2, 1_{p,q}; s+2, s)$$

where $1_{p,q} = diag(1_p, -1_q)$.

By [31, pp.289, (4.16),(4.18),(4.24)], we have

$$\eta(2\pi, T; s+2, s) = 2^{6s} e^{-t/2} |det(T)|^{2s} \xi(T, s).$$

□

Corollary 3.2.6. *Suppose that $sign(T) = (p, q)$ with $p + q = 3$. Then $W_T(e, s, \phi)$ is holomorphic at $s = 0$ with vanishing order*

$$ord_{s=0} W_T(e, s, \phi) \geq [\frac{q+1}{2}].$$

Proof. By Proposition 3.2.5, we know that

$$W_T(e,s,\phi) \sim \frac{\Gamma_p(s-\frac{3-p}{2})\Gamma_q(s+\frac{q+1}{2})}{\Gamma_3(s+2)\Gamma_3(s)} \int_F e^{-(aW+bW^*)} det(1+W)^{2s}$$
$$\times \frac{1}{\Gamma_p(s-\frac{3-p}{2})}\zeta_p(ZaZ; s+2, s-\frac{3-p}{2})\frac{1}{\Gamma_q(s+\frac{q+1}{2})}\zeta_q(Z'bZ'; s, s+\frac{q+1}{2})dw$$

where $\sim$ means up to nowhere vanishing entire function. Lemma 3.2.4 implies that the latter two factors in the integral are holomorphic in the whole $\mathbb{C}$. Thus we obtain that

$$ord_{s=0}W_T(e,s,\phi) \geq ord_{s=0}\frac{\Gamma_p(s-\frac{3-p}{2})\Gamma_q(s+\frac{q+1}{2})}{\Gamma_3(s+2)\Gamma_3(s)} = [\frac{q+1}{2}].$$

□

Remark 6. 1. The same argument also applies to higher rank Whittaker integral. More precisely, let V be the $n+1$-dimensional positive definite quadratic space and ϕ_0 be the standard Gaussian $e^{-2\pi tr(x,x)}$ on V^n. Then for T non-singular we have

$$order_{s=0}W_T(e,s,\phi_0) \geq ord_{s=0}\frac{\Gamma_p(s-\frac{n-p}{2})\Gamma_q(s+\frac{q+1}{2})}{\Gamma_n(s+\frac{n+1}{2})\Gamma_n(s)} = [\frac{n-p+1}{2}] = [\frac{q+1}{2}].$$

And it is easy to see that when $T>0$ (namely, represented by V), $W_T(e,0,\phi_0)$ is non-vanishing. One immediately consequence is that: $W_T(e,s,\phi_0)$ vanishes with order precisely one at $s=0$ only if the quadratic space with signature $(n-1,2)$ represents T. We will see by concrete computation for $n=3$ that the formula above actually gives the exact order of vanishing at $s=0$. It should be true for general n but we have not tried to verify this.

Proposition 3.2.7. *When $T>0$, we have*

$$W_T(e,0,\phi) = \frac{\pi^{12}}{2^{3/2}\Gamma_3(2)}e^{-2\pi T}.$$

Proof. Near $s=0$, we have

$$\begin{aligned}
&\eta(2\pi, T; s+2, s)\\
=&e^{-2\pi T}\int_{x>0} e^{-2\pi x} det(x+2T)^s det(x)^{s-2}dx\\
=&e^{-2\pi T}\left(\int_{x>0} e^{-2\pi x} det(2T)^s det(x)^{s-2}dx + O(s)\right)\\
=&e^{-2\pi T}\left(det(2T)^s(2\pi)^{-3s}\Gamma_3(s)+O(s)\right)
\end{aligned}$$

Note that

$$\Gamma_3(s+2)\Gamma_3(s) = \pi^3\Gamma(s+2)\Gamma(s+\frac{3}{2})\Gamma(s+1)\Gamma(s)\Gamma(s-\frac{1}{2})\Gamma(s-1).$$

This has a double pole at $s = 0$. Thus when $s = 0$, we obtain

$$W_T(e, s, \phi) = \frac{\pi^{12}}{2^{3/2}\Gamma_3(2)} e^{-2\pi T}.$$

□

3.3 Indefinite Whittaker integrals

Now we consider a non-definite T. We will find certain nice integral representations of the central derivative of the Whittaker integral $W_T(e, s, \phi)$ in the sequel when the sign of T is $(p, q) = (1, 2)$ or $(2, 1)$ respectively.

Case $(p, q) = (1, 2)$

Proposition 3.3.1. *Suppose that $4\pi T \sim diag(a, -b)$, $b = diag(b_1, b_2)$. Then we have*

$$W_T'(e, 0, \phi) = -\frac{\kappa(0)}{2\sqrt{\pi}\Gamma_3(2)} e^{t/2} \int_{\mathbb{R}^2} e^{-(a(1+w^2)+b_1(1+w_1^2)+b_2(1+w_2^2))} \zeta_2(diag(z_1, z_2), 0, \frac{3}{2})$$
$$\times \big(a(1 + w^2) - 1\big)\, dw_1 dw_2.$$

where (z_1, z_2) are the two eigenvalues of $b(1 + w'w)$ and $w^2 = w_1^2 + w_2^2$.

Proof. Recall by Prop. 3.2.5

(3.3.1) $$W_T(e, s, \phi) = \kappa(s)\Gamma_3(s + 2)^{-1}\Gamma_3(s)^{-1}2^{6s}e^{-t/2}|det(T)|^{2s}\xi(T, s)$$

where

$$\xi(T, s) = \int_{\mathbb{R}^2} e^{-(aW+bW^*)} det(1 + W)^{-2s} \zeta_p(ZaZ; s + 2, s - \frac{3 - p}{2})$$
$$\times \zeta_q(Z'bZ'; s, s + \frac{q + 1}{2})dw.$$

When $(p, q) = (1, 2)$, $\zeta_1(ZaZ; s + 2, s - 1)$ has a simple pole at $s = 0$. We here recall a fact that will be used frequently later, namely $\zeta_1(z; \alpha, \beta)$ has a recursive property ([31, pp. 282,(3.14)])

(3.3.2) $$\beta\zeta_1(z, \alpha, \beta) = z\zeta_1(z, \alpha, \beta + 1) - (\alpha - 1)\zeta_1(z, \alpha - 1, \beta + 1).$$

Using repeatedly

$$(s - 1)\zeta_1(z, s + 2, s - 1) = z\zeta_1(z, s + 2, s) - (s + 1)\zeta_1(z, s + 1, s),$$

$$s\zeta_1(z, s + 2, s) = z\zeta_1(z, s + 2, s + 1) - (s + 1)\zeta_1(z, s + 1, s + 1),$$

$$s\zeta_1(z, s + 1, s) = z\zeta_1(z, s + 1, s + 1) - s\zeta_1(z, s, s + 1),$$

we obtain the residue at $s = 0$

$$Res_{s=0}\zeta_1(z, s+2, s-1) = -(z^2\zeta_1(z, 2, 1) - 2z\zeta_1(z, 1, 1)).$$

It is easy to see that

$$\zeta_1(z, 1, 1) = \int_{\mathbb{R}_+} e^{-zx} dx = \frac{1}{z}$$

and

$$\zeta_1(z, 2, 1) = \int_{\mathbb{R}_+} e^{-zx}(x+1) dx = \frac{1}{z} + \frac{1}{z^2}.$$

Thus, we have

$$Res_{s=0}\zeta_1(z, s+2, s-1) = -z + 1.$$

Suppose that $w = (w_1, w_2)$ and $b = (b_1, b_2)$. Note that $\Gamma_3(s)$ has a double pole at $s = 0$ with leading Laurent coefficient $2\Gamma(1/2)s^{-2} = 2\sqrt{\pi}s^{-2}$. And $Tr(t) = a + b_1 + b_2$:

$$W_T'(e, 0, \phi_\infty) = -\frac{\kappa(0)}{2\sqrt{\pi}\Gamma_3(2)} e^{t/2} \int_F e^{-(a(1+w^2)+b_1(1+w_1^2)+b_2(1+w_2^2))}\zeta_2(diag(z_1, z_2), 0, \frac{3}{2})$$
$$\times (ZaZ - 1)\, dw_1 dw_2.$$

Finally note that $Z = (1 + w^2)^{1/2}$.

$\square$

The next result involves the exponential integral Ei defined by

$$-Ei(-z) = \int_0^\infty \frac{e^{-z(1+t)}}{1+t} dt = e^{-z}\zeta_1(z, 0, 1), \quad z \in \mathbb{R}_+. \tag{3.3.3}$$

It satisfies

$$\frac{d}{dz} Ei(z) = \frac{e^z}{z}$$

and

$$Ei(z) = \gamma + log(-z) + \int_0^z \frac{e^t - 1}{t} dt$$

where γ is the Euler constant. Then it is easy to see that $Ei(z)$ has logarithmic singularity near 0.

Lemma 3.3.2. *For simplicity, we will denote*

$$F(w_1, w_2) = e^{-(b_1(1+w_1^2)+b_2(1+w_2^2))}\zeta((z_1, z_2), 0, 3/2) = e^{-(z_1+z_2)}\zeta((z_1, z_2), 0, 3/2). \tag{3.3.4}$$

Then we have

$$W_T'(e, 0, \phi) = \frac{\kappa(0)}{8\sqrt{\pi}\Gamma_3(2)} e^{t/2-a} \left(\int_{\mathbb{R}^2} Ei(-aw^2)(2w_1F_1 + 2w_2F_2 + (1 + w^2)\Delta F) dw - 4\pi F(0) \right)$$

where $F_i = \frac{\partial}{\partial w_i} F$ *and* $\Delta = \frac{\partial^2}{\partial w_1^2} + \frac{\partial^2}{\partial w_1^2}$ *is the Laplace operator.*

Proof. Note that

$$\Delta e^{-aw^2} = 4a(aw^2 - 1)e^{-aw^2}$$

$$\nabla Ei(-aw^2) = \frac{2e^{-aw^2}}{w^2}(w_1, w_2)$$

and

$$\Delta Ei(-aw^2) = -4ae^{-aw^2}.$$

We may thus rewrite our integral as

$$\begin{aligned}
&\int_{\mathbb{R}^2} e^{-a(1+w^2)} F(w)(a(1+w^2) - 1)dw_1 dw_2 \\
&= \int_{\mathbb{R}^2} e^{-a(1+w^2)}(aw^2 - 1)F(w)dw_1 dw_2 + \int_{\mathbb{R}^2} ae^{-a(1+w^2)} F(w)dw_1 dw_2 \\
&= 1/(4a) \int \Delta e^{-a(1+w^2)} F(w)dw - 1/4e^{-a} \int \Delta Ei(-aw^2)F(w)dw.
\end{aligned}$$

Applying Stokes theorem and noting that the function $Ei(z)$ has a logarithmic singularity near $z = 0$, the second term is equal to:

$$-1/4e^{-a}\left(\int Ei(-aw^2)\Delta F dw - lim_{r\to 0}\int_{C_r} \nabla Ei(-aw^2)F(w)nds\right)$$

where C_r is the circle of radius r centered at the origin. It is not hard to verify that this is simplified as

$$-1/4e^{-a}\left(\int Ei(-aw^2)\Delta F dw - 4\pi F(0)\right).$$

Again apply Stokes to the first term:

$$-1/(4a)\int \nabla e^{-a(1+w^2)} \cdot \nabla F = 1/2 \int e^{-a(1+w^2)}(w_1 F_1 + w_2 F_2)dw.$$

Note that $\nabla Ei(-aw^2) = \frac{2e^{-aw^2}}{w^2}(w_1, w_2)$, this term is

$$1/4e^{-a}\int \nabla Ei(-aw^2) \cdot (w^2 F_1, w^2 F_2).$$

Apply Stokes again:

$$-1/4e^{-a}\int Ei(-aw^2)(2w_1 F_1 + 2w_2 F_2 + w^2 \Delta F)dw.$$

Putting together:

$$\begin{aligned}
&\int_{\mathbb{R}^2} e^{-a(1+w^2)} F(w)(a(1+w^2) - 1)dw_1 dw_2 \\
&= -\frac{1}{4}e^{-a}\left(\int Ei(-aw^2)(2w_1 F_1 + 2w_2 F_2 + (1+w^2)\Delta F)dw - 4\pi F(0)\right).
\end{aligned}$$

□

In the following we want to evaluate $F(w)$ (3.3.4) and its various derivatives. First we deduce an integral expression of $\zeta_2(diag(z_1,z_2);0,\frac{3}{2})$ (recall (3.2.3)).

Lemma 3.3.3. *For $z=(z_1,z_2)\in Sym_2(\mathbb{C})_+$, we have*

$$\zeta_2(diag(z_1,z_2);0,\frac{3}{2}) = 2\int_{x>0}\int_{y>0} e^{-z_1x-z_2y}(x+1)^{-1}(y+1)^{-1}\frac{\sqrt{xy}}{\sqrt{(x+y+1)}}dxdy.$$

Proof. By definition $\zeta_2(diag(z_1,z_2);0,\frac{3}{2})$ is given by

$$\int_{x>0}\int_{y>0} e^{-z_1x-z_2y}\int_{|t|<\sqrt{xy}}((x+1)(y+1)-t^2)^{-3/2}dtdxdy.$$

Substitute $t\to t(x+1)^{1/2}(y+1)^{1/2}$

$$\int_{x>0}\int_{y>0} e^{-z_1x-z_2y}(x+1)^{-1}(y+1)^{-1}\int_{|t|<\frac{\sqrt{xy}}{\sqrt{(x+1)(y+1)}}}(1-t^2)^{-3/2}dtdxdy.$$

It is easy to calculate the inner integral

$$2[t(1-t^2)^{-1/2}]|_0^{\frac{\sqrt{xy}}{\sqrt{(x+1)(y+1)}}} = 2\frac{\sqrt{xy}}{\sqrt{(x+y+1)}}.$$

$\square$

Lemma 3.3.4. *We have*

$$\frac{\partial}{\partial w_1}F(w) = -4\Gamma(3/2)e^{-z_1-z_2}\int_{\mathbb{R}_+} e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}(\frac{w_1}{1+w^2}x+b_1w_1)dx \tag{3.3.5}$$

and

$$F_{11}(w) := \frac{\partial^2}{\partial w_1^2}F = -4\Gamma(3/2)e^{-z_1-z_2}\int_{\mathbb{R}_+} e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}A_{11}dx \tag{3.3.6}$$

where

$$\begin{aligned}A_{11} &= -2b_1w_1(\frac{w_1}{1+w^2}x+b_1w_1) + \frac{(1+w^2)-2w_1^2}{(1+w^2)^2}x+b_1\\ &+(-3/2)\frac{2b_1b_2w_1+2b_1w_1x}{(x+z_1)(x+z_2)}(\frac{w_1}{1+w^2}x+b_1w_1).\end{aligned}$$

Similar formula for w_2.

Proof. By Lemma 3.3.3, we have

$$\frac{\partial}{\partial w_1}F(w) = -2\frac{\partial z_1}{\partial w_1}\int e^{-z_1(1+x)-z_2(1+y)}\frac{\sqrt{xy}}{(y+1)\sqrt{1+x+y}}dxdy - 2\frac{\partial z_2}{\partial w_1}...$$

where we omit the similar term for z_2 and the integral is taken over $x, y \in \mathbb{R}_+$. And all integrals below are taken over $\mathbb{R}_+$ which we hence omit.

Let us compute the integral. Substitute $x \mapsto x(1+y)$:

$$\int e^{-z_1x-z_2y}\frac{\sqrt{xy}}{(y+1)\sqrt{1+x+y}}dxdy = \int e^{-z_1x(1+y)-z_2y}\frac{\sqrt{x(1+y)y}}{(y+1)\sqrt{1+x(1+y)+y}}(1+y)dxdy$$

which can be simplified:

$$\int e^{-z_1x-y(z_1x+z_2)}\frac{\sqrt{xy}}{\sqrt{1+x}}dxdy.$$

Substitute $y \mapsto y(z_1x+z_2)^{-1}$ and separate variables:

$$\int e^{-z_1x}\frac{\sqrt{x}}{\sqrt{1+x}(z_1x+z_2)^{3/2}}dx\int e^{-y}y^{1/2}dy.$$

Substitute $x \mapsto xz_1^{-1}$:

$$\frac{1}{z_1}\Gamma(3/2)\int e^{-x}\frac{\sqrt{x}}{\sqrt{x+z_1}(x+z_2)^{3/2}}dx.$$

Thus we have

$$\frac{\partial}{\partial w_1}F(w) = -2\Gamma(3/2)e^{-z_1-z_2}\int e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}\left(\frac{\partial}{\partial w_1}ln(z_1z_2)x + \frac{\partial}{\partial w_1}(z_1+z_2)\right)dx.$$

Note that $z_1z_2 = b_1b_2(1+w^2), z_1+z_2 = b_1(1+w_1^2)+b_2(1+w_2^2)$

$$\frac{\partial}{\partial w_1}ln(z_1z_2) = \frac{2w_1}{1+w^2}, \quad \frac{\partial}{\partial w_1}(z_1+z_2) = 2b_1w_1.$$

From this we deduce further that

$$F_{11}(w) = -4\Gamma(3/2)e^{-z_1-z_2}\int e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}A_{11}dx$$

where

$$A_{11} = -2b_1w_1\left(\frac{w_1}{1+w^2}x+b_1w_1\right) + \frac{(1+w^2)-2w_1^2}{(1+w^2)^2}x + b_1 + \left(-\frac{3}{2}\right)\frac{2b_1b_2w_1+2b_1w_1x}{(x+z_1)(x+z_2)}\left(\frac{w_1}{1+w^2}x+b_1w_1\right).$$

Similarly,

$$F_{22}(w) = -4\Gamma(3/2)e^{-z_1-z_2}\int e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}A_{22}dx$$

where

$$A_{22} = -2b_2w_2\left(\frac{w_2}{1+w^2}x+b_2w_2\right) + \frac{(1+w^2)-2w_2^2}{(1+w^2)^2}x + b_2 + \left(-\frac{3}{2}\right)\frac{2b_1b_2w_2+2b_2w_2x}{(x+z_1)(x+z_2)}\left(\frac{w_2}{1+w^2}x+b_2w_2\right).$$

$\square$

Proposition 3.3.5. *We have*

$$W_T'(e,0,\phi_\infty) = \frac{\kappa(0)}{8\sqrt{\pi}\Gamma_3(2)} e^{t/2-a}\left(-4\pi e^{-b_1-b_2}\zeta_2((b_1,b_2),0,3/2)+\xi(T)\right)$$

where

$$\xi(T) = -4\Gamma(3/2)\int_{\mathbb{R}^2} Ei(-aw^2)e^{-z_1-z_2}\Big(\int_{\mathbb{R}} e^{-u^2}\frac{-2(z_1+z_2-1-b_1-b_2)}{(u^2+z_1)^{1/2}(u^2+z_2)^{1/2}}du$$
$$+\int_{\mathbb{R}} e^{-u^2}\frac{(2z_1z_2-2b_1b_2-z_1-z_2)u^2+2z_1z_2(z_1+z_2-1-b_1-b_2)}{(u^2+z_1)^{1/2}(u^2+z_2)^{3/2}}du\Big)dw.$$

Proof. Recall that we have

$$W_T'(e,0,\phi) = \frac{\kappa(0)}{8\sqrt{\pi}\Gamma_3(2)} e^{t/2-a}\left(\int_{\mathbb{R}^2} Ei(-aw^2)(2w_1F_1+2w_2F_2+(1+w^2)\Delta F)dw - 4\pi F(0)\right).$$

By Lemma 3.3.4, we obtain that

$$\Delta F(w) = -4\Gamma(3/2)e^{-z_1-z_2}\int e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}A dx$$

where

$$A = -2\frac{b_1w_1^2+b_2w_2^2}{1+w^2}x - 2(b_1^2w_1^2+b_2^2w_2^2) + \frac{2}{(1+w^2)^2}x + b_1+b_2$$
$$+(-\frac{3}{2})\frac{2}{(x+z_1)(x+z_2)}\Big(\frac{b_1w_1^2+b_2w_2^2}{1+w^2}x^2+\frac{b_1b_2w^2}{1+w^2}x+(b_1^2w_1^2+b_2^2w_2^2)x+b_1b_2(b_1w_1^2+b_2w_2^2)\Big).$$

And

$$2(w_1F_1(w)+w_2F_2(w)) = -4\Gamma(3/2)e^{-z_1-z_2}\int e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}B dx$$

where

$$B = \frac{2w^2x}{1+w^2}+2(b_1w_1^2+b_2w_2^2).$$

Thus

$$(2(w_1F_1(w)+w_2F_2(w))+(1+w^2)\Delta F)(-4\Gamma(3/2))^{-1} = e^{-z_1-z_2}\int e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}C dx$$

where C is given by

$$-2(b_1w_1^2+b_2w_2^2-1)x-(b_1-b_2)(w_1^2-w_2^2)+(b_1(1+w_1^2)+b_2(1+w_2^2))+2b_1b_2w^2-2(b_1-b_2)^2w_1^2w_2^2$$
$$-2(b_1w_1^2+b_2w_2^2)(b_1(1+w_1^2)+b_2(1+w_2^2))-3(b_1-b_2)^2w_1^2w_2^2\frac{x}{(x+z_1)(x+z_2)}.$$

In the integral above we substitute $x \mapsto u^2$. Then our goal is to compare the integral in the RHS of the above with:

$$\int_{\mathbb{R}} e^{-u^2}\left(\frac{-2(z_1+z_2-1-b_1-b_2)}{(u^2+z_1)^{1/2}(u^2+z_2)^{1/2}}+\frac{(2z_1z_2-2b_1b_2-z_1-z_2)u^2+2z_1z_2(z_1+z_2-1-b_1-b_2)}{(u^2+z_1)^{1/2}(u^2+z_2)^{3/2}}\right)du$$

which is also equal to

$$\int e^{-u^2}\frac{-2(b_1w_1^2+b_2w_2^2-1)u^4-2(b_1w_1^2+b_2w_2^2)(z_1+z_2)u^2+(2b_1b_2w^2+z_1+z_2)u^2}{(u^2+z_1)^{1/2}(u^2+z_2)^{3/2}}du.$$

Note that we have changed the domain of integration from $x \in \mathbb{R}_+$ to $u \in \mathbb{R}$.

Thus it suffices to prove that the following integral vanishes:

$$(b_1-b_2)^2\int Ei(-aw^2)w_1^2w_2^2e^{-z_1-z_2}\int e^{-u^2}\frac{2u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}+\frac{3u^4}{((u^2+z_1)(u^2+z_2))^{5/2}}dudw$$
$$+(b_1-b_2)\int Ei(-aw^2)(w_1^2-w_2^2)e^{-z_1-z_2}\int e^{-u^2}\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}dudw.$$

By the definition $Ei(-aw^2) = -\int_1^\infty e^{-aw^2u}u^{-1}du$, it suffices to prove that the following integral vanishes

$$(b_1-b_2)\int e^{-aw^2-b_1w_1^2-b_2w_2^2}w_1^2w_2^2\int e^{-u^2}\frac{2u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}+\frac{3u^4}{((u^2+z_1)(u^2+z_2))^{5/2}}dudw$$
$$+\int e^{-aw^2-b_1w_1^2-b_2w_2^2}(w_1^2-w_2^2)\int e^{-u^2}\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}dudw.$$

We substitute $X = w_1^2+w_2^2$ and $Y = w_1^2-w_2^2$. Then we have

$$dXdY = 2w_1w_2dw_1dw_2 = \sqrt{X^2-Y^2}dw_1dw_2$$

and

$$\int e^{-aw^2-b_1w_1^2-b_2w_2^2}(w_1^2-w_2^2)\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}dw$$
$$=\int_{X\geq 0}\int_{-X\leq Y\leq X} e^{-(a+b_1/2+b_2/2)X-(b_1-b_2)Y/2}Y\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}\frac{dY}{\sqrt{X^2-Y^2}}dX.$$

We apply integration by parts to the inner integral

$$-\int_{X\geq 0}e^{-(a+b_1/2+b_2/2)X}\int_{-X\leq Y\leq X}e^{-(b_1-b_2)Y/2}\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}d\sqrt{X^2-Y^2}dX$$
$$=\int_{X\geq 0}e^{-(a+b_1/2+b_2/2)X}\int_{-X\leq Y\leq X}\sqrt{X^2-Y^2}e^{-(b_1-b_2)Y/2}\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}$$
$$\left(-\frac{b_1-b_2}{2}-\frac{3}{2}\frac{b_1-b_2}{2}\frac{u^2}{(u^2+z_1)(u^2+z_2)}\right)dYdX.$$

We may simplify it and plug back

$$
\begin{aligned}
& -(b_1-b_2)/2\int_{X\geq 0} e^{-(a+b_1/2+b_2/2)X}\int_{-X\leq Y\leq X}\sqrt{X^2-Y^2}e^{-(b_1-b_2)Y/2}\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}} \\
& (2+\frac{3u^2}{(u^2+z_1)(u^2+z_2)})dY\,dX \\
& = -(b_1-b_2)\int e^{-aw^2-b_1w_1^2-b_2w_2^2}w_1^2w_2^2(\frac{2u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}+\frac{3u^4}{((u^2+z_1)(u^2+z_2))^{5/2}})dw.
\end{aligned}
$$

This proves the desired vanishing result.

Finally note that $F(0)=e^{-b_1-b_2}\zeta_2((b_1,b_2),0,3/2)$ and we complete the proof of Proposition 3.3.5.

□

Case $(p,q)=(2,1)$

Lemma 3.3.6. *$\zeta_2([z_1,z_2];s+2,s-\frac{1}{2})$ has a simple pole at $s=0$ with residue given by*

$$
\frac{\sqrt{\pi}}{2}\left(\int_{\mathbb{R}} e^{-u^2}\frac{(4z_1z_2-(z_1+z_2))u^2+2z_1z_2(z_1+z_2-1)}{(u^4+u^2(z_1+z_2)+z_1z_2)^{3/2}}du+\int_{\mathbb{R}} e^{-u^2}\frac{4z_1z_2-2z_1-2z_2+2}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}}du\right).
$$

Proof. By [31, pp. 283], we have an integral representation when $Re(s)>1$

$$
\begin{aligned}
& \zeta_2([z_1,z_2];s+2,s-\frac{1}{2}) \\
& = \int_{\mathbb{R}} e^{-z_2w^2}(1+w^2)^{2s-1/2}\zeta_1(z_1+z_2w^2,s+2,s-1/2)\zeta_1(z_2(1+w^2),s+3/2,s-1)dw.
\end{aligned}
$$

Use (3.3.2):

$$
(s-1/2)\zeta_1(z,s+2,s-1/2)=z\zeta_1(z,s+2,s+1/2)+(-s-1)\zeta_1(z,s+1,s+1/2)
$$

It is easy to compute

$$
\zeta_1(z,2,1/2)=\int_{\mathbb{R}_+} e^{-zx}(1+x)x^{-1/2}dx=z^{-1/2}\Gamma(1/2)+z^{-3/2}\Gamma(3/2)
$$

and

$$
\zeta_1(z,1,1/2)=\int_{\mathbb{R}_+} e^{-zx}x^{-1/2}dx=z^{-1/2}\Gamma(1/2).
$$

Therefore we obtain

$$
\zeta_1(z,2,-1/2)=-\Gamma(1/2)z^{-1/2}(2z-1).
$$

Use (3.3.2) again:

$$(s-1)\zeta_1(z, s+3/2, s-1) = z\zeta_1(z, s+3/2, s) + (-s-1/2)\zeta_1(z, s+1/2, s).$$

We may obtain

$$\begin{aligned}
&Res_{s=0}\zeta_1(z, s+3/2, s-1)\\
=&-z(z\zeta_1(z, 3/2, 1) - \frac{1}{2}\zeta_1(z, 1/2, 1)) + \frac{1}{2}(z\zeta_1(z, 1/2, 1) + \frac{1}{2}\zeta_1(z, -1/2, 1))\\
=&-z^2\zeta_1(z, 3/2, 1) + z\zeta_1(z, 1/2, 1) + \frac{1}{4}\zeta_1(z, -1/2, 1).
\end{aligned}$$

Applying integration by parts to the first and third integrals, we may evaluate the sum:

$$Res_{s=0}\zeta_1(z, s+3/2, s-1) = -z + \frac{1}{2}.$$

Therefore we obtain the residue of $\zeta_2([z_1, z_2]; s+2, s-\frac{1}{2})$ as an integral

$$2\Gamma(1/2)\int_{\mathbb{R}} e^{-z_2w^2}(1+w^2)^{-1/2}(z_1+z_2w^2)^{-1/2}(z_1+z_2w^2-\frac{1}{2})(z_2(1+w^2)-\frac{1}{2})dw.$$

Substitute $u = z_2w^2$ to obtain

$$2\Gamma(1/2)\int_{\mathbb{R}_+} e^{-u}(z_1+u)^{-1/2}(z_2+u)^{-1/2}(u+z_1-1/2)(u+z_2-1/2)u^{-1/2}du.$$

Now let $A = z_1+z_2-1/2, B = -1/2$ so that $A+B = z_1+z_2-1$. Then we can split the integral into three pieces:

$$\int_{u\in\mathbb{R}} e^{-u^2}\frac{(u^2+z_1-1/2)(u^2+z_2-1/2)}{(z_1+u^2)^{1/2}(z_2+u^2)^{1/2}}du = I + II + III$$

where

$$I = \int_{\mathbb{R}} e^{-u^2}\frac{u^4+Au^2}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}}du$$

$$II = \int_{\mathbb{R}} e^{-u^2}\frac{Bu^2-1/4}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}}du$$

and

$$III = \int_{\mathbb{R}} e^{-u^2}\frac{1}{4}\frac{4z_1z_2-2z_1-2z_2+2}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}}du.$$

Now, we rewrite the first integral and apply integration by parts

$$I = -\frac{1}{2}\int_{\mathbb{R}} \frac{u^3 + Au}{(u^4 + u^2(z_1+z_2) + z_1z_2)^{1/2}} de^{-u^2}$$
$$= \frac{1}{2}\int_{\mathbb{R}} e^{-u^2}\left(\frac{3u^2 + A}{(u^4 + u^2(z_1+z_2) + z_1z_2)^{1/2}} + \frac{(u^3+Au)(-\frac{1}{2})(4u^3 + 2u^2(z_1+z_2))}{(u^4 + u^2(z_1+z_2) + z_1z_2)^{3/2}}\right) du$$

which can be simplified

$$\int_{\mathbb{R}} e^{-u^2}\frac{\frac{1}{2}u^2 + \frac{1}{4}}{(u^4 + u^2(z_1+z_2) + z_1z_2)^{1/2}} du + \frac{1}{2}\int_{\mathbb{R}} e^{-u^2}\frac{(A(z_1+z_2) - z_1^2 - z_2^2)u^2 + 2z_1z_2A - z_1z_2(z_1+z_2)}{(u^4 + u^2(z_1+z_2) + z_1z_2)^{3/2}} du.$$

Notice that the first term cancels II. Plugging $A = z_1 + z_2 - 1/2$ into the above, we obtain

$$\int_{u\in\mathbb{R}} e^{-u^2}\frac{(u^2 + z_1 - 1/2)(u^2 + z_2 - 1/2)}{(z_1 + u^2)^{1/2}(z_2+u^2)^{1/2}} du$$
$$= \frac{1}{4}\int_{\mathbb{R}} e^{-u^2}\frac{(4z_1z_2 - (z_1+z_2))u^2 + 2z_1z_2(z_1+z_2-1)}{(u^4 + u^2(z_1+z_2) + z_1z_2)^{3/2}} du + \frac{1}{4}\int_{\mathbb{R}} e^{-u^2}\frac{4z_1z_2 - 2z_1 - 2z_2 + 2}{(u^4 + u^2(z_1+z_2) + z_1z_2)^{1/2}} du.$$

□

Proposition 3.3.7. *Suppose that $4\pi T \sim diag(a, -b)$, $a = diag(a_1, a_2)$. Then we have an integral representation*

$$W'_T(e, 0, \phi_\infty) = -\frac{\kappa(0)}{4\Gamma_3(2)} e^{t/2}\int_{\mathbb{R}^2} Ei(-b(1+w^2))e^{-a_1(1+w_1^2) - a_2(1+w_2^2)} dw$$
$$\left(\int_{\mathbb{R}} e^{-u^2}\frac{(4z_1z_2 - (z_1+z_2))u^2 + 2z_1z_2(z_1+z_2-1)}{(u^4 + u^2(z_1+z_2) + z_1z_2)^{3/2}} du + \int_{\mathbb{R}} e^{-u^2}\frac{4z_1z_2 - 2z_1 - 2z_2 + 2}{(u^4 + u^2(z_1+z_2) + z_1z_2)^{1/2}} du\right)$$

where z_1, z_2 are the two eigenvalues of ZaZ.

Proof. Recall by Prop. 3.2.5

$$W_T(e, s, \phi) = \kappa(s)\Gamma_3(s+2)^{-1}\Gamma_3(s)^{-1}2^{6s}e^{t/2}|det(T)|^{2s}\xi(T, s) \tag{3.3.7}$$

where

$$\xi(T, s) = \int_{\mathbb{R}^2} e^{-(aW + bW^*)} det(1+W)^{-2s}\zeta_2(ZaZ; s+2, s-\frac{1}{2})$$
$$\times \zeta_1(Z'bZ'; s, s+1) dw.$$

Note that

$$\zeta_1(z; 0, 1) = \int_{\mathbb{R}_+} e^{-zx}(x+1)^{-1} dx = -e^z Ei(-z).$$

Now the statement follows from the previous Lemma and that $\Gamma_3(s) \sim 2\sqrt{\pi}s^{-2}$.

□

4 Geometric Kernel function

In this section we construct the geometric kernel function for $\phi \in \mathscr{S}(\mathbb{B}^3)$ where $\mathbb{B}$ is an incoherent totally definite quaternion algebra over a totally real field F. We first recall the Gross–Schoen [12] cycles and their Bloch-Beilinson height pairing on the triple product of a curve. Then we review the generating series of Hecke operators and its modularity on product of Shimura curves associate to $\mathbb{B}$, see Proposition 4.3.1 following our previous paper [33]. The geometric kernel associate ϕ is then an automorphic form on the subgroup $\mathbb{G}$ of GL_2^3 of triples with same determinant. Finally, we decompose the geometric kernel function to series of local heights modulo some Eisenstein series and their derivations.

4.1 Gross–Schoen cycles

Let us first review Gross and Shoen's construction of the modified diagonal cycles in [12] and definitions of heights of Bloch [4], Beilinson [2, 3], and Gillet–Soulé [9]. Let k be a field and let X be a smooth, projective, and geometrically connected curve over k. Let $Y = X^3$ be the triple product of X over k and let $e = \sum a_i p_i$ be a divisor of degree $\sum a_i \deg p_i = 1$ such that some positive multiple ne is defined over k. Define the diagonal and the partially diagonal cycles with respect to base e as follows:

$$\Delta_{123} = \{(x, x, x) : x \in X\},$$

$$\Delta_{12} = \sum_i a_i \{(x, x, p_i) : x \in X\},$$

$$\Delta_{23} = \sum a_i \{(p_i, x, x) : x \in X\},$$

$$\Delta_{31} = \sum a_i \{(x, p_i, x) : x \in X\}$$

$$\Delta_1 = \sum_{i,j} a_i a_j \{(x, p_i, p_j) : x \in X\},$$

$$\Delta_2 = \sum_{i,j} a_i a_j \{(p_i, x, p_j) : x \in X\},$$

$$\Delta_3 = \sum_{i,j} a_i a_j \{(p_i, p_j, x) : x \in X\}.$$

Then define the Gross-Schoen cycle associated to e to be

$$\Delta_e = \Delta_{123} - \Delta_{12} - \Delta_{23} - \Delta_{31} + \Delta_1 + \Delta_2 + \Delta_3 \in \mathrm{Ch}^2(X^3)_{\mathbb{Q}}.$$

Gross and Schoen has shown that Δ_e is homologous to 0 in general, and that Δ_e it is rationally equivalent to 0 if X is rational, or elliptic, or hyperelliptic when e is a Weierstrass point.

Over a global field k, a natural invariant of Δ_e to measure the non-triviality of a homologically trivial cycle is the height of Δ_e which was conditionally constructed by Beilinson–Bloch

[2, 3, 4] and unconditionally by Gross–Schoen [12] for Δ_e. More precisely, assume that k is the fractional field of a discrete valuation ring R and that X has a regular, semi-stable model $\mathscr{X}$ over $S := \mathrm{Spec} R$. Then Gross-Schoen construct a regular model $\mathscr{Y}$ over S of $Y = X^3$ and show that the modified diagonal cycle Δ_e on Y can be extended to a codimension 2 cycle on $\mathscr{Y}$ which is numerically equivalent to 0 in the special fiber $\mathscr{Y}_s$.

If k is a function field of a smooth and projective curve B over a field, then Gross and Schoen's construction gives a cycle $\widehat{\Delta}_e$ with rational coefficients on a model $\mathscr{Y}$ of $Y = X^3$ over B. We can define the height of Δ_e as

$$\langle \Delta_e, \Delta_e \rangle = \widehat{\Delta}_e \cdot \widehat{\Delta}_e.$$

The right hand here is the intersection of cycles on $\mathscr{Y}$. This pairing does not depend on the choice of $\mathscr{Y}$ and the extension $\widehat{\Delta}_e$ of Δ_e.

If k is a number field, then we use the same formula to define the height for the arithmetical cycle

$$\widehat{\Delta} = (\widetilde{\Delta}_e, g)$$

Gillet–Soulé's arithmetic intersection theory [8] where

- $\widetilde{\Delta}_e$ is the Gross–Schoen extension of Δ_e over a model $\mathscr{Y}$ over $\mathrm{Spec} \mathscr{O}_k$;
- g is a Green's current on the complex manifold $Y(\mathbb{C})$ of the complex variety $Y \otimes_{\mathbb{Q}} \mathbb{C}$ for the cycle Δ_e: g is a current on $Y(\mathbb{C})$ of degree $(1,1)$ with singularity supported on $\Delta_e(\mathbb{C})$ such that the curvature equation holds:

  $$\frac{\partial\bar{\partial}}{\pi i} g = \delta_{\Delta_e(\mathbb{C})}.$$

  Here the right hand side denotes the Dirac distribution on the cycle $\Delta_e(\mathbb{C})$ when integrate with forms of degree $(2,2)$ on $Y(\mathbb{C})$.

More generally, let $t_i \in \mathrm{Ch}^1(X \times X)$ $(i = 1, 2, 3)$ be three correspondences on X, then $t = t_1 \otimes t_2 \otimes t_3 \in \mathrm{Ch}^3(X^3 \times X^3)$ is a correspondence of X^3, and we have a pairing

$$\langle \Delta_e, t_* \Delta_e \rangle. \tag{4.1.1}$$

Gross and Schoen have shown that

$$t \mapsto \langle \Delta_e, t_* \Delta_e \rangle$$

factor through the natural action of on the Jacobian $J = \mathrm{Jac}(X)$. In other words, we have a well defined linear functional:

$$\mathrm{End}(J)^{\otimes 3} \longrightarrow \mathbb{C}.$$

Generalities on correspondences

In following we give a slightly different description of correspondences which is more convenient than what given in the paper of Gross–Schoen. Let $\mathscr{Y}_1, \mathscr{Y}_2$ be two regular arithmetic schemes over the ring $\mathscr{O}_k$ of integers of a number field k of relative dimension n with generic fibers Y_1, Y_2. For any smooth subvariety $Z \in Z^*(Y_1 \times Y_2)$, let $\mathscr{Z}$ be the Zariski closure of Z in $\mathscr{Y}_1 \times_{\mathscr{O}_k} \mathscr{Y}_2$ with induced projections: $\pi_i : \mathscr{Z} \longrightarrow \mathscr{Y}_i$, we can define a correspondence on arithmetical Chow groups:

$$\widehat{Z}_* : \widehat{\mathrm{Ch}}^*(\mathscr{Y}_1) \longrightarrow \widehat{\mathrm{Ch}}^*(\mathscr{Y}_2)$$

by

$$Z_*\alpha = \pi_{2*}(\pi_1^*\alpha), \qquad \alpha \in \widehat{\mathrm{Ch}}^*(\mathscr{Y}_1).$$

Such a definition in general does not preserve the composition in the sense that for three regular schemes Y_i $(i = 1, 2, 3)$ and two correspondences $Z_1 \in Z^*(Y_1 \times Y_2$ and $Z_2 \in Z^*(Y_2 \times Y_3)$, the following identity holds

$$\widehat{Z_2 \circ Z_1} = \widehat{Z}_2 \circ \widehat{Z}_1$$

$Z_2 \circ Z_1$ is well-defined as a cycle by the formula

$$Z_1 \circ Z_2 = \pi_{13*}(\pi_{12}^* Z_2 \cdot \pi_{23}^* Z_1).$$

Lemma 4.1.1. *Assume that over any closed point $y_2 \in \mathscr{Y}_2$ either $\mathscr{Z}_1 \longrightarrow \mathscr{Y}_2$ or $\mathscr{Z}_2 \longrightarrow \mathscr{Y}_2$ is smooth. Then*

$$\widehat{Z_2 \circ Z_1} = \widehat{Z}_2 \circ \widehat{Z}_1$$

Proof. Under the assumption of the lemma, the cycle $\mathscr{Z}_3 := \mathscr{Z}_1 \times \mathscr{Y}_2 \mathscr{Z}_2$ is locally integral whose generic fiber $Z_2 \circ Z_1$ is the cycle image of the fiber product

$$Z_1 \times_{Y_2} Z_2 \longrightarrow Y_1 \times Y_2.$$

$$\begin{tikzcd}
 & & \mathscr{Z}_3 \arrow[dl, "f_1"'] \arrow[dr, "f_3"] & & \\
 & \mathscr{Z}_1 \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & & \mathscr{Z}_2 \arrow[dl, "h_2"'] \arrow[dr, "h_3"] & \\
\mathscr{Y}_1 & & \mathscr{Y}_2 & & \mathscr{Y}_3
\end{tikzcd}$$

It follows that the cycle image of $\mathscr{Z}_3$ on $\mathscr{Y}_1 \times_{\mathscr{O}_k} \mathscr{Y}_3$ is the Zariski closure of $\widehat{Z_2 \circ Z_1}$. Thus we can use $\mathscr{Z}_3$ to compute arithmetic correspondence:

$$\widehat{Z_2 \circ Z_1} = h_{3*} f_{3*} f_1^* g_1^*.$$

As the middle diagram is Cartisian, with one side étale,

$$f_{3*} f_1^* = h_2^* g_{2*}.$$

It follows that

$$\widehat{Z_2 \circ Z_1} = h_{3*} h_2^* g_{2*} g_1^* = \widehat{Z}_2 \circ \widehat{Z}_1.$$

$\square$

4.2 Shimura curves

In the following, we will review the theory of Shimura curves following our previous paper [34]. Let F be a totally real number field. Let $\mathbb{B}$ be a quaternion algebra over $\mathbb{A}$ with odd ramification set Σ including all archimedean places. Then for each open subset U of $\mathbb{B}_f^\times$ we have a Shimura curve X_U. The curve is not connected; its set of connected components is can be parameterized by $F_+^\times\backslash\mathbb{A}_f^\times/q(U)$. For two open compact subgroups $U_1\subset U_2$ of $\mathbb{B}_f^\times$, one has a canonical morphism $\pi_{U_1,U_2}:\quad X_{U_1}\longrightarrow X_{U_2}$ which satisfies the composition property. Thus we have a projective system X of curves X_U.

For any $x\in\mathbb{B}_f$, we also have isomorphism $T_x:\quad X_U\longrightarrow X_{x^{-1}Ux}$ which induces an automorphism on the projective system X and compatible with multiplication on $\mathbb{B}_f^\times$: $T_{xy}=T_x\cdot T_y$ if x and y are *coprime with respect to* U in the sense that for any finite place v of F, either x_v or y_v is in U_v. The induced actions are the obvious one on the sets of connected components after taking norm of U_i and x.

For each archimedean place τ of F, the associate analytic space at τ can be described as follows:

$$X_{U,\tau}^{\mathrm{an}}=B(\tau)_+^\times\backslash\mathscr{H}\times\mathbb{B}_f^\times/U\cup\{\mathrm{Cusps}\}$$

where $B(\tau)_+^\times$ is the group of totally positive elements in a quaternion algebra $B(\tau)$ over F with ramification set $\Sigma\setminus\{\tau\}$ with an action on $\mathscr{H}^\pm$ by some fixed isomorphisms

$$B(\tau)\otimes_\tau\mathbb{R}=M_2(\mathbb{R})$$

$$B(\tau)\otimes\mathbb{A}_f\simeq\mathbb{B}_f,$$

and where $\{\mathrm{Cusp}\}$ is the set of cusps which is non-empty only when $F=\mathbb{Q}$ and $\mathbb{B}_f=M_2(\mathbb{A}_f)$. All of these morphisms on X_U's has obvious description on complex manifolds $X_{U,\tau}(\mathbb{C})$. In this uniformization, the action T_x is given by right multiplication by x on component group $\mathbb{B}_f^\times$.

Modular interpretation

An important tool to study Shimura curves is to use modular interpretation. For a fixed archimedean place τ, the space $\mathscr{H}^\pm$ parameterizes Hodge structures on $V_0:=B(\tau)$ which has type $(-1,0)+(0,-1)$ (resp $(0,0)$) on $V_0\otimes_\tau\mathbb{R}$ (resp. $V_0\otimes_\sigma\mathbb{R}$ for other archimedean places $\sigma\neq\tau$). The non-cuspidal part of $X_{U,\tau}(\mathbb{C})$ parameterizes Hodge structure and level structures on a $B(\tau)$-module V of rank 1.

Due to the appearance of type $(0,0)$, the curve X_U does not parameterize abelian varieties unless $F=\mathbb{Q}$. To get a modular interpretation, we use an auxiliary imaginary quadratic extension K over F with complex embeddings $\sigma_K:K\longrightarrow\mathbb{C}$ for each archimedean places σ of F other than τ. These σ_K's induce a Hodge structure on K which has type $(0,0)$ on $K\otimes_\tau\mathbb{R}$ and type $(-1,0)+(0,-1)$ on $K\otimes_\sigma\mathbb{R}$ for all $\sigma\neq\tau$. Now the tensor product of Hodge structures on $V_K:=V\otimes_F K$ is of type $(-1,0)+(0,-1)$. In this way, X_U parameterizes some abelian varieties with homology group H_1 isomorphic to V_K. The construction makes X_U a curve over the reflex field for σ_K's:

$$K^\sharp = \mathbb{Q}\left(\sum_{\sigma\neq\tau}\sigma(x), \quad x\in K\right).$$

See our paper [34] for a construction following Carayol in the case $K = F(\sqrt{d})$ with $d \in \mathbb{Q}$ where $\sigma_K(\sqrt{d})$ is chosen independent of σ.

Integral model

The curve X_U has a canonical integral model $\mathscr{X}_U$ over $\mathscr{O}_F$ when U is included in a maximal order $\mathscr{O}_{\mathbb{B}}$ is fixed. In fact at each place v where $U_v = \mathscr{O}_{\mathbb{B}_v}^\times$, one can describe a integral model $\mathscr{X}_U$ when U^v is small using either a modular interpretation when v is split in $\mathbb{B}$ or the Drinfeld uniformization when v is not split in $\mathbb{B}$. The scheme $\mathscr{X}_U$ is regular over v and carries a formal $\mathscr{O}_{\mathbb{B}_v}$-module $\mathscr{V}$. Locally at a geometric point, $X_{U,v}$ is the universal deformation of the formal module. The model for general U with arbitrary U^v and $U_v = \mathscr{O}_{\mathbb{B}_v}$ can be constructed by taking quotient.

For general U_v, one takes $\mathscr{X}_U$ to be the integral closure of $\mathscr{X}_{U^0}$ in the function field of X_U for an $U^0 = U^v\mathscr{O}_{\mathbb{B}_v}^\times$ with U_v^0 a maximal compact subgroup of $\mathbb{B}_v^\times$ containing U_v.

If v is not split in $\mathbb{B}$, U^v is small, and $U_v = 1 + \pi_v^n\mathscr{O}_{\mathbb{B}_v}$ with $n \geq 1$, the scheme $\mathscr{X}_U$ is regular at points over v and parameterizes Drinfeld level structure on the formal module $\mathscr{V}$ over $\mathscr{X}_{U^0}$. If v is not split in $\mathbb{B}$, then $\mathscr{X}_U$ may not regular and does not have a natural modular interpretation.

Hodge class

The curve X_U has a *Hodge class* $L_U \in \mathrm{Pic}(X_U)\otimes\mathbb{Q}$ which is compatible with pull-back morphism and such that $L_U \simeq \omega_{X_U}$ when U is sufficiently small. We may extend this class to integral model $\mathscr{X}_U$ compatible with pull-backs. In this way, we need only describe the model locally over a place v where U_v maximal and U^v small. In this case, we simply take $\mathscr{L}_U$ to the relative dualising sheaf.

We also define a class $\xi_U \in \mathrm{Pic}(X_U)\otimes\mathbb{Q}$ which has degree 1 on each connected component and is proportional to $\mathscr{L}_U$.

4.3 Hecke correspondences and generating series

We want to define some correspondences on X_U, i.e., some divisor classes on $X_U \times X_U$. The projective system of surfaces $X_U \times X_U$ has an action by $\mathbb{B}_f^\times \times \mathbb{B}_f^\times$. Let K denote the open compact subgroup $K = U \times U$.

Hecke operators

For any double coset UxU of $U\backslash\mathbb{B}_f^\times/U$, we have a Hecke correspondence

$$Z(x)_U \in Z^1(X_U \times X_U)$$

defined as the image of the morphism

$$(\pi_{U\cap xUx^{-1},U},\pi_{U\cap x^{-1}Ux,U}\circ T_x):\qquad Z_{U\cap xUx^{-1}}\longrightarrow X_U^2.$$

In terms of complex points at a place of F as above, the Hecke correspondence $Z(x)_U$ takes

$$(z,g)\longrightarrow\sum_i(z,gx_i)$$

for points on $X_{U,\tau}(\mathbb{C})$ represented by $(z,g)\in\mathscr{H}^\pm\times\mathbb{B}_f$ where x_i are representatives of UxU/U.

Hodge class

On $M_K:=X_U\times X_U$, one has a *Hodge bundle* $\mathscr{L}_K\in\mathrm{Pic}(M_K)\otimes\mathbb{Q}$ defined as

$$\mathscr{L}_K=\frac{1}{2}(p_1^*\mathscr{L}_U+p_2^*\mathscr{L}_U).$$

Generating Function

Let $\mathbb{V}$ denote the orthogonal space $\mathbb{B}$ with quadratic form q. For any $x\in\mathbb{V}$, let us define a cycle $Z(x)_K$ on $X_U\times X_U$ as follows. This cycle is non-vanishing only if $q(x)\in F^\times$ or $x=0$. If $q(x)\in F^\times$, then we define $Z(x)_K$ to be the Hecke operator UxU defined in the last subsection. If $x=0$, then we define $Z(x)_K$ to be the push-forward of the Hodge class on the subvariety M_α which is union of connected components $X_\alpha\times X_\alpha$ with $\alpha\in F^\times\backslash\mathbb{A}^\times/F_{\infty,+}^\times\nu(U)$. Let $\widetilde{K}=\mathrm{O}(F_\infty)\cdot(U\times U)$ act on $\mathbb{V}$.

For $\phi\in\mathscr{S}(\mathbb{V})^{\widetilde{K}}$, we can form a generating series

$$Z(\phi)=\sum_{x\in\widetilde{K}\backslash\mathbb{V}}\phi(x)Z(x)_K.$$

It is easy to see that this definition is compatible with pull-back maps in Chow groups in the projection $M_{K_1}\longrightarrow M_{K_2}$ with $K_i=U_i\times U_i$ and $U_1\subset U_2$. Thus it defines an element in the direct limit $\mathrm{Ch}^1(M)_\mathbb{Q}:=\lim_K\mathrm{Ch}^1(M_K)$ if it absolutely convergent.

Let $\mathscr{S}(\mathbb{V})^{\mathrm{O}(V_\infty)}$ denote the subspace of $O(F_\infty)$ invariants in $\mathscr{S}(\mathbb{V})$. Let $\mathbb{H}=\mathrm{GSpin}(\mathbb{V})$ which can be identified with the subgroup of elements of pairs (b_1,b_2) in $\mathbb{B}^\times\times\mathbb{B}^\times$ with the same norm. Then $\mathscr{S}(\mathbb{V})$ has a Weil representation by $\mathbb{H}\times\mathrm{SL}_2(\mathbb{A})$. Define

$$Z(g,\phi)=Z_{r(g)\phi}.$$

By our previous paper [33], this series is absolutely convergent and is modular for $\mathrm{SL}_2(\mathbb{A})$:

$$Z(\gamma g,\phi):=Z(g,\phi)\tag{4.3.1}$$

Moreover, for any $h\in\mathbb{H}$,

$$Z(g,r(h)\phi)=\rho(h)Z(g,\phi).\tag{4.3.2}$$

where $\rho(h)$ denotes the pull-back morphism on $\mathrm{Ch}^1(M)$ by right translation of h_f.

In the following, we want to extend the above definition to $g \in \mathrm{GL}_2(\mathbb{A})$. First, we consider the extended Weil representation on $\mathscr{S}(\mathbb{V})$ by

$$\mathscr{R} = \left\{(b_1, b_2, g) \in \mathbb{B}^\times \times \mathbb{B}^\times \times \mathrm{GL}_2(\mathbb{A}) : \quad q(b_1 b_2^{-1}) = \det g\right\}$$

by

$$r(h, g)\phi(x) = |q(h)|^{-1} r(d(\det(g))^{-1} g)\phi(h^{-1}x).$$

Let $\mathrm{GL}_2(\mathbb{A})^+$ denote subgroup of $\mathrm{GL}_2(\mathbb{A})$ with totally positive determinant at archimedean places. For $g \in \mathrm{GL}_2(\mathbb{A}_F)^+$, define

$$Z(g, \phi) = \rho(h)^{-1} Z(r(g, h)\phi),$$

where h is an element in $\mathbb{B}^\times \times \mathbb{B}^\times$ with norm $\det g$. By (4.3.2), the definition here does not depend on the choice of h. The the following is the modularity of $Z(\phi)$:

Proposition 4.3.1. *The cycle $Z(g, \phi)$ is automorphic for $\mathrm{GL}_2(\mathbb{A})^+$: for any $\gamma \in \mathrm{GL}_2(F)^+$, $g \in \mathrm{GL}_2(\mathbb{A})$,*

$$Z(\gamma g, \phi) = Z(g, \phi).$$

Proof. Let $\gamma \in \mathrm{GL}_2(F)^+$ if suffices to show

$$Z(r(\alpha, \gamma)\phi) = Z(\phi)$$

where $\alpha \in \mathbb{B}^\times \times \mathbb{B}^\times$ has the norm $\det \gamma$. Write $\gamma_1 = d(\gamma)^{-1}\gamma$. By definition, the left is equal to

$$\begin{aligned}
\rho(\alpha)^{-1} Z(L(\alpha) r(\gamma_1)\phi) &= \sum_{x \in \widetilde{K} \backslash \mathbb{V}} r(\gamma_1)\phi(\alpha^{-1}x)\rho(\alpha)^{-1} Z(x)_K \\
&= \sum_{x \in K \cdot \mathrm{O}(F_\infty) \backslash \mathbb{V}} r(\gamma_1)\phi(\alpha^{-1}x) Z(\alpha^{-1}x)_K \\
&= \sum_{x \in \widetilde{K} \backslash \mathbb{V}} r(\gamma_1)\phi(x) Z(x)_K \\
&= Z(r(\gamma_1)\phi) = Z(\phi).
\end{aligned}$$

□

Notice that the natural embedding $\mathrm{GL}_2(\mathbb{A}_F)^+ \longrightarrow \mathrm{GL}_2(\mathbb{A}_F)$ gives bijective map

$$\mathrm{GL}_2(F)^+ \backslash \mathrm{GL}_2(\mathbb{A}_F)^+ \longrightarrow \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F)$$

thus we can extend the form $Z(g, \phi)$ uniquely an automorphic form on $\mathrm{GL}_2(\mathbb{A}_F)$.

Corollary 4.3.1. *For $g \in \mathrm{GL}_2(\mathbb{A}_F)$ define*

$$Z(g, \phi) = Z(\gamma h, \phi)$$

for some $\gamma \in \mathrm{GL}_2(F)$ such that $\gamma h \in \mathrm{GL}_2^+(\mathbb{A}_F)$. Then $Z(g, \phi)$ is automorphic for $\mathrm{GL}_2(\mathbb{A})$.

4.4 Heights of Gross–Schoen cycles

Let $\Delta_{U,\xi}$ be the Gross–Schoen cycle on X_U^3 which is obtained form the diagonal cycle by some modification with respect to the Hodge class ξ (the unique class in $Pic^1(X)_{\mathbb{Q}} = \lim_U Pic^1(X_U)_{\mathbb{Q}}$ that is $\mathbb{B}_f^\times$-invariant). Note that ξ_U is a degree one divisor on any component of X_U. And varying level structure U, the Hodge class ξ_U form a projective system hence the Gross–Schoen cycle$\Delta_{U,\xi}$ forms a projective system. It is shown in [12] that $\Delta_{U,\xi}$ is homologously trivial and a height pairing $\langle \Delta_{U,\xi}, T_*\Delta_{U,\xi}\rangle$ is well defined for any self-correspondence T of X_U^3. More generally, one has well-defined height pairing

$$\langle \Delta_{U,\xi}, \mathrm{Z}(\phi)\Delta_{U,\xi}\rangle$$

for a Hecker operator $\mathrm{T}(\phi)$ defined by a function ϕ in the space $\mathscr{S}((\mathbb{B}_f^\times)^3)$ of locally constant with compact support on $(\mathbb{B}_f^\times)^3$ invariant under $U^3 \times U^3$. Here $\mathscr{S}(\mathbb{B}_f^\times)$ has two actions by $\mathbb{B}_f^\times$ from left and right translations. In fact varying U, the Hecke operator $\mathrm{T}(\phi)$ forms an inductive system. The projection formula ensures that the above paring does not depends on the choice of the open compact U.

Note that the Hodge class ξ is invariant (up to torsion) under $\mathbb{B}_f^\times$-translation. And the diagonal cycle and various partial diagonals are automatically invariant under the diagonal $\Delta(\mathbb{B}_f^\times) \subset (\mathbb{B}_f^\times)^3$. It follows from the projection formula that the linear form, denoted by γ_f, defined by $\phi \mapsto \langle \Delta_{U,\xi}, \mathrm{T}(\phi)\Delta_{U,\xi}\rangle$ is $\mathbb{B}_f^\times \times \mathbb{B}_f^\times$-invariant:

$$\gamma_f \in \mathrm{Hom}_{\mathbb{B}_f^\times \mathbb{B}_f^\times}(\mathscr{S}(\mathbb{B}_f^\times)^{\otimes 3}, \mathbb{C}).$$

Moreover, the height pairing depends only on the action of $\mathrm{T}(\phi_i)$ on the weight 2 forms ([12], Prop. 8.3). In other words, the linear form γ_f factors through the natural $(\mathbb{B}_f^\times \times \mathbb{B}_f^\times)^3$-equivariant projection

$$\mathscr{S}(\mathbb{B}_f^\times)^{\otimes 3} \twoheadrightarrow \bigoplus_\pi \pi_f \otimes \widetilde{\pi}_f$$

where the sum is over the Jacquet-Langlands correspondences ρ on $\mathbb{B}^\times$ of all weight 2 cuspidal representation of $\mathrm{GL}_2(\mathbb{A})^3$. In particular, by restricting to the subspace $\pi_f \otimes \widetilde{\pi}_f$ for one π, we have a well-defined height pairing (still denoted by γ_f):

$$\gamma_f \in \mathrm{Hom}_{\mathbb{B}_f^\times \times \mathbb{B}_f^\times}(\pi_f \otimes \widetilde{\pi}_f, \mathbb{C}).$$

We multiply this with matrix coefficients pairing α_v at infinite places v to get a linear form:

$$\gamma \in \mathrm{Hom}_{\mathbb{B}^\times \times \mathbb{B}^\times}(\pi \otimes \widetilde{\pi}, \mathbb{C}). \tag{4.4.1}$$

Lemma 4.4.1. *Let $\phi \in \mathscr{S}(\mathbb{V}_f)$ be such that $\theta(\phi \otimes \varphi) = f \otimes \widetilde{f} \in \pi_f \otimes \widetilde{\pi}_f$. Then we have an equality in $CH^1(X \times X)$ modulo the space spanned by $e \times X, X \times e$ for all $e \in Pic(X)$:*

$$(Z(\phi), \varphi)_{Pet} = (*)\mathrm{T}_{f \otimes \widetilde{f}}.$$

Thus we can consider the generating function for a triple $\phi = \otimes_i \phi_i \in \mathscr{S}(\mathbb{V}^3)$ fixed by $\widetilde{K}^3$ for a compact open $U \subset \mathbb{B}_f^\times$:

$$\deg Z(g,\phi) := \langle \Delta_{U,\xi}, Z(g,\phi)\Delta_{U,\xi}\rangle, \quad g \in \mathrm{GL}_2^3(\mathbb{A}).$$

Now the main ingredient of our proof is reduced to the weak form of an arithmetic Siegel–Weil formula:

$$E'(g,0,\phi) \equiv 2\deg Z(g,\phi), \quad g \in \mathbb{G}(\mathbb{A})$$

where "$\equiv$" means modulo all forms on $\mathbb{G}(\mathbb{A})$ that is perpendicular to σ.

4.5 First decomposition

In this subsection, we are trying to give a first decomposition of the height pairing. First let us choose a decomposition of $\widehat{\Delta}$ by extend each term to an arithmetic class. Recall that we can write

$$\Delta_\xi = \Delta - \sum_{i\neq j} \xi_k \cdot \Delta_{i,j} + \sum_{i,j} \xi_i \cdot \xi_j \tag{4.5.1}$$

where

- two sums runs over non-ordered elements $i \neq j$ in $\{1,2,3\}$, k is the complement of $\{i,j\}$ in $\{1,2,3\}$;
- $\xi_k = \pi_k^*\xi$, $\Delta_{i,j}$ is the partial diagonal of elements $x = (x_1,x_2,x_3)$ with $x_i = x_j$;

Fix a good model $\mathscr{Y}$ of X^3 as in Gross–Schoen's paper. We extend the term in the right side of (4.5.1) to arithmetic cycles as follows:

- $\widehat{\Delta} = (\bar{\Delta}, g_\Delta)$ and $\widehat{\Delta}_{i,j} = (\bar{\Delta}_{i,j}, g_{\Delta_{i,j}})$ by taking Zariski closures $\bar{\Delta}$, $\bar{\Delta}_{i,j}$ and some Green currents g_Δ and $g_{\Delta_{i,j}}$;
- $\widehat{\xi}$ the extension ξ over $\mathscr{X}_U$ as described in §4.3, and a metric on ξ with curvature proportional to the Poincare metric in the complex uniformizations.

By Gross–Schoen there is a vertical cycle W, formed by cycles supported on $\mathscr{Y}$ and smooth forms on $Y(\mathbb{C})$ such that

$$\widehat{\Delta}_\xi = \widehat{\Delta} - \sum_{i\neq j} \widehat{\xi}_k \cdot \widehat{\Delta}_{i,j} + \sum_{i,j} \widehat{\xi}_i \cdot \widehat{\xi}_j + W \tag{4.5.2}$$

¿From our previous work on Gross–Zagier formula, for any $\phi \in \mathscr{S}(\mathbb{V})^{\mathrm{O}(F_\infty)}$, we have that

$$Z(g,\phi)\xi = E(g,\phi)\xi$$

where $E(g, \phi)$ is an Eisenstein series for the section $r(g)\phi(0)$, and that

$$Z(g, \phi)\widehat{\xi} = E(g, \phi)\xi + D(\phi)$$

where $D(\phi)$ is a derivation for $E(\phi)$. It follows that for $g \in \mathrm{GL}_2(\mathbb{A})^3_+$, $\phi \in \mathscr{S}(\mathbb{V}^3)^{\mathrm{O}(F_\infty)^3}$, the series of the intersection numbers

$$\langle \Delta_\xi, \quad Z(g, \phi)\Delta_\xi \rangle$$

is equal to

$$\langle \widehat{\Delta}, \quad Z(g, \phi)(\widehat{\Delta} + W) \rangle$$

modulo some partial series and their derivation for variables g_i in $g = (g_1, g_2, g_3)$. If for some finite place v, $\phi = \phi^v \otimes \phi_v$ with ϕ_v supported on regular elements, then Δ and $Z(g, \phi)\Delta$ are disjoint. Thus the above arithmetic intersection can be decomposed into a sum of local intersections:

$$\langle \widehat{\Delta}, \quad Z(g, \phi)(\widehat{\Delta} + W) \rangle = \sum_v \langle \widehat{\Delta}, \quad Z(g, \phi)(\widehat{\Delta} + W) \rangle_v \tag{4.5.3}$$

Triple product of Hecke operators

With notion as in the last section, let v be finite place of F where the $\mathscr{X}_U$ is smooth over v. In the following, we want to express this intersection as an triple intersection of Hecke operators a good place. Let let $\bar{Z}(\phi)$ be the Zariski closure of $Z(\phi)$ on $\mathscr{X}_U^6$. Then we have

$$(\bar{\Delta} \cdot Z(\phi)\bar{\Delta})_v = (\pi_1^* \bar{\Delta} \cdot \bar{Z}(\phi) \cdot \pi_2^* \bar{\Delta})_v = \deg i^* \bar{Z}(\phi)$$

where

$$i : \mathscr{X}_U^2 \longrightarrow \mathscr{X}_U^6, \quad (x, y) \mapsto (x, x, x, y, y, y).$$

After a linear combination, it suffices to consider the situation where $\phi = \phi_1 \otimes \phi_2 \otimes \phi_3$. In this case, the above gives the triple product of three Hecke operators on $\mathscr{X}_U^2$:

$$(\bar{\Delta}, Z(\phi)\bar{\Delta})_v = (Z(\phi_1) \cdot Z(\phi_2) \cdot Z(\phi_3))_v.$$

Such a decomposition formula holds even for archimedean place and bad places when we use adelic metrics. More precisely, the cycle $Z(\phi_i)$ can be extended into an arithmetic cycle $\widehat{Z}(\phi_i)$ which is admissible with respect to bundle $\widehat{\xi}$, see our previous paper [36]. Then we still have

$$\langle \widehat{\Delta}_\xi, Z(g, \phi)\widehat{\Delta}_\xi \rangle \equiv (\widehat{Z}(\phi_1) \cdot \widehat{Z}(\phi_2) \cdot \widehat{Z}(\phi_3)) \tag{4.5.4}$$

modulo partial Eisenstein series and their derivations. The right hand side has a canonical decomposition into a sum of local intersections when ϕ has a regular support at some place of F.

5 Local triple height pairings

In this section, we want to compute the local triple height pairings of Hecke operators at the unramified places and archimedean places.

For unramified place, we first study the modular interpretation of Hecke operators and reduce the identity to work of Gross–Keating on formal groups.

For archimedean places, we ....

5.1 Reduction of Hecke operators

In this section, we would like to study reduction of Hecke operators. For an $x \in \mathbb{V}$ with positive norm in F, the cycle $Z(x)_K$ is the graph of the Heck operator given by the cost UxU. In term of analytic correspondences on X_U, $Z(x)_K$ is the correspondences defined by maps:

$$Z(x)_K \simeq X_{U\cap xUx^{-1}} \longrightarrow X_U \times X_U$$

Moduli interprstation at an archimedean place

First let us give some moduli interpretation of Hecke operators at an archimedean place τ. Write B a quaternion algebra over F with ramification set $\Sigma \setminus \{\tau\}$. Fix an isomorphism $\mathbb{B}^\tau \simeq B \otimes \mathbb{A}^\tau$. Recall from §51 in our Asia journal paper, that the curve X_U parameterizes the isomorphism classes of triples $(V, h, \bar{\kappa})$ where

1. V is a free B-module of rank 1;
2. h is an embedding $\mathbb{S} \longrightarrow \mathrm{GL}_B(V_\mathbb{R})$ which has trivial component at τ_i for $i > 1$;
3. $\bar{\kappa}$ is a $\mathrm{Isom}(\widehat{V}_0, \widehat{V})/U$, where $V_0 = B$ as a left B-module.

If we decompose $UxU = \coprod x_iU$, then $Z(x)_K$ sends one object $(V, h, \bar{\kappa})$ to sum of $(V, h, \overline{\kappa x_i})$. It is easy to see that this class represents the isomorphism class of $(V_i, h_i, \bar{\kappa}_i)$ such that there is an isomorphism $f_i : (V_i, h_i) \longrightarrow (V, h)$ of such that the following two diagrams are commutative:

(5.1.1)
$$\begin{tikzcd} \widehat{V}_0 \arrow[r, "\kappa_i"] \arrow[d, "x_i"] & \widehat{V}_i \arrow[d, "\widehat{f}_i"] \\ \widehat{V}_0 \arrow[r, "\kappa"] & \widehat{V} \end{tikzcd}$$

Thus we may write abstractly,

(5.1.2)
$$Z(x)_K(V, h, \bar{\kappa}) = \sum_i (V_i, h_i, \bar{\kappa}_i).$$

Notice replace κ and κ_i by equivalent classes, we may assume that $x_i = x$. Thus the subvariety $Z(x)_K$ of M_K parameterizes the triple:

$$(V_1, h_1, \bar{\kappa}_1), \qquad (V_2, h_2, \bar{\kappa}_2), \quad f$$

where the first two are objects as described as above for $\bar{k}_1$ and $\bar{k}_2$ level structures modulo $U_1 := U \cap xUx^{-1}$ and $U_2 = U \cap x^{-1}Ux$ respectively, and $f : (V_2, h_2) \longrightarrow (V_1, h_1)$ such that:

$$\begin{tikzcd} \widehat{V}_0 \arrow[r, "\kappa_2"] \arrow[d, "x"] & \widehat{V}_2 \arrow[d, "\widehat{f}"] \\ \widehat{V}_0 \arrow[r, "\kappa_1"] & \widehat{V}_1 \end{tikzcd} . \tag{5.1.3}$$

Now we want to describe the above moduli interpretation with an integral structure. We assume that U is compact and then is included in a maximal open compact subgroup of form $\widehat{\mathscr{O}}_B = \mathscr{O}_{\mathbb{B}}$, where $\mathscr{O}_B$ is a maximal order of B. Let $V_{0,\mathbb{Z}} = \mathscr{O}_B$ as an $\mathscr{O}_B$-lattice in V_0. Then for any triple $(V, h, \bar{\kappa})$ we obtain a triple $(V_{\mathbb{Z}}, h, \bar{\kappa})$ with $V_{\mathbb{Z}} = \kappa(V_{0\mathbb{Z}})$ which satisfies the analogous properties as above. In fact, M_U parameterizes such integral triples. The Hecke operator $Z(x)_K$ has the following expression:

$$Z(x)_K(V_{\mathbb{Z}}, h, \bar{\kappa}) = \sum_i (V_{i\mathbb{Z}}, h_i, \kappa_i)$$

where $V_{i\mathbb{Z}} = \kappa_i(V_{0\mathbb{Z}})$. We can't replace terms in the above diagram by integral lattices as f_i and x_i only define quasi-isogeny:

$$f_i \in \mathrm{Hom}_{\mathscr{O}_B}(V_{i\mathbb{Z}}, V_{\mathbb{Z}}) \otimes F, \qquad x_i \in \widehat{B} = \mathrm{End}_B(\widehat{V}_{0,\mathbb{Z}}) \otimes F.$$

When U is sufficiently small, we have universal objects $(V_U, h, \bar{k})$, $(V_{U,\mathbb{Z}}, h, \bar{\kappa})$. We will also consider the divisible $\mathscr{O}_B$-module $\widetilde{V}_U = \widehat{V}_U/\widehat{V}_{U,\mathbb{Z}}$. The subvariety $Z(x)_K$ also have a universal object $f : V_{U_2} \longrightarrow V_{U_1}$.

Goes back to curves over F, the rational object does not makes sense, but the local system $\widehat{V}$ and $\widehat{V}_{\mathbb{Z}}$ make sense as $\mathbb{B}_f$ and $\mathscr{O}_{\mathbb{B}_f}$ modules respectively. The Hecke operator parameterizes certain morphism $\widehat{f} : \widehat{V}_{U_2} \longrightarrow \widehat{V}_{U_1}$.

Modular interpretation at an finite place

Now we want to study the extension of $Z(x)_K$ to an integral model $\mathscr{X}_U$ over a finite place v of F. We assume that $U = U_v \times U^v$. Recall from §5.3 in our Asia journal paper, the prime to v-part of $(\widehat{V}_U, \bar{\kappa})$ extends to an etalé system over $\mathscr{X}_U$, but the v-part extends to a system of special divisible $\mathscr{O}_{\mathbb{B}_v}$-module of dimension 2, height 4, with Drinfeld level structure:

$$(\mathscr{V}, \bar{\alpha}).$$

We would like to give an moduli interpretation for the integral model $\mathscr{Z}(x)_K$ of $Z(x)_K$. First of all the isogeny $f : \widehat{V}_{U_2} \longrightarrow \widehat{V}_{U_1}$ induces the same quasi-isogeny on divisible $\mathscr{O}_{\mathbb{B}_f}$-modules. For prime to v-part, this is the same as over generic fiber. We need to describe the quasi-isogeny on formal modules. First lets us assume that $U_v = \mathscr{O}_{\mathbb{B}_v}^\times$ is maximal.

If v is not split in $\mathbb{B}$, then $U_{1v} = U_{2v} = U_v$. Thus condition on f_v on the generic fiber is just required to have order equal to $\mathrm{ord}(x^{-1})$. Thus $\mathscr{Z}(x)_K$ will parameterizes quasi-isogeny of pairs whose order at v has order x. Recall from §5.3 in our Asia journal paper that the notion of quasi-isogeny as quasi-isogeny of divisible module which can be lifted to the generic fiber.

If v is split in $\mathbb{B}$, then we may choose an isomorphism $\mathscr{O}_{\mathbb{B}_v} = M_2(\mathscr{O}_v)$. Then the formal module $\mathscr{V}$ is a direct sum $\mathscr{E} \oplus \mathscr{E}$ where $\mathscr{E}$ is a divisible $\mathscr{O}_F$-module of dimension 1 and height 2. By replacing x by an element in $U_v x U_v$ we may assume that x_v is diagonal: $x_v = \begin{pmatrix} \pi_v^c & \\ & \pi_v^d \end{pmatrix}$ with $c, d \in \mathbb{Z}$ and $c \leq d$. It is clear that the condition on f on the generic fiber is a composition of a scalar multiplication by π_v^c (as a quasi-isogeny) and a isogeny with kernel isomorphic to the cyclic modulo $\mathscr{O}_v/\pi^{d-a}\mathscr{O}_v$. Thus the scheme $\mathscr{Z}(x)_K$ parameterizes quasi-isogeny f of geometric points type (c, d) in the following sense:

1. the v-component $\pi^{-c} f_v : \mathscr{E}_2 \longrightarrow \mathscr{E}_1$ is an isogeny;

2. the kernel of $\pi_v^{-c} f_v$ is cyclic of order $d - c$ in the sense that it is the image of a homomorphism $\mathscr{O}_v/\pi^{d-c}\pi_v \longrightarrow \mathscr{E}_2$.

We also call such a quasi-isogeny *a type* x_v. Notice that the number a, b can be defined without reference to U_v. Indeed, a is the minimal integer such that $\pi^{-a} x_v$ is integral over $\mathscr{O}_v$ and that $a + b = \mathrm{ord}(\det x_v)$.

For a geometric point in M_K with formal object $\mathscr{E}_1, \mathscr{E}_2$, by Serre–Tate theory, the formal neighborhood $\mathscr{D}$ is the product of universal deformations $\mathscr{D}_i$ of $\mathscr{E}_i$. The divisor of $\mathscr{Z}(x)_K$ in this neighborhood is defined as the sum of the universal deformation of quasi-isogenies. In the following, we want to study the behaviors of this divisor in a formal neighborhood of a pair of surpersingular points on M_K when $U = U_v U^v$ with U_v maximal.

Recall from §5.4 in our Asia journal paper, all supersingular points are isogenous to each other. Fix one of the super singular point P_0 representing the triple $(\mathscr{V}_0, \widetilde{V}_0^v, \bar{\kappa}_0^v)$. Let $B = \mathrm{End}^0(P_0)$ which is a quaternion algebra obtained from $\mathbb{B}$ by changing invariants at v. We may use κ_0 to identify $\widetilde{V}_0 = \widehat{V}_0/\widehat{V}_{0\mathbb{Z}}$. The action of $(B \otimes \mathbb{A}_f^v)^\times$ and $(\mathbb{B}_f^v)^\times$ both acts on the set of structures. We may use κ_0 to identify them. In this way, the set $\mathscr{X}_U^{ss}$ of supersingular point is identified with

$$B_0 \backslash (B \otimes \mathbb{A}_f^v)^\times / U^v$$

so that the element $g \in (B \otimes \mathbb{A}_f^v)^\times$ represents the objects

$$(\mathscr{V}_0, \widehat{V}_0^v, gU^v),$$

where B_0 means the subgroup of $B^\times$ of elements with order 0 at v. The set $\mathscr{Z}(x)_{K,v}^{ss}$ of supersingular points on the cycle $Z(x)_K$ represents the isogeny $f : P_2 \longrightarrow P_1$ of two supersingular points of level $U_1 = U \cap xUx^{-1}$ and $U_2 = U \cap x^{-1}Ux$. In terms of triple as above, we have the following conditions that $\mathscr{Z}(x)_K^{ss}$ represents triples (g_1, g_2, f) of elements $g_i \in (B \otimes \mathbb{A}_f^v)^\times / U_i$ and $f \in B^\times$ with following properties

$$g_1^{-1} f^v g_2 = x^v, \qquad \mathrm{ord}_v(\det(x_v)) = \mathrm{ord}_v(q(f_v)). \tag{5.1.4}$$

Two triples (g_1, g_2, f) and (g_1', g_2', f') are equivalent if there is a $\gamma_i \in B^\times$ such that

$$\gamma_i g_i = g_i', \qquad \gamma_1 f \gamma_2^{-1} = f'. \tag{5.1.5}$$

By (6.4), the norms of g_1 and g_2 in the same class modulo $F_+^\times$. Thus by (6.5) we may modify them so that they have the same norm. Thus in term of orthogonal space $V = (B, q)$ and

$$H = \mathrm{GSpin}(V) = \{(g_1, g_2) \in B^\times, \nu(g_1) = \nu(g_2)$$

$$(g_1, g_2)x = g_1 x g_2^{-1}, \qquad g_i \in B^\times, x \in V,$$

we may rewrite condition (6.4) as

$$x^v = g^{-1} f^v, \qquad g = (g_1, g_2) \in H(\mathbb{A}_f^v) \tag{5.1.6}$$

It is clear that the equivalent class of (g_1, g_2, f) is completely determined by x^v. Indeed, since norm of x is positive, we have an element $f \in B$ with the same norm as x. Then there is a $g \in H(\mathbb{A}_f^v)$ such that $x = g^{-1} f^v$ in $\widehat{V}^v$.

Let $\mathscr{H}_v$ be the universal deformation of $\mathscr{V}_0$, then the union of universal deformation of surpersingular points is given by

$$B_0 \backslash \mathscr{H}_v \times (B \otimes \mathbb{A}_f^v)^\times / U^v$$

Notice that $\mathscr{H}_v$ is a formal scheme over $\mathscr{O}_v^{\mathrm{ur}}$. Thus the universal deformation of M_K at its supersingular points are given by

$$H(F)_0 \backslash \mathscr{D}_v \times H(\mathbb{A}_f^v)/K^v.$$

where $\mathscr{D}_v = \mathscr{H}_v \otimes_{\mathscr{O}_v^{\mathrm{ur}}} \mathscr{H}_v$. For every $f \in (B')^\times$, let $\mathscr{D}_f(c, d)$ be the divisor of $\mathscr{D}$ defined by universal deformation of f of type (c, d). Let H_f be the stabilizer of f then for any $g \in H(\mathbb{A}_f^v)$, we can define divisor

$$\begin{aligned} \mathscr{Z}(f, g, a, b)_K =& H(F)_0 \backslash H(F)_0 (\mathscr{D}_f(c, d) \times H_f(\mathbb{A}_f^v) g) K^v / K^v \\ \simeq & H_f(F) \backslash \mathscr{D}_f(c, d) \times H_f(\mathbb{A}_f^v)/K_f \end{aligned}$$

$K_f = H_f(\mathbb{A}_f^v) \cap g K_v g^{-1}$. As the equivalent class of the pair of (f, g) is completely determined by x^v and a, b is completely determined by x_v, we also write this cycle as $\mathscr{Z}(x_v, x^v)_K$.In this way, we have

$$\mathscr{Z}^{ss}(x)_{K,v} = \mathscr{Z}(x_v, x^v)_K \tag{5.1.7}$$

5.2 Local intersection at unramified place

In this section, we want to study the local intersection at a finite place v which is split in $\mathbb{B}$.

We still work on $\mathbb{H} = \mathrm{GSpin}(\mathbb{V})$. Let x_1, x_2, x_3 be three vectors in $K\backslash\mathbb{V}_f$ such that the cycles $Z(x_i)_K$ intersects properly in an integral model $\mathscr{M}_K$ of M_K. This means that there are no $k_i \in K$ such that the space

$$\sum Fk_ix_i$$

is two dimensional with totally positive definite intersection pairing.

First let us consider the case where U_v is maximal. We want to compute the intersection index at a geometric point (P_1, P_2) in the spacial fiber over a finite prime v of F. The non-zero intersection of the three cycles will imply that there are three quasi-isogenies $f_i : P_2 \longrightarrow P_1$ with type determined by x_i's. Notice that P_1 is ordinary (resp. supersingular) if and only if P_2 is ordinary (resp. supersingular).

If they both are ordinary, then we have canonical liftings $\widetilde{P}_i$ to CM-points on the generic fiber. All f_i can be also lifted to quasi-isogenies of $\widetilde{f}_i : \widetilde{P}_2 \longrightarrow \widetilde{P}_1$. This will contradict to the assumption that the three cycles $Z(x_i)_K$ have no intersection. This follows that P_i's are supersingular points.

Now lets us treat the case of supersingular intersection. By (6.7), we know that $Z(x_i)_K$ has an extension $\mathscr{Z}(f_i, g_i, c_i, d_i)_K$ on the formal neighborhood of surpersingular points:

$$H(F)_0\backslash\mathscr{D} \times H(\mathbb{A}_f^v)/K^v.$$

Here $c_i, d_i \in \mathbb{Z}$ such that $\begin{pmatrix} \pi^{c_i} & \\ & \pi^{d_i} \end{pmatrix} \in U_v x_{iv} U_v$, and $(f_i, g_i) \in B \times H(\mathbb{A}_f^v)$ such that $g_i^{-1}(f_i) = x_i^v$ in $\mathbb{V}_f^v$. If these three has nontrivial intersection at a supersingular point represented by $g \in H(F)_0\backslash H(\mathbb{A}_f^v)/K^v$, then we can write $g_i = gk_i$ with some $k_i \in K^v$. The intersection scheme $Z(k_1x_1, k_2x_2, k_3x_3)_K$ is represented by

$$Z(k_1x_1, k_2x_2, k_3x_3)_K = [\mathscr{D}_{f_1}(c_1, d_1) \cdot \mathscr{D}_{f_2}(c_2, d_2) \cdot \mathscr{D}_{f_3}(c_3, d_3) \times g]$$

on D, here $f = (f_i) \in (V)^3$ and $c = (c_i), d = (d_i) \in \mathbb{Z}^3$. As this intersection is proper, one has that the space generated by f_i's is three dimensional and positive definite. Notice that $g \in H(\mathbb{A}_f^v)/K^v$ is completely determined by the condition $g^{-1}f_i \in K^v x_i^v$. Thus we have that

$$(Z(x_1)_K \cdot Z(x_2)_K \cdot Z(x_3)_K)_v = \sum_{kx^v \in K^v\backslash(Kx_1^v, Kx_2^v, Kx_3^v)} \deg Z(k_1x_1, k_2x_2, k_3x_3)_K$$

where sum runs through cosets such that $k_ix_i^v$ generated a subspace of dimension 3.

In the following, we let us compute the intersection at v for cycles $Z(\phi_i)$ for $\phi_i \in \mathscr{S}(\mathbb{V})$. Assume that $\phi_i(x) = \phi_i^v(x^v)\phi_{iv}(x_v)$. By the above discussion, we see that

$$\begin{aligned} Z(\phi_1) \cdot Z(\phi_2) \cdot Z(\phi_3) &= \prod_{i=1}^{3} \sum_{x_i \in \widetilde{K}\backslash\mathbb{V}} \phi_i(x_i) Z(x_i)_K \\ &= \sum_{x^v \in \widetilde{K}^3\backslash(\mathbb{V}^v)^3_+} \sum_{x_v \in K_v^3\backslash(\mathbb{V}_v)^3_{x^v}} \phi(x) \deg Z(x)_K \\ &= \sum_{x^v \in \widetilde{K}^v\backslash(\mathbb{V}^v)^3_+} \phi^v(x^v) m(x^v, \phi_v) \end{aligned}$$

where $(\widehat{V})^3_+$ denote the set of elements $x^v \in (\widehat{V}^v)^3$ such that the intersection matrix of x_i^v as a symmetric elements in $M_3(\mathbb{A}_f^v)$ takes entries in F, $(V_v)^3_{x^v}$ denote the set of elements x_{i_v} with norm equal to the norm of x_i^v, and

$$m(x^v, \phi_v) = \sum_{x_v \in K_v^3 \backslash (\mathbb{V}_v)^3_{x^v}} \phi_v(x_v) \deg Z(x^v, x_v)_K.$$

In order to compare with theta series, let us rewrite the intersection in terms of space $V = B$. Notice that every x^v can be written as $x^v = g^{-1}(f)$ with $f \in (V)^3_+$ of elements with non-degenerate intersection matrix $T(x)$, then we have

$$Z(\phi_1) \cdot Z(\phi_2) \cdot Z(\phi_3) = \sum_{f \in H(F) \backslash V_+^3} \sum_{g \in H(\mathbb{A}^v)/\widetilde{K}^v} \phi^v(g^{-1}f) m(f, \phi_v)_K$$

where for $f \in (V_v)^3$

$$m(f, \phi_v) = \sum_{x_v \in K_v^3 \backslash (\mathbb{V}_v)^3_{x^v}} \phi_v(x_v) \deg Z(f, x_v)_K.$$

This is a pseudo-theta series if $m(\phi_v)$ has no singularity over $f \in (V_v)^3$.

In the following we want to deduce a formula for the intersection using work of Gross-Keating. For a element $f \in (B)_v$ with integral norm, let $\mathscr{T}_f$ denote the universal deformation divisor on $\mathscr{D}$ of the isogeny $f : \mathscr{V} \longrightarrow \mathscr{V}$. We extend this definition to arbitrary f by setting $\mathscr{T}_f = 0$ if f is not integral. Then we have the following relation:

$$\mathscr{D}_f(c, d) = \mathscr{T}_{\pi^{-c} f} - \mathscr{T}_{\pi^{-c-1} f}.$$

Indeed, for any $f \in \pi \mathscr{O}_B$, there is an embedding from $\mathscr{T}_{f/\pi}$ to $\mathscr{T}_f$ by taking any deformation $\varphi : \mathscr{E}_1 \longrightarrow \mathscr{E}_2$ to $\pi\varphi$. The complement are exactly the deformation with cyclic kernel. It follows that $\deg Z(x^v, x_v)$ is an alternative sum of intersection of Gross–Keating's cycles:

$$\deg Z(x^v, x_v) = \sum_{\epsilon_i \in \{0,1\}} (-1)^{\epsilon_1 + \epsilon_2 + \epsilon_3} \mathscr{T}_{\pi^{-c_1-\epsilon_1} f_1} \mathscr{T}_{\pi^{-c_2-\epsilon_2} f_2} \mathscr{T}_{\pi^{-c_3-\epsilon_3} f_3}$$

Theorem 5.2.1 (Gross-Keating, [10])**.** *Assume that ϕ_v is the characteristic function of $\mathscr{O}^3_{B,v}$. Then for $f \in (V'_v)^3$, the intersection number $m(f, \phi_v)$ depends only on the moment $T = T(f)$ and*

$$m(f, \phi_v) = \nu(T(f)).$$

Comparison

In this subsection we will relate the global v-Fourier coefficient of the analytic kernel function with the local intersection of triple Hecke correspondences when the Shimura curve has good reduction at v.

Recall that we have a decomposition of $E'(g,0,\phi)$ according to the difference of Σ_T and Σ:

$$E'(g,0,\phi) = \sum_v E'_v(g,0,\phi) + E'_{sing}(g,0,\phi) \tag{5.2.1}$$

where

$$E'_v(g,0,\phi) = \sum_{\Sigma_T=\Sigma(v)} E'_T(g,0,\phi) \tag{5.2.2}$$

and

$$E'_{sing}(g,0,\phi) = \sum_{T,\det(T)=0} E'_T(g,0,\phi).$$

Theorem 5.2.2. *Assume that ϕ_v is the characteristic function of $\mathscr{O}^3_{\mathbb{B}_v}$. And let S be the set of places outside which everything is unramified. Assume that for $w \in S$, ϕ_w is supported in $\mathbb{V}^3_{w,reg}$. Then for $g = (g_1, g_2, g_3) \in \mathbb{G}$ such that $g_{i,v} = 1$ for $v \in S$, we have an equality*

$$\deg(Z(g,\phi)_v) = E'_v(g,0,\phi).$$

Proof. By our choice of ϕ, there is no self-intersection in $(Z(\phi_1)\cdot Z(\phi_2)\cdot Z(\phi_3))_v$.

$$\begin{aligned}\deg(Z(g,\phi))_v &= \sum_{x^v\in(\widetilde{K}^v)^3\backslash(\mathbb{V}^v)^3_+} r(g^v)\phi^v(x^v)m(x^v, r(g_v)\phi_v)\\ &= \sum_{\Sigma(T)=\Sigma(v)}\prod_{w\neq v}\int_{(\mathbb{B}^3_v)_T} r(g^v)\phi_w(x_w)dx_w\cdot m_T(r(g_v)\phi_v)\end{aligned}$$

where

$$m_T(\phi_v) = \sum_{x_v\in K_v^3\backslash(\mathbb{B}_v)^3_{\mathrm{diag}(T)}} \phi_v(x_v)\deg Z_T(x_v)_K,$$

where the sum is over elements of $\mathbb{B}^3_v$ with norms equal to diagonal of T, and the cycle $Z_T(x_v)$ is equal to $Z(x^v, x_v)$ with $x^v \in (\mathbb{V}^v)$ with *non-singular* moment matrix T. Comparing with Whittaker function of Eisenstein series, we are left to prove the following equality

$$W'_{T,v}(g_v,0,\phi_v) = m_T(r(g_v)\phi_v). \tag{5.2.3}$$

By Gross-Keating, this is true when $g_v = e$ is the identity element. We will reduce the general g_v to this known case.

Suppose that

$$g_v = d(\nu)n(b)m(a)k$$

for b, a are both diagonal matrices and k in the standard maximal compact subgroup of $\mathbb{G}$. Then it is easy to see that the Whittaker function obeys the rule:

$$W'_{T,v}(g_v,0,\phi_v) = \psi(\nu Tb)|\nu|^{-3}|det(a)|^2 W'_{\nu aTa}(e,0,\phi_v).$$

On the intersection side, we have the similar formula:

$$\begin{aligned} m_T(r(g)\phi_v) =& |\nu|^{-3}\sum_{x_v} r(g_1)\phi_v(x_v)\deg Z_{\nu T}(x_v)_K \\ =& \psi_{\nu T}(b)|\nu|^{-3}|\det a|^2\sum_{x_v}\phi_v(xa)\deg Z_{\nu T}(x_v)_K \end{aligned}$$

where $h_v \in G_v$ with $\nu(h_v) = \nu^{-1}$ and the sum runs over all x_v with norm $\nu \cdot diag(T)$.

By our definition of cycles, for diagonal matrix a, we have

$$Z_{\nu T}(x) = Z_{\nu a T a}(xa).$$

It follows that

$$m_T(r(g)\phi_v) = \psi_{\nu T}(b)|\nu|^{-3}|\det a|^2 m_{\nu a T a}(\phi_v).$$

□

5.3 Archimedean height

Let B be the Hamilton quaternion and let ϕ be the standard Gaussian. Let $B' = M_{2,\mathbb{R}}$ be the matrix algebra. Let $x = (x_1, x_2, x_3) \in B'^3$ with non-singular moment matrix $Q(x)$ and let $g_i = g_{x_i}$ be a Green's function of D_{x_i}. Define the star product

$$\Lambda(x) = \int_{\mathscr{H}^2} g_1 * g_2 * g_3. \tag{5.3.1}$$

Then $\Lambda(x)$ depends only on the moment $Q(x) \in Sym_3(\mathbb{R})$ (with signature either $(1,2)$ or $(2,1)$ since B' has signature $(2,2)$). Hence we simply write it as $\Lambda(\frac{1}{4\pi}Q(x))$ (note that we need to shift it by a multiple 4π).

We will consider a Green's function of logarithmic singularity which we call *pre-Green function* since it does not give the admissible Green's function. Their difference will be discussed later.

Now we specify our choice of pre-Green functions. For $x \in B'$ consider a function $D = \mathscr{H}^2 \to \mathbb{R}_+$ defined by

$$s_x(z) := q(x_z) = 2\frac{(x,z)(x,\bar{z})}{(z,\bar{z})}.$$

In terms of coordinates $z = \begin{pmatrix} z_1 & -z_1z_2 \\ 1 & -z_2 \end{pmatrix}$ and $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$s_x(z) = \frac{(-az_2 + dz_1 - b + cz_1z_2)(-a\bar{z}_2 + d\bar{z}_1 - b + c\bar{z}_1\bar{z}_2)}{-(z_1 - \bar{z}_1)(z_2 - \bar{z}_2)}.$$

We will consider the pre-Green function of D_x on D given by

$$g_x(z) := \eta(s_x(z))$$

where we recall that
$$\eta(t) = Ei(-t) = -\int_1^\infty e^{-tu}\frac{du}{u}.$$

In the following we want to compute the star product for a non-singular moment $4\pi T = Q(x)$. Our strategy is close to that of [19], namely by steps: in the first step we will establish a $SO(3)$-invariance of $\Lambda(T)$ which simplifies the computation to the case T is diagonal; in the second step we compute $\Lambda(T)$ when T is diagonal and we compare the result with the derivative of the Whittaker integrals $W'_T(e, s, \phi)$.

Step one: $SO(3)$-invariance

The following lemma is a special case of a more general result of Kudla-Millson. For convenience we give a proof here.

Lemma 5.3.1. *Let $\omega_x = \partial\overline{\partial} g_x$. For any $(x_1, x_2) \in V^2$, the $(2,2)$-form $\omega_{x_1} \wedge \omega_{x_2}$ on $\mathscr{H}^2$ is invariant under the action of $SO(2)$ on V^2.*

Proof. Let $k \in SO(2)$ be the matrix $\begin{pmatrix} cos\theta & sin\theta \\ -cos\theta & sin\theta \end{pmatrix}$ and for simplicity, we denote $c = cos\theta$ and $s = sin\theta$. Let $x = cx_1 + sx_2$ and $y = -sx_1 + cx_2$. Then, by the formula
$$\begin{aligned} e^{s_x(z)}\omega_x &= s_x(z)\partial log s_x(z)\overline{\partial} log s_x(z) - \partial\overline{\partial} log s_x(z) \\ &= \frac{(x,z)(x,\overline{z})}{(z,\overline{z})}\Big(\frac{\partial(x,z)}{(x,z)} - \partial log(z,\overline{z})\Big)\Big(\frac{\overline{\partial}(x,\overline{z})}{(x,\overline{z})} - \overline{\partial} log(z,\overline{z})\Big) - \partial\overline{\partial} log(z,\overline{z}) \end{aligned}$$
and similar formula for ω_y, we have that
$$e^{s_x(z)+s_y(z)}\omega_x \wedge \omega_y(z) = A + B \wedge \partial\overline{\partial} log(z,\overline{z}) + \partial\overline{\partial} log(z,\overline{z})\partial\overline{\partial} log(z,\overline{z})$$
where
$$\begin{aligned} A = &\frac{(x,z)(x,\overline{z})}{(z,\overline{z})}\Big(\frac{\partial(x,z)}{(x,z)} - \partial log(z,\overline{z})\Big)\Big(\frac{\overline{\partial}(x,\overline{z})}{(x,\overline{z})} - \overline{\partial} log(z,\overline{z})\Big) \\ \wedge &\frac{(y,z)(y,\overline{z})}{(z,\overline{z})}\Big(\frac{\partial(y,z)}{(y,z)} - \partial log(z,\overline{z})\Big)\Big(\frac{\overline{\partial}(y,\overline{z})}{(y,\overline{z})} - \overline{\partial} log(z,\overline{z})\Big) \end{aligned}$$
and
$$\begin{aligned} B = &\frac{(x,z)(x,\overline{z})}{(z,\overline{z})}\Big(\frac{\partial(x,z)}{(x,z)} - \partial log(z,\overline{z})\Big)\Big(\frac{\overline{\partial}(x,\overline{z})}{(x,\overline{z})} - \overline{\partial} log(z,\overline{z})\Big) \\ + &\frac{(y,z)(y,\overline{z})}{(z,\overline{z})}\Big(\frac{\partial(y,z)}{(y,z)} - \partial log(z,\overline{z})\Big)\Big(\frac{\overline{\partial}(y,\overline{z})}{(y,\overline{z})} - \overline{\partial} log(z,\overline{z})\Big). \end{aligned}$$

It is easy to see that
$$\begin{aligned} B = &(s_x(z) + s_y(z))\partial log(z,\overline{z})\overline{\partial} log(z,\overline{z}) - \partial((x,\overline{z})(x,z) + (y,\overline{z})(y,z))\overline{\partial} log(z,\overline{z})/(z,\overline{z}) \\ &- \overline{\partial}((x,\overline{z})(x,z) + (y,\overline{z})(y,z))\partial log(z,\overline{z})/(z,\overline{z}) + (\partial(x,z)\overline{\partial}(x,\overline{z}) + \partial(y,z)\overline{\partial}(y,\overline{z}))/(z,\overline{z}). \end{aligned}$$

Now it is easy to see that the above sum is invariant since the following two terms are respectively invariant

$$(x,\overline{z})(x,z)+(y,\overline{z})(y,z), \quad (\partial(x,z)\overline{\partial}(x,\overline{z})+\partial(y,z)\overline{\partial}(y,\overline{z})).$$

Now we come to A:

$$(z,\overline{z})^2A=\partial(x,z)\overline{\partial}(x,\overline{z})\partial(y,z)\overline{\partial}(y,\overline{z})-((y,z)\partial(x,z)-(x,z)\partial(y,z))\overline{\partial}(x,\overline{z}))\overline{\partial}(y,\overline{z}))\partial log(z,\overline{z})$$
$$-((y,\overline{z})\overline{\partial}(x,\overline{z})-(x,\overline{z})\overline{\partial}(y,\overline{z}))\partial(x,z))\partial(y,z))\overline{\partial}log(z,\overline{z})$$
$$+(\partial(x,z)\overline{\partial}(x,\overline{z})+\partial(y,z)\overline{\partial}(y,\overline{z}))\partial log(z,\overline{z})\overline{\partial}log(z,\overline{z}).$$

This is invariant since the following four terms are respectively invariant

$$\partial(x,z)\overline{\partial}(x,\overline{z}), \quad \partial(y,z)\overline{\partial}(y,\overline{z}),$$
$$(y,z)\partial(x,z)-(x,z)\partial(y,z), \quad (y,\overline{z})\overline{\partial}(x,\overline{z})-(x,\overline{z})\overline{\partial}(y,\overline{z}).$$

This completes the proof. □

Proposition 5.3.2 (Invariance under $SO(3)$)**.** *The local archimedean height pairing* $\Lambda(T)$ *is invariant under* $SO(3)$*, i.e.,*

$$\Lambda(T)=\Lambda(kTk^t), \quad k\in SO(3).$$

Proof. Note that the group $SO(3)$ is generated by matrices of the form $\begin{pmatrix}1 & & \\ & cos\theta & sin\theta \\ & -cos\theta & sin\theta\end{pmatrix}$ and subgroup of even permutation of S_3, the symmetric group. Thus it suffices to prove that

$$\Lambda(x_1,x_2,x_3)=\Lambda(x,y,x_3)$$

for $x=cx_1+sx_2$ and $y=-sx_1+cx_2$ where $c=cos\theta, s=sin\theta$.

Further, since $g^*\omega_x=\omega_{g^{-1}x}$ for $g\in Aut(\mathscr{H}^2)$, we can assume that $x_3=\sqrt{a}\begin{pmatrix}1 & \\ & \pm1\end{pmatrix}$ depending on the sign of $det(x_3)$. Then $Z_{x_3}=\Delta(\mathscr{H})$ is the diagonal embedding of $\mathscr{H}$ if $det(x_3)>0$, otherwise $Z_{x_3}=\emptyset$.

By definition,

$$\Lambda(x,y,x_3)=\int_{\mathscr{H}^2}g_{x_3}(z)\omega_x(z)\wedge\omega_y(z)+\int_{Z_{x_3}}g_x*g_y|_{Z_{x_3}}.$$

Now the first term is invariant by Lemma above and the second term is either zero (when $det(x_3)<0$) or has been treated in the work of Kudla ([19]) when x,y generates a plane of signature $(1,1)$. The left case is when x,y generates a negative definite plane. In this case the proof of Kudla still applies. This completes the proof. □

Remark 7.

1. The proof of $SO(2)$-invariance in [19] is in deed very difficult though elementary.
2. Similarly, by induction we can prove invariance for $SO(n+1)$ for V of signature $(n,2)$.

Step two: star product

It turns out that for the convenience of computation, it is better to employ the bounded domain $\mathbb{D}^2$ where $\mathbb{D} = \{z \in \mathbb{C} || z| < 1\}$ is the unit disk. We have an explicit biholomorphic isomorphism from $\mathbb{D}^2 \to \mathscr{H}^2$ given by

$$(z_1, z_2) \mapsto (i\frac{1+z_1}{1-z_1}, i\frac{1+z_2}{1-z_2}).$$

Then, using the bounded model $\mathbb{D}^2$, we can express

$$s_x(z) = \frac{|(ai-b-c-di)z_1z_2 + (ai+b-c+di)z_1 + (-ai+b-c-di)z_2 + (-ai-b-c+di)|^2}{4(1-|z_1|^2)(1-|z_2|^2)}.$$

We first compute several differentials which will be used later on.

Lemma 5.3.3. *Let $a_i \in \mathbb{R}_+$ and $x_i \in B', i = 1, 2, 3, 4,$ be the following four elements*

$$x_1 = \sqrt{a_1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_2 = \sqrt{a_2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$x_3 = \sqrt{a_3}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x_4 = \sqrt{a_4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We will shorten $s_1(z) := s_{x_i}(z)$. Then we have

$$s_1(z) = a_1\frac{|z_1 - z_2|^2}{(1-|z_1|^2)(1-|z_1|^2)}, \quad s_2(z) = a_2\frac{|1 - z_1z_2|^2}{(1-|z_1|^2)(1-|z_1|^2)}$$

$$s_3(z) = a_3\frac{|1 + z_1z_2|^2}{(1-|z_1|^2)(1-|z_1|^2)}, \quad s_4(z) = a_4\frac{|z_1 + z_2|^2}{(1-|z_1|^2)(1-|z_1|^2)}.$$

Moreover, we have

$$e^{s_1(z)}\partial\overline{\partial}Ei(-s_1(z)) = \left(a_1 + a_1\frac{|z_1 - z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 1\right)\frac{dz_1 \wedge d\overline{z}_1}{(1-|z_1|^2)^2} + ...$$
$$- a_1\frac{(1-\overline{z}_1z_2)^2}{(1-|z_1|^2)(1-|z_2|^2)}\frac{dz_1 \wedge d\overline{z}_2}{(1-|z_1|^2)(1-|z_2|^2)} + ...$$

where we the omitted terms can be easily recovered by the symmetry of z_1, z_2. Similarly we have

$$e^{s_2(z)}\partial\overline{\partial}Ei(-s_2(z)) = \left(a_2\frac{|\overline{z}_1 - z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 1\right)\frac{dz_1 \wedge d\overline{z}_1}{(1-|z_1|^2)^2} + ...$$
$$- a_2\frac{(\overline{z}_1 - z_2)^2}{(1-|z_1|^2)(1-|z_2|^2)}\frac{dz_1 \wedge d\overline{z}_2}{(1-|z_1|^2)(1-|z_2|^2)} + ...$$

$$e^{s_3(z)}\partial\overline{\partial}Ei(-s_3(z)) = \left(a_3\frac{|\overline{z}_1+z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 1\right)\frac{dz_1\wedge d\overline{z}_1}{(1-|z_1|^2)^2} + ...$$
$$a_3\frac{(\overline{z}_1+z_2)^2}{(1-|z_1|^2)(1-|z_2|^2)}\frac{dz_1\wedge d\overline{z}_2}{(1-|z_1|^2)(1-|z_2|^2)} + ...$$

$$e^{s_4(z)}\partial\overline{\partial}Ei(-s_4(z)) = \left(a_4 + a_4\frac{|z_1+z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 1\right)\frac{dz_1\wedge d\overline{z}_1}{(1-|z_1|^2)^2} + ...$$
$$+ a_4\frac{(1+\overline{z}_1z_2)^2}{(1-|z_1|^2)(1-|z_2|^2)}\frac{dz_1\wedge d\overline{z}_2}{(1-|z_1|^2)(1-|z_2|^2)} + ...$$

And moreover

$$\partial\overline{\partial}Ei(-s_1(z))\wedge\partial\overline{\partial}Ei(-s_4(z)) = e^{-s_1(z)-s_4(z)}\frac{dz_1\wedge d\overline{z}_1}{(1-|z_1|^2)^2}\wedge\frac{dz_2\wedge d\overline{z}_2}{(1-|z_2|^2)^2}$$
$$\left(4a_1a_4\frac{(1-|z_1z_2|^2)^2}{(1-|z_1|^2)^2(1-|z_2|^2)^2} - 2a_1\frac{|1+\overline{z}_1z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 2a_4\frac{|1-\overline{z}_1z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} + 2\right).$$

$$\partial\overline{\partial}Ei(-s_2(z))\wedge\partial\overline{\partial}Ei(-s_3(z)) = e^{-s_2(z)-s_3(z)}\frac{dz_1\wedge d\overline{z}_1}{(1-|z_1|^2)^2}\wedge\frac{dz_2\wedge d\overline{z}_2}{(1-|z_2|^2)^2}$$
$$\left(4a_2a_3\frac{(|z_1|^2-|z_2|^2)^2}{(1-|z_1|^2)^2(1-|z_2|^2)^2} - 2a_2\frac{|\overline{z}_1-z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 2a_3\frac{|\overline{z}_1+z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} + 2\right).$$

Proof. Simple but tedious computation.

□

We also need

Lemma 5.3.4 (Change of variables)**.** *Define a diffeomorphism between* $\mathbb{D}^2$ *and* $\mathbb{C}^2\simeq\mathbb{R}^4$ *by* $(z_1,z_2)\mapsto(w_1,w_2)$ *where* $w_i = u_i+\sqrt{-1}v_i$ *and*

$$u_i = \frac{x_i}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}},\quad v_i = \frac{y_i}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}.$$

Then the Jacobian is given by

$$\frac{\partial(u_1,v_1,u_2,v_2)}{\partial(x_1,y_1,x_2,y_2)} = -\frac{1-|z_1|^2|z_2|^2}{(1-|z_1|^2)^3(1-|z_2|^2)^3}.$$

Moreover we have

$$\frac{dx_1dy_1dx_2dy_2}{(1-|z_1|^2)^2(1-|z_2|^2)^2} = -\frac{du_1dv_1du_2dv_2}{\sqrt{(1+|w_1|^2+|w_2|^2)^2-4|w_1|^2|w_2|^2}}.$$

Proof. Let $\lambda = \frac{1}{(1-|z_1|^2)(1-|z_2|^2)}$. Note that

$$u_1^2 + v_1^2 = \lambda - \frac{1}{1-|z_2|^2}$$

and similarly

$$u_2^2 + v_2^2 = \lambda - \frac{1}{1-|z_1|^2}.$$

This shows that λ satisfies a quadratic equation

$$\lambda^2 - (1 + |w_1|^2 + |w_2|^2)\lambda + |w_1|^2|w_2|^2 = 0.$$

Denote its two roots by $\lambda_1 > \lambda_2$. Since $|z_i| < 1$, a careful check shows that $\lambda = \lambda_1$ is the larger one of its two roots. Moreover, we have

$$\frac{1 - |z_1|^2|z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} = \lambda - \lambda^{-1}|w_1|^2|w_2|^2 = \lambda_1 - \lambda_2 = \sqrt{\Delta}$$

where Δ is the discriminant of the quadratic equation above.

$\square$

Theorem 5.3.5. *We have for $T \in Sym_3(\mathbb{R})$ with signature either $(1, 2)$ or $(2, 1)$,*

$$W'_{T,\infty}(e, 0, \phi) = -\frac{\kappa(0)}{\sqrt{\pi}\Gamma_3(2)} e^{-2\pi T}\Lambda(T).$$

In particular, everything depends only on the eigenvalues of T (presumedly not obvious).

Proof. By Proposition 5.3.2, we may assume that T is a diagonal matrix.

We first treat the case $(p, q) = (2, 1)$ and let's assume that $4\pi T = (a_1, a_4, -b)$. And we may choose x_i as in Lemma 5.3.3 as long as we take $a_2 = b$. Then by the same lemma, $\Lambda(T)$ is given by the integral

$$\Lambda(T) = \int_{\mathbb{D}^2} Ei(-b\frac{|1 + z_1z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)})e^{-s_1(z)-s_4(z)} \frac{dz_1 \wedge d\overline{z}_1}{(1-|z_1|^2)^2} \wedge \frac{dz_2 \wedge d\overline{z}_2}{(1-|z_2|^2)^2}$$
$$(4a_1a_4\frac{(1-|z_1z_2|^2)^2}{(1-|z_1|^2)^2(1-|z_2|^2)^2} - 2a_1\frac{|1+\overline{z}_1z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 2a_4\frac{|1-\overline{z}_1z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} + 2)$$

Now let us make the substitution

$$u_1 = \frac{x_1 + x_2}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}, \quad u_2 = \frac{x_1 - x_2}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}$$
$$v_1 = \frac{y_1 + y_2}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}, \quad v_2 = \frac{y_1 - y_2}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}.$$

From Lemma 5.3.4 we may calculate the Jacobian of our substitutions

$$\Lambda(T) = \frac{1}{4}\int_{\mathbb{R}^4} Ei(-b(1+u_1^2+v_2^2))e^{-a_1(u_2^2+v_2^2)-a_4(u_1^2+v_1^2)}$$
$$(4a_1a_4\Delta - 2a_1(1+u_1^2+v_1^2) - 2a_4(1+u_2^2+v_2^2)+2)\frac{du_1dv_1du_2dv_2}{\sqrt{\Delta}}$$

which we may rearrange as

$$\frac{1}{4}\int_{\mathbb{R}^2} Ei(-b(1+u_1^2+v_2^2))e^{-a_4v_2^2-a_1u_1^2}dv_2du_1\int_{\mathbb{R}^2} e^{-a_4u_2^2-a_1v_1^2}$$
$$(4a_1a_4\Delta - 2a_1(1+u_1^2+v_1^2) - 2a_4(1+u_2^2+v_2^2)+2)\frac{dv_1du_2}{\sqrt{\Delta}}.$$

Here

$$\Delta = 1+u_1^2+v_1^2+u_2^2+v_2^2+(u_1u_2+v_1v_2)^2. \tag{5.3.2}$$

Comparing with Proposition 3.3.7, it suffices to prove that the integral

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^2} e^{-a_4y_2^2-a_1y_1^2}(4a_1a_4\Delta - 2a_1(1+x_1^2+y_1^2) - 2a_4(1+x_2^2+y_2^2)+2)\frac{dy_1dy_2}{\sqrt{\Delta}} \tag{5.3.3}$$

is equal to

(5.3.4)
$$\int_{\mathbb{R}} e^{-u^2}\frac{4AB-2A-2B+2}{((u^2+A)(u^2+B))^{1/2}}du + \int_{\mathbb{R}} e^{-u^2}\frac{(4AB-A-B)u^2+2AB(A+B-1)}{((u^2+A)(u^2+B))^{3/2}}du.$$

Here note that we rename the variables u_i, v_i to x_i, y_i and they should not be confused with the real/imaginary part of z_i (coordinates of the bounded domain $\mathbb{D}$). And A, B are the two eigenvalues (as the z_1, z_2 in Prop. 3.3.7) of 2×2 matrix $(1+xx')^{1/2}a(1+xx')^{1/2}$ for x be the column vector $(x_1,x_2)^t$and

$$\Delta = (1+x'x)(1+y'y) - (x'y)^2.$$

Now notice that the 2×2 matrix $(1+xx')^{-1} = 1 - \frac{1}{1+x'x}xx'$. We have

$$1 + y'(1+xx')^{-1}y = 1 + y'y - \frac{y'xx'y}{1+x'x} = \frac{\Delta}{1+x'x}.$$

Substitute $y \mapsto (1+xx')^{1/2}y$, the integral (5.3.3) is reduced

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^2} e^{-y'(1+xx')^{1/2}a(1+xx')^{1/2}y}(4AB(1+y'y) - 2y'(1+xx')^{1/2}a(1+xx')^{1/2}y - 2A-2B+2)\frac{dy_1dy_2}{\sqrt{1+y'y}}.$$

Now make another substitution $y \mapsto ky$ where $k \in SO(2)$ is such that

$$(1+xx')^{1/2}a(1+xx')^{1/2} = k'diag(A,B)k$$

we obtain:

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^2} e^{-y'diag(A,B)y}(4AB(1+y'y)-2y'diag(A,B)y-2A-2B+2)\frac{dy_1dy_2}{\sqrt{1+y'y}}. \tag{5.3.5}$$

Using the integral

$$\int_{x\in\mathbb{R}} e^{-Ax^2}dx = \frac{1}{\sqrt{A}}\Gamma(1/2) = \sqrt{\pi}\frac{1}{\sqrt{A}}$$

we may rewrite the integral (5.3.5) as

$$\int_{\mathbb{R}^3} e^{-w^2(1+y_1^2+y_2^2)-Ay_1^2-By_2^2}(4AB(1+y_1^2+y_2^2)-2(Ay_1^2+By_2^2)-2A-2B+2)dwdy_1dy_2.$$

We interchange the order of integrals

$$\int_{\mathbb{R}} e^{-w^2}\int_{\mathbb{R}^2} e^{-y_1^2(w^2+A)-y_2^2(w^2+B)}((4AB-2A)y_1^2+(4AB-2B)y_2^2-4AB-2A-2B+2)dy_1dy_2dw.$$

Now we can integrate against y_1, y_2 and it is easy to verify that we arrive at the integral (5.3.4). This finishes the proof when $(p,q)=(2,1)$.

We now treat the slightly harder case $(p,q)=(1,2)$. Assume that $4\pi T=(a,-b_1,-b_2)$ and we may take $a_4=a, b_1=a_3, b_2=a_2$ as in Lemma 5.3.3. Then the same substitution as before yields that $\Lambda(T)$ is the sum of two terms:

$$\begin{aligned}&\frac{1}{4}\int_{\mathbb{R}^4} Ei(-a(u_1^2+v_1^2))e^{-b_1(1+u_1^2+v_2^2)-b_2(1+u_2^2+v_1^2)}\\ &\times(4b_1b_2(u_1u_2+v_1v_2)^2-2b_1(u_1^2+v_2^2)-2b_2(u_2^2+v_1^2)+2)\frac{du_1du_2dv_1dv_2}{\sqrt{\Delta}}.\end{aligned}$$

and (note that $s_4(z)$ has zeros along the divisor defined $z_1+z_2=0$ on $\mathbb{D}^2$)

$$\int_{\mathbb{D}} Ei(-s_2(z,-z))\partial\overline{\partial}Ei(-s_3(z,-z)).$$

By Proposition 3.3.5, the Whittaker integral also breaks into two pieces. It is easy to prove that the first one matches the second term above. In deed this already appeared in the work [23, Thm. 5.2.7, (ii)]. It suffices to prove that the integral

$$\int_{\mathbb{R}^2} e^{-b_1v_2^2-b_2u_2^2}(4b_1b_2(u_1u_2+v_1v_2)^2-2b_1(u_1^2+v_2^2)-2b_2(u_2^2+v_1^2)+2)\frac{du_2dv_2}{\sqrt{\Delta}} \tag{5.3.6}$$

is equal to

$$\sqrt{\pi}\int_{\mathbb{R}} e^{-u^2}\left(\frac{-2(A+B-1-b_1-b_2)}{(u^2+A)^{1/2}(u^2+B)^{1/2}}+\frac{(2AB-2b_1b_2-A-B)u^2+2AB(A+B-1-b_1-b_2)}{(u^2+A)^{1/2}(u^2+B)^{3/2}}\right)du. \tag{5.3.7}$$

Here A, B are the two eigenvalues of $(1+w'w)^{1/2}b(1+w'w)^{1/2}$ where $w=(u_1,v_1)$.

Similar to the previous case we may rewrite the integral (5.3.6) as

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-x^2(1+u_1^2+v_1^2)} I(x)dx \tag{5.3.8}$$

where

$$I(x)=\int_{\mathbb{R}^2} e^{-(b_1+x^2)v_2^2-(b_2+x^2)u_2^2-x^2(u_1u_2+v_1v_2)^2}(4b_1b_2(u_1u_2+v_1v_2)^2-2b_1(u_1^2+v_2^2)-2b_2(u_2^2+v_1^2)+2)du_2dv_2.$$

We now want to make the exponent in $I(x)$ as a linear combination of only square terms (we point out that the same idea also works for the previous case). This suggests to make the substitution

$$y_1=u_1u_2+v_1v_2,\quad y_2=\sqrt{\frac{b_2+x^2}{b_1+x^2}}v_1u_2-\sqrt{\frac{b_1+x^2}{b_2+x^2}}u_1v_2.$$

Then we have

$$u_2=\eta^{-1}(-\sqrt{\frac{b_1+x^2}{b_2+x^2}}u_1y_1-v_1y_2),\quad v_2=\eta^{-1}(-\sqrt{\frac{b_2+x^2}{b_1+x^2}}v_1y_1+u_1y_2)$$

where

$$\eta=\frac{(b_1+x^2)u_1^2+(b_2+x^2)v_1^2}{\sqrt{(b_1+x^2)(b_2+x^2)}}.$$

And

$$y_1^2+y_2^2=((b_2+x^2)u_2^2+(b_1+x^2)v_2^2)(\frac{u_1^2}{b_2+x^2}+\frac{v_1^2}{b_1+x^2}).$$

Then after the substitution we obtain

$$I(x)=\int_{\mathbb{R}^2} e^{-(\frac{(b_1+x^2)(b_2+x^2)}{(b_1+x^2)u_1^2+(b_2+x^2)v_1^2}+x^2)y_1^2-\frac{(b_1+x^2)(b_2+x^2)}{(b_1+x^2)u_1^2+(b_2+x^2)v_1^2}y_2^2}(Cy_1^2+Dy_2^2+E)\eta^{-1}dy_1dy_2$$

where

$$\begin{cases} C=4b_1b_2+\eta^{-2}(-2b_1\frac{b_2+x^2}{b_1+x^2}v_1^2-2b_2\frac{b_1+x^2}{b_2+x^2}u_1^2)\\ D=\eta^{-2}(-2b_1u_1^2-2b_2v_1^2)\\ E=-2b_1u_1^2-2b_2v_1^2+2.\end{cases}$$

Moreover let's denote

$$\begin{aligned}F=&(b_1+x^2)(b_2+x^2)+((b_1+x^2)u_1^2+(b_2+x^2)v_1^2)x^2\\ =&(1+u_1^2+v_1^2)x^4+(b_1(1+u_1^2)+b_2(1+v_1^2))x^2+b_1b_2.\end{aligned}$$

Now we may fold the integrals against y_1, y_2 to obtain

$$I(x)=\pi F^{-1/2}\left(CF^{-1}((b_1+x^2)u_1^2+(b_2+x^2)v_1^2)/2+D\frac{(b_1+x^2)u_1^2+(b_2+x^2)v_1^2}{(b_1+x^2)(b_2+x^2)}/2+E\right)$$

which can be simplified as

$$\pi F^{-1/2}E + \pi F^{-3/2}\left(2b_1b_2(u_1^2+v_1^2) - b_1(1+u_1^2) - b_2(1+v_1^2))x^2 + 2b_1b_2(b_1u_1^2+b_2v_1^2-1)\right).$$

Plug back to the integral (5.3.8) and make a substitution $u = x(1+u_1^2+v_1^2)$. Therefore, we have proved that the integral (5.3.6) is equal to

$$\pi\int_{\mathbb{R}} e^{-u^2}\frac{-2b_1u_1^2 - 2b_2v_1^2 + 2}{(u^4 + (b_1(1+u_1^2)+b_2(1+v_1^2))u^2 + b_1b_2(1+u_1^2+v_1^2))^{1/2}}du$$
$$+\pi\int_{\mathbb{R}} e^{-u^2}\frac{(2b_1b_2(u_1^2+v_1^2) - b_1(1+u_1^2) - b_2(1+v_1^2))u^2 + 2b_1b_2(1+u_1^2+v_1^2)(b_1u_1^2+b_2v_1^2-1)}{(u^4 + (b_1(1+u_1^2)+b_2(1+v_1^2))u^2 + b_1b_2(1+u_1^2+v_1^2))^{3/2}}du$$

This is clearly equal to the integral (5.3.7). We then complete the proof. □

Comparison

In this section, assume that $\tau|\infty$ and $\psi_\tau(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$. Recall that the generating function is defined for $g \in GL_2^+(\mathbb{A})$

$$Z(g,\phi) = \sum_{x\in\widehat{V}/K} r(g_{1f})\phi(x)Z(x)_K W_{T(x)}(g_\infty)$$

where the sum runs over all admissible classes. And for our fixed embedding $\tau : F \hookrightarrow \mathbb{C}$ we have an isomorphism of $\mathbb{C}$-analytic varieties (as long as K is neat):

$$Y_{K,\tau}^{an} \simeq G(F)\backslash D \times G(\mathbb{A}_f)/K \cup \{\text{cusp}\}$$

where, for short, $G = G(\tau)$ is the nearby group.

For $x_i \in V, i = 1,2,3$, we define the archimedean height

$$(Z(x_1,h_1)_K \cdot Z(x_2,h_2)_K \cdot Z(x_3,h_3)_K)_\infty := g_{x_1,h_1K} * g_{x_2,h_2K} * g_{x_3,h_3K}$$

where the Green function is defined as follows: for $[z,h'] \in G(F)\backslash D \times G(\mathbb{A}_f)/K$

$$g_{x,hK}([z,h']) = \sum_{\gamma\in G(F)/G_x(F)} \gamma^*[\eta(s_x(z)1_{G_x(\widehat{F})hK}(h'))].$$

For an admissible class $x \in \widehat{V}$ we will denote by g_x its Green function.

Theorem 5.3.6. *Let $g = (g_1,g_2,g_3) \in \mathbb{G}_{\mathbb{A}} = GL_2^{+,3}(\mathbb{A})$ and T be non-singular of sign $(1,2)$ or $(2,1)$. And assume that ϕ_v is supported on non-singular locus at some finite place v. Then the archimedean contribution $(Z(g_1,\phi_1)\cdot Z(g_2,\phi_2)\cdot Z(g_3,\phi_3))_\infty$ is given by*

$$\sum_{x=(x_i)\in G(F)\backslash V(v)(F)^3}\left(\int_{G(F)}\phi(hx)dh\right)\left(\int_D *_{i=1}^3 g_{x_i}(z)dz\right).$$

Proof. First we consider $g = (g_1, g_2, g_3) \in SL_2^3(\mathbb{A})$. Afterwards we extend this to $GL_2^+(\mathbb{A})$.

By definition we have

$$Z(g,\phi)_\infty = \sum_{x=(x_i)\in(K\backslash\widehat{V})^3} \phi(x) W_{T(x_\infty)}(g_\infty)\left(\int_{G(F)\backslash D\times G(\mathbb{A}_f)/K} *_{i=1}^3 g_{x_i}(z,h')d[z,h']\right)$$

where the sum is over all admissible classes.

Note that

$$\gamma^*[\eta(s_x(z)1_{G_x(\widehat{F})hK}(h'))] = \eta(s_{\gamma^{-1}x}(z)1_{G_{\gamma 1 x}(\widehat{F})\gamma^{-1}hK}(h')).$$

For a fixed triple (x_i), the integral is nonzero only if there exists a $\gamma \in G(F)$ such that

$$\gamma h' \in G_{\gamma_i^{-1}x_i}(\widehat{F})\gamma_i^{-1}h_i K \Leftrightarrow \gamma_i^{-1}h_i \in G_{\gamma_i^{-1}x_i}(\widehat{F})\gamma h' K.$$

Observe that the sum in the admissible classes can be written as $x_i \in G(F)\backslash V(F)$ and $h_i \in G_{x_i}(\widehat{F})\backslash G(\widehat{F})/K$. Here we denote for short $V = V(v)$ that is the nearby quadratic space ramified at $\Sigma(v)$. Thus we may combine the sum $x_i \in G(F)\backslash V(F)$ with $\gamma_i \in G(F)/G_{x_i}(F)$ and combine the sum over $\gamma \in G(F)$ with the quotient $G(F)\backslash D \times G(\mathbb{A}_f)/K$:

$$\sum_{x\in G(F)\backslash V(F)^3}\left(\int_{h'\in G(\widehat{F})/K}\phi(h'x)dh'\right)\left(\int_D *_{i=1}^3 g_{x_i}(z)dz\right).$$

Here we have used the fact that $G_x = \{1\}$ if $T(x)$ is non-singular and we are assuming that ϕ_v is supported in the non-singular locus at some finite place v.

It is routine to extend the result to GL_2^+.

□

Holomorphic projection

In the rest of this subsection we will calculate the holomorphic projection of $E'(g, 0, \phi)$. By the lemma above, we need to calculate the integral

$$\alpha_s(T) := \int_{\mathbb{R}_+^3} W_T'(\phi, \begin{pmatrix} y^{1/2} & \\ & y^{-1/2}\end{pmatrix}, 0)det(y)^{1+s}e^{-2\pi Ty}\frac{dy}{det(y)^2}$$

where $y = diag(y_1, y_2, y_3)$ and $T \in Sym_3(\mathbb{R})$ with positive diagonal $diag(T) = t = diag(t_1, t_2, t_3)$.

Note that when $t > 1$ and $Re(s) > -1$, we have an integral representation of the Legendre function of the second kind:

$$Q_s(t) = \int_{\mathbb{R}_+}\frac{du}{(t+\sqrt{t^2-1}cosh\, u)^{1+s}} = \frac{1}{2}\int_1^\infty \frac{(x-1)^s dx}{x^{1+s}(\frac{t-1}{2}x+1)^{1+s}}.$$

And the admissible pairing at archimedean place will be given by the constant term at $s = 0$ of (the regularized sum of) $Q_s(1 + 2s_{\gamma x}(z)/q(x))$.

Consider another function for $t > 1, Re(s) > -1$:

$$P_s(t) := \frac{1}{2}\int_1^\infty \frac{dx}{x(\frac{t-1}{2}x+1)^{1+s}}.$$

Then obviously we have

$$Q_0(t) = P_0(t).$$

One may use either of the three functions (i.e., Ei, Q_s and P_s) to construct Green's functions. And Theorem 5.3.5 shows that to match the analytic kernel function, the function Ei is the right choice; while the admissible pairing requires to use Q_s. The following proposition relates Ei to P_s and hence to Q_s by the coincidence $Q_0 = P_0$.

Proposition 5.3.7. *Let $x \in M^3_{2,\mathbb{R}}$ such that $T = T(x)$ is non-singular and has positive diagonal. Then we have*

$$\alpha_s(T) = det(t)^{-1}\left(\frac{\Gamma(s+1)}{(4\pi)^{1+s}}\right)^3 \int_{\mathbb{D}} \eta_s(x_1) * \eta_s(x_2) * \eta_s(x_3)$$

where

$$\eta_s(x,z) := P_s(1 + 2\frac{s_x(z)}{q(x)})$$

is a Green's function of $\mathbb{D}_x$.

Proof. Firstly by the definition we have

$$\alpha_s(T) = \int_{\mathbb{R}^3_+} det^2(y^{1/2})W'_{\sqrt{y}T\sqrt{y}}(\phi, e, 0)det(y)e^{-2\pi Ty}det(y)^s\frac{dy}{y^2}$$

which is equal to

$$\int_{\mathbb{R}^3_+} W'_{\sqrt{y}T\sqrt{y}}(\phi, e, 0)e^{-2\pi Ty}det(y)^s dy.$$

If we modify $x \in M^3_{2,\mathbb{R}}$ with moment $T = T(x)$ to a new $x' = (x'_i)$ with $x'_i = x_i/q(x_i)^{1/2}$, we have $T(x') = t^{-\frac{1}{2}}Tt^{-\frac{1}{2}}$ (so that the diagonal are all 1). By Theorem 5.3.5 we have (after substitution $y \to yt$)

$$\alpha_s(T) = det(t)^{-1-s}\int_{\mathbb{R}^3_+} \Lambda(y^{\frac{1}{2}}T(x')y^{\frac{1}{2}})e^{-4\pi y}det(y)^s dy.$$

By the definition of $\Lambda(T)$, this is the same as

$$det(t)^{-1-s}\int_{\mathbb{R}^3_+} \{*_{i=1}^3 \eta(y_i^{\frac{1}{2}}x'_i; z), 1\}_D e^{-4\pi y}det(y)^s dy$$

where

$$\{*_{i=1}^3 \eta(y_i^{\frac{1}{2}}x'_i; z), 1\}_D = \int_D *_{i=1}^3 \eta(y_i^{\frac{1}{2}}x'_i; z).$$

It is easy to see that one can interchange the star product and integral over y. We thus obtain

$$\alpha_s(T) = det(t)^{-1-s}\{*_{i=1}^3 \int_{\mathbb{R}_+} \eta(y_i^{\frac{1}{2}}x_i; z)e^{-4\pi y}y_i^s dy_i, 1\}_D.$$

Now we compute the inner integral:

$$\begin{aligned}
&\int_{\mathbb{R}_+} \eta(y^{\frac{1}{2}}x; z)e^{-4\pi y}y^s dy \\
=&\int_{\mathbb{R}_+} Ei(-4\pi y s_x(z))e^{-4\pi y}y^s dy \\
=&\int_{\mathbb{R}_+}\int_1^\infty e^{-4\pi y s_x(z)u}\frac{1}{u}du e^{-4\pi y}y^s dy \\
=&\frac{\Gamma(s+1)}{(4\pi)^{1+s}}\int_1^\infty \frac{1}{u(1+s_x(z)u)^{1+s}}du \\
=&\frac{\Gamma(s+1)}{(4\pi)^{1+s}}P_s(1+2s_x(z)).
\end{aligned}$$

□

Based on the decomposition of $E'(g,0,\phi)$ in sec. 2.8, we can have a decomposition of its holomorphic projection, denoted by $E'(g,0,\phi)_{hol}$:

$$E'(g,0,\phi)_{hol} = \sum_v \sum_{T,\Sigma(T)=\Sigma(v)} E'_T(g,0,\phi)_{hol} \tag{5.3.9}$$

where we only change $E'_T(g,0,\phi)$ to $E'_T(g,0,\phi)_{hol}$ when $\Sigma(T)=\Sigma(v)$ for v is one archimedean place and in this case

$$E'_T(g,0,\phi)_{hol} = W_T(g_\infty)m_v(T)W_{T,f}(g_f,0,\phi_f)$$

where $m(T)$ is the star product of $P_s(1+2s_x(z)/q(x))$ for x with moment T.

Note that all equalities above are for $g\in\mathbb{G}$ with $g_v=1$ when $v\in S$, the finite set of non-archimedean places where ϕ_v is ramified of sufficiently large order.

Summarizing all above, we have

Theorem 5.3.8. *Assume that for at least two v where $\phi_v\in\mathscr{S}(V^3_{v,reg})$. Then for $g\in\mathbb{G}$ with $g_w=1$ for $w\in S_f$, the set of finite place outside which ϕ_v is spherical, Then we have*

$$Z(g,\phi)_v = \sum_{T,\Sigma(T)=\Sigma(v)} E'_T(g,0,\phi)_{hol}.$$

Proof. Under the assumption, all nonsingular coefficients vanish on both sides! The comparison of non-singular coefficients are done by Theorem 5.3.5 and the holomorphic projection above. □

5.4 Local intersection at ramified places

In this subsection, we want to describe the local height pairing of Gross–Schoen cycles on the triple product of a Shimura curve at bad places. Our treatment is complete only if v is non-split in the corresponding quaternion algebra. Some further treatment needs to treat so called non surpersingular local intersection.

Let ${}_vV$ be the coherent quadratic space attached to the quaternion algebra ramified at $\Sigma(v)$.

Theorem 5.4.1. *Assume that $\phi_v \in \mathscr{S}(V_v^3)$ is ramified of sufficiently larger order. Then there exists an $\phi'_v \in \mathscr{S}({}_vV_v^3)$ such that for $\phi^{(v)} = \phi^v \otimes \phi'_v \in \mathscr{S}({}_vV_{\mathbb{A}}^3)$ we have*

$$deg Z_\phi(g)_v = E(g, 0, \phi^{(v)})$$

when $g \in \mathbb{G}_{\mathbb{A}}$ has $g_w = 1$ for $w \in S$ (in particular for $w = v$).

A first decomposition of global heights

Let F be a totally real field with adeles $\mathbb{A}$ and let $\mathbb{B}$ be an incoherent quaternion algebra over $\mathbb{A}$ which is totally definite at archimedean places. For U an open compact group of $\mathbb{B}_f^\times$ then we have Shimura curve X_U and Gross–Schoan cycles Δ_ξ on X_U^3 with respect to the Hodge class ξ:

$$\Delta_\xi = \Delta - \sum_{i<j} \Delta_{ij} \cdot \pi_k^*\xi + \sum_{i<j} \pi_i^*\xi \cdot \pi_j^*\xi$$

where k is the complement of i, j and $\Delta_{i,j}$ is the partial diagonal $x_i = x_j$.

Let $\phi \in \mathscr{S}(\mathbb{V}^3)$ invariant under $\widetilde{K}^3$ with $\mathrm{O}(F_\infty) \times U^2$, then we have a well-defined generating series of Hecke operators

$$Z(\phi)_U = \sum_{x \in \mathrm{O}(F_\infty)^3 U^6 \backslash \mathbb{V}^3} \phi(x) Z(x)_U$$

and the generating series of height pairing

$$\langle \Delta_\xi, Z(\phi)_U \Delta_\xi \rangle.$$

Recall that this height pairing is taken on a good model of X_U^3 or more precisely, the subvariety Y of points in the same connected components. This height pairing can be computed using *regular model* $\mathscr{Y}$ of Y. In fact, Gross–Schoen shows that this can be done using a good semistable model: there is a unique arithmetic class $\widehat{\Delta}_\xi$ perpendicular to any vertical to any vertical classes.

$$\langle \Delta_\xi, Z(\phi)_U \Delta_\xi \rangle = (\widehat{\Delta}_\xi, Z(\phi)_U \widehat{\Delta}_\xi)$$

Using de Jong's alternation, this classes can be defined over any regular model.

The Shimura variety Y is attached to the subgroup $\mathbb{G}$ of $\mathbb{B}^3$ consists of triples (b_i) with the same reduced norms. Assume that $U = U^v U_v$ with U^v sufficiently small. One good model $\mathscr{Y}$ can be chosen by the following process:

1. Let $U^0 = U^v U_v^0$ be an open compact subgroup of $\mathbb{B}_f^\times$ such that U_v^0 is maximal. Then corresponding Shimura curve X^0 has a regular model $\mathscr{X}^0$.
2. Take a normalization $\mathscr{X}'$ of $\mathscr{X}^0$ in some base change X_L for some finite extension.
3. Blowing up $\mathscr{X}'$ at its singular points sufficiently many times to get a regular semistable model $\mathscr{X}$.
4. Blowing up components of $\mathscr{X}^3$ in any given order to get a good model $\widetilde{\mathscr{X}^3}$ for X_L^3.
5. The union of diagonal connected components in $\widetilde{\mathscr{X}^3}$ gives a good model $\mathscr{Y}$.

Let $\widehat{\xi}$ be the canonical arithmetic extension of ξ and let $\widehat{\Delta}_{i,j}$ be an arithmetic classes with admissible currents at archimedean places. Then there is an unique arithmetic extension $\widehat{\Delta}$ such that

$$\widehat{\Delta}_\xi = \widehat{\Delta} - \sum_{i<j} \widehat{\Delta}_{ij} \cdot \pi_k^* \widehat{\xi} + \sum_{i<j} \pi_i^* \widehat{\xi} \cdot \pi_j^* \widehat{\xi}$$

Since $\widehat{\Delta}_\xi$ is perpendicular to vertical cycles, in the intersection pairing, we may replace the first $\widehat{\Delta}_\xi$ by *any arithmetic class* $\widehat{\Delta}_\xi^0$. For our purpose, we take this class in the fom

$$\widehat{\Delta}_\xi^0 = \widehat{\Delta}^0 - \sum_{i<j} \widehat{\Delta}_{ij}^0 \cdot \pi_k^* \widehat{\xi} + \sum_{i<j} \pi_i^* \widehat{\xi} \cdot \pi_j^* \widehat{\xi}$$

with $\widehat{\Delta}^0$ and $\widehat{\Delta}_{ij}^0$ are *arbitrary arithmetic extensions* of Δ and $\Delta_{i,j}$. By adjoint property of Hecke operators:

$$(\widehat{Z}_1, Z(\phi)_U \widehat{Z}_2) = (Z(\phi^\vee)_U \widehat{Z}_1, \widehat{Z}_2)$$

where $\phi^\vee(x) = \phi(\bar{x})$,

$$(\widehat{\Delta}_\xi, Z(\phi)\widehat{\Delta}_\xi) = (\widehat{\Delta}_\xi^0, Z(\phi)_U \widehat{\Delta}_\xi) = (\widehat{\Delta}^0, Z(\phi)\widehat{\Delta}) + \sum_k (Z(\phi)\pi_k^* \widehat{\xi}, D_k) + \sum_k (Z(\phi^\vee)\pi_k^* \widehat{\xi}, D_k')$$

where D_k and D_k' are some cycles independent of ϕ. By computation in Xinyi's draft, the last two terms are sums Eisenstein series and their derivations in variables $g_k \in \mathrm{GL}_2(\mathbb{A})$. Thus modulo these partial derivations of Eisenstein series, it remains to compute the intersection

$$(\widehat{\Delta}^0, Z(\phi)_U \widehat{\Delta}).$$

Assume that ϕ has regular support at some finite places. Then Δ and support of $Z(\phi)_U \Delta$ are disjoint. In this case, we have local decomposition:

$$(\widehat{\Delta}, Z(\phi)_U \widehat{\Delta}^0) = \sum_v (\widehat{\Delta}^0, Z(\phi)_U \widehat{\Delta})_v.$$

Once a model $\mathscr{Y}$ is chosen, we may choose $\widehat{\Delta}^0$ to be $(\bar{\Delta}, g_\Delta)$ formed by the Zariski closure $\bar{\Delta}$ of Δ and the Green's current of $\widehat{\Delta}$. In this case the difference $V = \widehat{\Delta} - \widehat{\Delta}$ is a vertical 2 cycle supported on the singular fibers of $\mathscr{Y}$. The local intersection $(\widehat{\Delta}, Z(\phi)_U\widehat{\Delta}^0)_v$ has been computed explicit for archimedean and good finite places; and the results match with computation from analytic kernels. The aim of this subsection is to estimate the local intersection at a ramified finite place v. Using decomposition,

$$(\widehat{\Delta}^0, Z(\phi)_U\widehat{\Delta})_v = (\bar{\Delta}, Z(\phi)_U\bar{\Delta})_v + (Z(\phi)_U\bar{\Delta}, V)_v,$$

it suffices to estimate the intersection of $Z(\phi)_U\bar{\Delta}$ with horizontal cycle $\bar{\Delta}$ or a vertical cycle C. Notice the horizontal intersections only happen at supersingular points or superspcial point.

Recall that local intersection is at a finite place v can be defined as follows. Let W be the subvariety of $Y \times Y$ consisting of pair of elements in the same connected component. Let $\mathscr{W}$ be a good model obtained form $\mathscr{Y} \times \mathscr{Y}$ by successive blowing up as in Gross–Schoen's paper. Let $\mathscr{Z}(\phi)$ be the generating series

$$\mathscr{Z}(\phi)_K = \sum_{x \in \widetilde{K}_E \backslash \mathbb{V}_E} \phi(x)\mathscr{Z}(x)_K$$

where $\mathscr{Z}(x)_K$ is the Zariski closures of $Z(x)_K$ in $\mathscr{W}$. Then the local intersection is defined by the intersection on $\mathscr{W}$ of the three cycles:

$$(\bar{\Delta}, Z(\phi)_K\widehat{\Delta}) = (\pi_1^*\bar{\Delta}, \mathscr{Z}(\phi)_K, \pi_2^*\widehat{\Delta})_v$$

where π_i are the projections of $\mathscr{W}$ to two facts.

Horizontal intersection at a non-split prime

In this section, we assume that v is ramified in $\mathbb{B}$. We want to study local horizontal intersections.

Let B be the quaternion algebra over F with ramification set $\Sigma \cup \{v\}$ and For each positive integer n, let G_n denote the subgroup of $(B^\times)^n$ of n-tuple of elements (b_i) with same norms. Then Y and W are Shimura variety associated to the group G_3 and G_6 respectively. The intersection $\pi_1^*\Delta$ and $\pi_2^*\Delta$ is a Shimura variety associated to the group G_2 embedded into G_6 by

$$(b_1, b_2) \mapsto (b_1, b_1, b_1, b_2, b_2, b_2).$$

We let $\mathbb{G}_n$ denote the subset of $(\mathbb{B}^\times)^n$ of n-tuples of elements b_i with the same norm. Then we can identify $G_n(\mathbb{A}^v)$ with $\mathbb{G}_n^v$. Let U_n denote the intersection of U^n and $\mathbb{G}_n$ in $(\mathbb{B}_f^\times)^n$.

The formal completion of $\mathscr{W}^0$ at these supersingular points can be identified with

$$\widehat{\mathscr{W}^0} = G_6(F)_0\backslash\widetilde{\Omega^6} \times G_6(\mathbb{A}_f^v)/K^v = G_6(F)\backslash\mathscr{D}_6^0 \times G_6(\mathbb{A}_f^v)/U_6^v$$

where

- $G_6(F)_0$ is the subgroup of $G_6(F)$ of elements of $G_6(F)$ with norm of order 0 at v;

- Ω is Drinfeld upper-half plane which admits an action by $G_1(F_v)_0$;
- $\widetilde{\Omega}^6$ is certain resolution of singularity;
- $\mathscr{D}_6^0 = G_6(F_v) \times_{G_6(F_v)_0} \widetilde{\Omega^6}$ which admits an action by $G_6(F_v)$.

Let $\mathscr{D}_6$ be a the pull-back of $\mathscr{D}_6^0$ on $\mathscr{W}$. Then the completion of $\mathscr{W}$ at its supersinular locus is

$$\widehat{\mathscr{W}} = G_6(F)\backslash\mathscr{D}_6 \times G_6(\mathbb{A}_f^v)/U_6^v.$$

The cycle $\pi_1^*\bar{\Delta} \times \pi_2^*\bar{\Delta}$ is represented by a subvariety

$$(\bar{\Delta}_2)_v = G_2(F)\backslash\mathscr{D}_2 \times G_2(\mathbb{A}_f^v)/U_6^v.$$

For each pair of elements $(x, y) \in \mathbb{B}_v^3 \times B_v^3$ with norms satisfying $q(x_i) = q(y_i) \neq 0$, we can define a subvariety $\mathscr{D}(x, y)$ of $\mathscr{D}_6$ which continuously depend on x and y. The Hecke operator $\mathscr{Z}(x)_K$ can be represented by

$$\mathscr{D}(x_v, y) \times g, \qquad g^{-1}(y) = x^v,$$

$$y \in B^3, \qquad g \in G_{6,y}(\mathbb{A}_f^v)\backslash G_6(\mathbb{A}_f^v)$$

where $g^{-1}(y)$ means action of G_2^3 on B_v^3 as usual:

$$g^{-1}(y)_i = (g_i^{-1} y_i g_{i+3}), \qquad i = 1, 2, 3.$$

Let $m(x, y)$ denote the intersection of $\mathscr{D}(x, y)$ with $\mathscr{D}_2$ in $\mathscr{D}_6$. Then $m(x, y)$ is a locally constant functions in x and y. The horizontal local intersection at v is given by

$$(\mathscr{Z}(x) \cdot \pi_1^*\bar{\Delta} \cdot \pi_2^*\bar{\Delta}) = \sum_{g^{-1}y=x} m(x, y), \qquad y \in B^3, g \in G_{2,y}(\mathbb{A}_f^v)\backslash G_2(\mathbb{A}_f^v).$$

Assume that $\phi = \phi^v \otimes \phi_v$. Then the generating series of horizontal local intersection is given by

$$\begin{aligned}(\mathscr{Z}(\phi) \cdot \pi_1^*\bar{\Delta} \cdot \pi_2^*\Delta) &= \sum_{x \in \widehat{U}_2^3\backslash\mathbb{V}^3} \phi(x)(\mathscr{Z}(x) \cdot \pi_1^*\bar{\Delta} \cdot \pi_2^*\bar{\Delta}) \\ &= \sum_x \phi(x) \sum_{g^{-1}y=x} m(x, y) \\ &= \sum_{y \in V^3} \sum_{g \in G_{2,y}(\mathbb{A}_f^v)\backslash G_2(\mathbb{A}_f^v)} \phi^v(g^{-1}y) m(y, \phi_v)\end{aligned}$$

where

$$m(y, \phi_v) = \sum_{x \in \mathbb{B}_v^3} \phi(x_v) m(x, y), \qquad q(x_i) = q(y_i).$$

Lemma 5.4.1. *Fix a ϕ_v. The function $y \longrightarrow m(y, \phi_v) \neq 0$ only if the moment metrix is supported in a compact subset of $\mathrm{Sym}_3(F_v)$.*

Proof. We need only prove the lemma for the function $y \longrightarrow m(x, y)$ when an x is fixed. In this case, the cycle $\mathscr{D}_2$ and $\mathscr{D}(x, y)$ has non-empty intersection only if they have non-empty intersection in the minimal level, and only if any two of the graphs $\Gamma(y_i)$ of the isomorphisms $y_i : \Omega \longrightarrow \Omega$ have a non- intersection in the generic fiber $\mathbb{P}^1(\mathbb{C}_v) - \mathbb{P}^1(F_v)$. Or in the other words, the morphism $y_i y_j^{-1}$ does not have a fixed point in $\mathbb{P}^1(F_v)$. This will implies that $y_i \bar{y}_j = y_i y_j^{-1} q(y_j)$ is elliptic in the sense it generates a quadratic subfield E_{ij} in B_v over F_v. Recall that in a quadratic field, an element t is integral only if its norm is integral. If $2n \geq -\mathrm{ord}(q(t))$, then $\pi_v^n t$ has integral norm, thus $\mathrm{tr}(\pi_v^n t)$ is integral. Take $n = -[\mathrm{ord}(q(t))/2]$, then we get

$$\mathrm{ordtr}(t) \geq -[\mathrm{ord}(q(t))/2].$$

Since $q(y_i) = q(x_i)$, we thus obtain that entries of $Q(y)$ has an estimate

$$\mathrm{ord}(\mathrm{tr}(y_i \bar{y}_j)) \geq -[\mathrm{ord}(x_i \bar{x}_j)/2].$$

This shows that $Q(y)$ is in a compact subset of $\mathrm{Sym}_3(F_v)$. □

Vertical intersection at a non-split prime

Now let us to compute vertical local intersection at a split prime v:

$$(\bar{\Delta}, Z(\phi)V)_v$$

where V is a two dimensional vertical cycle on $\mathscr{Y}$. First let us assume that V is irreducible and represented by (S, g') for S a vertical cycle on $\mathscr{D}_3$ and $g' \in G_3(\mathbb{A}_f^v)$. Then $\pi_1^* V \cdot \pi_2^* \bar{\Delta}$ is represented by

$$(\pi_1^* S \cdot \pi_2^* \mathscr{D}_1 \times (g', g''), \qquad g'' \in G_1(\mathbb{A}_f^v), \quad q(g'') = q(g').$$

For $x \in \mathbb{B}_v^3$ and $y \in B_v^3$ with the same and nonzero norm, let

$$\nu(x, y) = (\pi_1^* S \cdot \pi_2^* \mathscr{D}_1 \cdot \mathscr{D}(x, y)).$$

Then we have

$$(\mathscr{Z}(x) \cdot \pi_1^* V \cdot \pi_2^* \bar{\Delta}) = \sum_{g^{-1} y = x^v} \nu(x_v, y)$$

where $g = (g', g'')$ as above.

Assume that $\phi = \phi^v \otimes \phi_v$. The generating series of local vertical heights is given by

$$\begin{aligned}(\mathscr{Z}(\phi) \cdot \pi_1^* V \cdot \pi_2^* \bar{\Delta}) &= \sum_x \phi(x) \sum_{g^{-1} y = x^v} \nu(x_v, y) \\ &= \sum_{y \in V^3} \sum_{g'' \in G_1(\mathbb{A}_f^v)_{q(g')}} \phi^v(g'^{-1} y g'') \nu(y, \phi_v)\end{aligned}$$

where subscript $q(g')$ means elements with norm $q(g')$, and

$$\nu(y, \phi_v) = \sum_{x \in \mathbb{B}_v^3} \phi(x_v)\nu(x_v, y), \qquad q(x_i) = q(y_i)$$

Thus we obtain a pseudo-theta series if we can show that $\nu(y, \phi_v)$ has a compact support as a function of y.

Lemma 5.4.2. *The function $\nu(y, \phi_v)$ has compact support in y.*

Proof. Assume that S has image in Ω^3 included into $A_1 \times A_2 \times \times A_3$, where A_i's are irreducible components of special component of Ω. To show the compactness of this function, we need only study the compactness of the set of elements $y = (y_i)$'s such that the Zariski closure of graph of multiplication of y on $\Omega^3 \times \Omega^3$ passing through $A \times B \times C$. Since $\mathbb{B}_v^\times$ also has a natura action on the special components of Ω, the non-empty intersection implies that there is a irreducible component A_0 such that

$$A_i = y_i A_0.$$

Recall that after fixing an isomorphism $B_v \simeq M_2(F_v)$, the irreducible components in the special component of Ω are parameterized by homothety classes of lattices in F_v^2. The difference of two tuples y's is a tuple $g = (g_1, g_2, g_3)$ of three elements $g_i \in B_v^\times$ with norm 1 such that $g_i A_i = A_i$. This last equation implies that $g_i \in \mathrm{SL}_2(\Lambda_i)$ where Λ_i is a lattice in F_v^2 corresponding to A_i. This follows the set of y's is compact when x is fixed. Since ϕ_v has compact support, this shows that the function $\nu(y, \phi_v)$ has compact support. □

Intersection at split prime

A point on $\mathscr{W} \otimes \mathscr{O}_v$ is called supersingular if its image in $(\mathscr{X}^0)^6$ is a supersingular point. Fix one supersingular point on $\mathscr{X}^0$. Then the set $\mathscr{W}_v^{0,ss}$ of super singular points can be identified with

$$\mathscr{W}_{v,ss}^0 = G_6(F)_0 \backslash G_6(\mathbb{A}_f^v)/U_3^v = G_6(F)\backslash G_6(\mathbb{A}_f)/U_6^0.$$

More precisely, the formal completion of $\mathscr{W}^0$ at these supersingular points can be identified with

$$\widehat{\mathscr{W}_{ss}^0} = G_6(F)_0 \backslash \Omega^6 \times G_6(\mathbb{A}_f^v)/K^v = G_6(F)\backslash \mathscr{D}_6^0 \times G_6(\mathbb{A}_f^v)/U_6^v$$

where

- $G_6(F)_0$ is the subgroup of $G_6(F)$ of elements of $G_6(F)$ with norm of order 0 at v;
- Ω is a universal deformation domain of a supersingular point which admits an action by $G_1(F_v)_0$;
- $\mathscr{D}_6^0 = G_6(F_v) \times_{G_6(F_v)_0} \mathscr{D}^6$ which admits an action by $G_6(F_v)$.

Let $\mathscr{D}_6$ be a the pull-back of $\mathscr{D}_6^0$ on $\mathscr{W}$. Then the completion of $\mathscr{W}$ at its supersinular locus is

$$\widehat{\mathscr{W}_{ss}} = G_6(F)\backslash \mathscr{D}_6 \times G_6(\mathbb{A}_f^v)/U_6^v.$$

The cycle $\pi_1^*\bar{\Delta} \times \pi_2^*\bar{\Delta}$ is represented by a subvariety

$$(\bar{\Delta}_2)_v^{ss} = G_2(F)\backslash \mathscr{D}_2 \times G_2(\mathbb{A}_f^v)/U_6^v.$$

For each pair of elements $(x, f) \in \mathbb{B}_v^3 \times B_v^3$ with norms satisfying $q(x_i) = q(f_i) \neq 0$, we can define a subvariety $\mathscr{D}(x, f)$ of $\mathscr{D}_6$ which continuously depend on x and f. The Hecke operator $\mathscr{Z}(x)_K$ can be represented by

$$\mathscr{D}(x, y) \times g, \qquad g^{-1}(y) = x^v,$$

$$y \in B^3, \qquad g \in G_{6,y}(\mathbb{A}_f^v)\backslash G_6(\mathbb{A}_f^v)$$

where $g^{-1}(y)$ means action of G_2^3 on B_v^3 as usual:

$$g^{-1}(y)_i = (g_i^{-1} y_i g_{i+3}), \qquad i = 1, 2, 3.$$

Let $m(x, y)$ denote the intersection of $\mathscr{D}(x, y)$ with $\mathscr{D}_2$ in $\mathscr{D}_6$. Then $m(x, y)$ is a locally constant functions in x and y. The horizontal local intersection at v is given by

$$(\mathscr{Z}(x) \cdot \pi_1^*\bar{\Delta} \cdot \pi_2^*\bar{\Delta})^{ss} = \sum_{g^{-1}y = x} m(x, y), \qquad y \in B^3, g \in G_{2,y}(\mathbb{A}_f^v)\backslash G_2(\mathbb{A}_f^v).$$

Assume that $\phi = \phi^v \otimes \phi_v$. Then the generating series of horizontal local intersection is given by

$$\begin{aligned}
(\mathscr{Z}(\phi) \cdot \pi_1^*\bar{\Delta} \cdot \pi_2^*\Delta)^{ss} &= \sum_{x \in \widehat{U}_2^3\backslash \mathbb{V}^3} \phi(x)(\mathscr{Z}(x) \cdot \pi_1^*\bar{\Delta} \cdot \pi_2^*\bar{\Delta})^{ss} \\
&= \sum_{x} \phi(x) \sum_{g^{-1}y = x} m(x, y) \\
&= \sum_{y \in V^3} \sum_{g \in G_{2,y}(\mathbb{A}_f^v)\backslash G_2(\mathbb{A}_f^v)} \phi^v(g^{-1}y) m(y, \phi_v)
\end{aligned}$$

where

$$m(y, \phi_v) = \sum_{x \in \mathbb{B}_v^3} \phi(x_v) m(x, y), \qquad q(x_i) = q(y_i).$$

Since B_v is anisotropic, the equation $q(x_i) = q(y_i)$ for $x \in \mathrm{supp}(\phi_v)$ shows that the function $m(y, \phi_v)$ has a compact support. In other word, the last sum is a pseudo-theta series.

Vertical intersection at a split place

Now let us to compute vertical local intersection at a split prime v:

$$(\bar{\Delta}, Z(\phi)V)_v$$

where V is a two dimensional vertical cycle on $\mathscr{Y}$. First let us assume that V is irreducible. We divide V according the dimension of its image αV in the morphism $\alpha : \mathscr{Y} \longrightarrow \mathscr{X}^3$:

$$V = \begin{cases} \text{ordinary} & \text{if } \dim \alpha V = 2 \\ \text{supersingular} & \text{if } \dim \alpha V = 0 \\ \text{mixed} & \text{if } \dim \alpha V = 1 \end{cases}$$

For a general cycle V as a linear combination $\sum a_i V_i$ of irreducible cycles V_i, we say that V has a type as listed as above if every V_i has this type.

Lemma 5.4.3. *If V has a type as above, then $Z(x)V$ can be represented by a cycle with the same type.*

Proof. We need only treat the case where V is irreducible. Let $\bar{Z}(x)$ be the Zariski closure of $Z(x)$ in $(\mathscr{X}^3) \times (\mathscr{X}^3)$. Then the projection $\beta : \mathscr{Y}^2 \longrightarrow \mathscr{X}^6$ sends $\mathscr{Z}(x)$ to $\bar{Z}(x)$. Let $\pi_i : \mathscr{Z}(x) \longrightarrow \mathscr{Y}$ $(i = 1, 2)$ and $p_i : \bar{Z}(x) \longrightarrow \mathscr{X}^3$ be projections. Then we have the following commutative diagram:

$$\begin{tikzcd} & \mathscr{Z}(x) \arrow[dl, "\pi_1"'] \arrow[d, "\beta"] \arrow[dr, "\pi_2"] & \\ \mathscr{Y} \arrow[d, "\alpha"'] & \bar{Z} \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \mathscr{Y} \arrow[d, "\alpha"] \\ \mathscr{X}^3 & & \mathscr{X}^3 \end{tikzcd} .$$

Recall that $\pi_1^* V$ can be represented by a linear combination $\sum_{i \in I} a_i V_i$ so that each V' has image $\pi_1 V' = V$. The cycle $Z(x)V$ is represented by $\sum_{j \in J} a_j [\pi_2 V_j]$ where J is a subset of j's so that $\pi_2(V_j)$ has dimension 2. Applying α to these cycles, we have

$$\alpha \pi_2 V_j = p_2 \beta V_j.$$

Since both p_1 and p_2 are finite and flat,

$$\dim p_2 \beta V_j = \dim \beta V_j = \dim \pi_1 \beta V_j = \dim \alpha \pi_1 V_j = \dim \alpha V.$$

□

Same proof as in superspecial case shows that the generating series of local intersection of supersingular vertical components gives a pseudo-theta series.

6 Proof of Main Theorem

In this section we will finish proving the main result of this paper. Note that we need to prove

$$deg Z_\phi \equiv E'(\cdot, 0, \phi)_{hol} \quad mod\, \mathscr{A}_{coh}(\mathbb{G})$$

where $\mathscr{A}_{coh}(\mathbb{G})$ is the subspace of $\mathscr{A}(\mathbb{G})$ generated by restrictions of $E(\cdot, 0, \phi)$ for $\phi \in \mathscr{S}(V_{\mathbb{A}}^3)$ for all possible *coherent* $V_{\mathbb{A}}$.

Firstly we compile established facts. Note that the test functions are chosen as follows:

1. For $v|\infty$, we have chosen ϕ_v to the standard Gaussian.
2. For those v where both π_v and ψ_v are unramified, we choose ϕ_v to be the characteristic function of $\mathscr{O}_{B,v}^3$.
3. Let S be the finite set of the rest of all (non-archimedean) places. And for $v \in S$, we choose ϕ_v to be ramified of sufficiently large order depending on ψ_v.

Now by the decomposition of $E'(\cdot, 0, \phi)_{hol}$ (equation 5.3.9), we have for $g \in \mathbb{G}$ with $g_v = 1$ when $v \in S$:

(6.0.1)
$$E'(g, 0, \phi)_{hol} = \sum_v E'_v(g, 0, \phi)_{hol}$$

$$E'_v(g, 0, \phi)_{hol} = \sum_{T, \Sigma(T) = \Sigma(v)} E'_T(g, 0, \phi)_{hol}$$

where the sum runs only over non-singular T by the vanishing of singular coefficients for such g. And when $\Sigma(T) = \Sigma(v)$ for $v|\infty$, we have

$$E'_T(g, 0, \phi)_{hol} = W_T(g_\infty) m_v(T) W_{T,f}(g_f, 0, \phi_f)$$

where $m_v(T)$ is the star product of $P_s(1 + 2s_x(z)/q(x))$ for x with moment T.

On the intersection side, we also have a decomposition

$$deg Z_\phi(g) = \sum_v deg Z_\phi(g)_v.$$

And we have proved the following comparison:

1. When $v|\infty$, by Theorem 5.3.8, we have for $g \in \mathbb{G}$ with $g_v = 1$ when $v \in S$
$$deg Z_\phi(g)_v = E'_v(g, 0, \phi)_{hol}.$$
2. When $v < \infty$ and v is outside S, by Theorem 5.2.2, we have for $g \in \mathbb{G}$ with $g_v = 1$ when $v \in S$
$$deg Z_\phi(g)_v = E'_v(g, 0, \phi) = E'_v(g, 0, \phi)_{hol}.$$

3. When $v \in S$, by Theorem 5.4.1, we have for $g \in \mathbb{G}$ with $g_v = 1$ when $v \in S$

$$deg Z_\phi(g)_v = E^{(v)}(g)$$

for some $E^{(v)} \in \mathscr{A}_{coh}(\mathbb{G})$. And by Proposition 2.7.3, we have for g as above

$$E'_v(g, 0, \phi) = 0.$$

To sum up, we have an automorphic form

$$F(g) = deg Z_\phi(g) - E'(g, 0, \phi)_{hol} - \sum_{v \in S} E^{(v)}(g) \in \mathscr{A}(\mathbb{G})$$

with the property that for all $g \in \mathbb{G}$ with $g_v = 1$ when $v \in S$:

$$F(g) = 0.$$

But it is easy to see that for a finite set S of non-archimedean places, $\mathbb{G}_F \mathbb{G}^S$ is dense in $\mathbb{G}_{\mathbb{A}}$. Therefore we must have

$$F(g) \equiv 0$$

for all $g \in \mathbb{G}_{\mathbb{A}}$! This proves that

$$deg Z_\phi(g) \equiv E'(g, 0, \phi)_{hol} \quad mod\, \mathscr{A}_{coh}(\mathbb{G}).$$

7 Application to cycles and rational points

Let X_i $(i = 1, 2, 3)$ be three smooth, projective, and geometrically connected curves over a number field k. Let $W = \mathrm{Ch}^2(X_1 \times X_2 \times X_3)^{00}$ be the subgroup of codimension 2 cycles z such that $\pi_{ij*}z = 0$ in $\mathrm{Ch}^1(X_i \times X_j)$ for the projections π_{ij} to $X_i \times X_j$. Then it can be shown that every cycle in W is homologically trivial. The conjecture of Beilinson and Bloch says that W is finitely generated with rank

$$\dim_{\mathbb{Q}} W_{\mathbb{Q}} = \mathrm{ord}_{s=2} L(s, M)$$

where M is the motive $\otimes_i H^1(X)$ which has ℓ-adic realization $M_\ell := \otimes_i H^1(X_{i\bar{k}}, \mathbb{Q}_\ell)$. In the following, we would like to give a refinement using correspondences on X_i.

In fact, let W^1 denote the subgroup of W generated by $\pi_{ij}^* z$ with $z \in \mathrm{Ch}^2(X_i \times X_j)$ with trivial image in $\mathrm{Alb}(X_i \times X_j)$. Then it is conjectured that $W^1 = 0$. Right $\widetilde{W} = W/W^1$. Then one can show that the natural action of $\otimes_i \mathrm{Ch}^1(X_i \times X_i)$ on cycles stabilizes W and W^1 and that the induced action on $\widetilde{W}$ factors through the quotient $\otimes \mathrm{NS}(X_i)$ where $\mathrm{NS}(X_i \times X_i)$ is the Neron–Severi group of $X_i \times X_i$ and also $\mathrm{End}(\mathrm{Jac}(X_i))$. Since $\mathrm{NS}(X_i \times X_i) \otimes \mathbb{Q}$ is semisimple, $\otimes \mathrm{NS}(X_i \times X_i)$ is also semisimple. Thus its representation can be decomposed into irreducible representation using idempotent t such that

$$D_t := t(\otimes_i \mathrm{NS}(X_i \times X_i) \otimes \mathbb{Q})t$$

is a division algebra over its center E_t. Our refined Beilinson–Bloch conjecture is as follows: for each embedding $\tau : E_t \longrightarrow \mathbb{C}$

$$\dim_{E_t} t\widetilde{W}_{\mathbb{Q}} = \mathrm{ord}_{s=2} L(s, tM, \tau).$$

Here the L-series of right hand is an Euler product over primes $\wp$ of k of local factors $L_\wp(s, tM, \tau) = P_\wp^\tau(\mathrm{N}(\wp)^s)^{-1}$, where $P_\wp^\tau(T)$ is a complex base change of a polynomial over E_t defined as the determinant of $1 - \mathrm{Frob}_\wp \cdot T$ on the $tM_\ell^{I_\wp}$ as $\mathbb{Q}_\ell \otimes E_t$-module for any ℓ different than the characteristic of $\wp$ and inert in E_t. One consequence of the conjecture is that $\mathrm{ord}_{s=1} L(s, tM, \tau)$ does not depend on τ. As Neron–Tate height pairing on the Mordell–Weil group of rational points of an abelian variety, the group $\widetilde{W}$ has height pairing defined using arithmetical intersection theory. The volume of $\widetilde{W}$ should be related to the leading term in the Taylor expansion of $L(s, tM) = \prod_\tau L(s, tM, \tau)$ at $s = 2$.

Now we go back our projective limit X of Shimura curves. For each open compact subgroup U, let W_U denote the group $\mathrm{Ch}^2(X_U \times X_U \times X_U)^{00}$, and let W denote the direct limit of W_U. Then W admits an action by $(\mathbb{B}^\times)^3$. The action on $W_{\mathbb{C}}$ factors through the action of $\oplus\pi\otimes\widetilde{\pi}$, where π are Jacquet–Langlands correspondences of cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_E)$ of parallel weight 2. Thus $W_{\mathbb{C}}$ is decompose into isotropic pieces of π:

$$W_{\mathbb{C}} = \oplus_\pi W(\pi) \otimes \pi$$

where

$$W(\pi) = \mathrm{Hom}_{\mathbb{B}_E^\times}(\pi, W_{\mathbb{C}}).$$

The Beilinson–Bloch conjecture says

$$\dim_{\mathbb{C}} W(\pi) = \mathrm{ord}_{s=1/2} L(s, \sigma).$$

One consequence of this conjecture is that $\mathrm{ord}_{s=1/2} L(s, \sigma)$ if we replace σ by a conjugate σ^τ for any $\tau \in \mathrm{Aut}(\mathbb{C})$, where σ^τ is the unique cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_E)$ whose eigenvalues are τ-conjugate of those of π. Indeed, let L denote the subfield of $\mathbb{C}$ generated by Hecke eigenvalues of π. Then π is the base change of absolutely irreducible representation π^0 over an L-vector spaces. For each $\tau \in \mathrm{Aut}(\mathbb{C})$ we get a complex representation π^τ by base change $\tau|_L$ whose Jacquet–Langlands correspondence is σ^τ. Define

$$W(\pi^0) = \mathrm{Hom}_{\mathbb{B}_E^\times}(\pi^0, W)$$

as $\mathbb{Q}(\mathbb{B}_E^\times)$-modules. Then $W(\pi^0)$ is a vector space over L and

$$W(\pi^\tau) = W(\pi^0) \otimes_\tau \mathbb{C}.$$

Thus $\dim_{\mathbb{C}} W(\pi^\tau) = \dim_L W(\pi^0)$ which is independent of τ.

A more precisely conjecture should relate the regulator of a lattice in $W(\pi^0)$ with leading term of $L(s, \pi^0) := \prod_\tau L(s, \pi^\tau)$. The regulate should be defined as the discriminant of a bilinear pairing on $W(\pi^0)$ and $W(\widetilde{\pi}^0)$ as follows. On W, there is a height pairing induced by the height pairings of W_U's with weight $\mathrm{vol}(U_E)$. The invariance of this pairing under the action of $\mathbb{B}_E^\times$ further induces a bilinear pairing between $W(\pi^0)$ and $W(\widetilde{\pi}^0)$ as follows such that for α and β in these spaces respectively, and $u \in \pi^0$ and $v \in \widetilde{\pi}^0$:

$$\langle \alpha, \beta \rangle \langle u, v \rangle = \langle \alpha(u), \beta(v) \rangle.$$

With the terminology as above the Gross–Schoen cycles Δ_ξ define a map

$$\widetilde{\pi} \longrightarrow W(\pi), \quad \widetilde{f} \mapsto W_{\widetilde{f}}, \qquad W_{\widetilde{f}}(v) = \mathrm{T}(v \otimes \widetilde{f})\Delta_\xi.$$

Now for $f \in \pi$ and $\widetilde{\pi}$, their pairing is computed as follows: for any $v \in \pi, \widetilde{v} \in \widetilde{\pi}$, we have

$$\begin{aligned} \langle W_{\widetilde{f}}, W_f \rangle \langle v, \widetilde{v} \rangle =& \langle \mathrm{T}(v \otimes \widetilde{f})\Delta_\xi, \mathrm{T}(\widetilde{v} \otimes f)\Delta_\xi \rangle \\ =& \langle \mathrm{T}(f \otimes \widetilde{v}) \circ \mathrm{T}(v \otimes \widetilde{f})\Delta_\xi, \Delta_\xi \rangle \\ =& \langle \mathrm{T}(f \otimes \widetilde{f})\Delta_\xi, \Delta_\xi \rangle \langle v, \widetilde{v} \rangle \end{aligned}$$

Thus we have

$$\langle W_{\widetilde{f}}, W_f \rangle = \langle \mathrm{T}(f \otimes \widetilde{f})\Delta_\xi, \Delta_\xi \rangle = \gamma_\pi(f \otimes \widetilde{f}).$$

The main theorem in the last section is equivalent to

$$\langle W_{\widetilde{f}}, W_f \rangle = \frac{\zeta_F(2)^2 L'(1/2, \sigma)}{2L(1, \sigma, ad)} \alpha_\pi(f \otimes \widetilde{f}).$$

There are some consequences:

1. $L'(1/2, \sigma) = 0$ if and only if it is zero for all conjugates of σ;

2. assume that σ is unitary, then we take $\widetilde{f} = \bar{f}$. The Hodge index conjecture implies $L'(1/2, \sigma) \geq 0$. This is an consequence of the Riemann Hypothesis.

3. assume that σ is unitary the height pairing is positive definite, then W_f are proportional to each other.

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