

# On Volumes of Arithmetic Line Bundles II

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September 13, 2009

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## 1 Introduction

This paper uses convex bodies to study line bundles in the setting of Arakelov theory. The treatment is parallel to [Yu2], but the content is independent.

The method of constructing a convex body in Euclidean space, now called “Okounkov body”, from a given algebraic linear series was due to Okounkov [Ok1, Ok2], and was explored systematically by Kaveh–Khovanskii [KK] and Lazarsfeld–Mustața [LM]. Many important results of algebraic geometry can be derived from convex geometry through the bridge that the volume of the convex body gives the volume of the linear series.

Let  $K$  be a number field,  $\mathcal{X}$  be an arithmetic variety of relative dimension  $d$  over  $O_K$ , and  $\overline{\mathcal{L}}$  be a hermitian line bundle over  $\mathcal{X}$ . There are two important arithmetic invariants  $\hat{h}^0(\overline{\mathcal{L}})$  and  $\chi(\overline{\mathcal{L}})$ . Their growth under tensor powers are measured respectively by  $\text{vol}(\overline{\mathcal{L}})$  and  $\text{vol}_\chi(\overline{\mathcal{L}})$ . In [Yu2], we have introduced the Okounkov body  $\Delta(\overline{\mathcal{L}}) \subset \mathbb{R}^{d+1}$  of  $\overline{\mathcal{L}}$ , whose volume computes  $\text{vol}(\overline{\mathcal{L}})$ . It is a natural arithmetic analogue of the construction in [LM].

In the current paper, we use the usual Okounkov body  $\Delta(\mathcal{L}_K) \subset \mathbb{R}^d$  of the generic fibre  $\mathcal{L}_K$  viewed as a line bundle on the projective variety  $\mathcal{X}_K$ . Then we introduce the Chebyshev

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\*The author is fully supported by a research fellowship of the Clay Mathematics Institute.

transform  $c[\overline{\mathcal{L}}]$  of  $\overline{\mathcal{L}}$ , which is a convex function on  $\Delta(\mathcal{L}_K)$ . We show that the Lebesgue integral of  $c[\overline{\mathcal{L}}]$  on  $\Delta(\mathcal{L}_K)$  gives  $\text{vol}_X(\overline{\mathcal{L}})$  under some boundedness condition. For example, it is true in the cases of toric varieties. We conjecture that it is true in the general case that the generic fibre  $\mathcal{L}_K$  is big.

The construction of  $c[\overline{\mathcal{L}}]$  is inspired by the work of Nyström [Ny], and can be viewed as the global case of Nyström's work. For each place  $v$  of  $K$ , we introduce a local Chebyshev transform  $c_v[\overline{\mathcal{L}}]$ , which is a real-valued convex function on  $\Delta(\mathcal{L}_K)$  depending on the  $v$ -adic metric of  $\mathcal{L}_K$  induced by the model  $\overline{\mathcal{L}}$ . The archimedean case is exactly Nyström's construction, and the non-archimedean case is analogous. Then the global Chebyshev transform is defined by  $c[\overline{\mathcal{L}}] = \sum_v c_v[\overline{\mathcal{L}}]$ .

## 1.1 Algebraic case

We first review the construction of [LM]. Instead of flag of subvarieties, we will use local coordinates as in [Ny]. It actually gives a generic infinitesimal flag in the sense of [LM].

Let  $X$  be a projective variety of dimension  $d$  over any base field  $K$ . Fix a regular rational point  $x_0 \in X(K)$  which exists by replacing  $K$  by a finite extension if possible. Let  $t = (t_1, \dots, t_d)$  be a system of parameters at  $x_0$ . In another word,  $t_1, \dots, t_d$  is a minimal set of generators of the maximal ideal of the local ring  $O_{X, x_0}$ .

Let  $L$  be a line bundle over  $X$ . Fix a local section  $s_0$  of  $L$  around  $x_0$  which does not vanish at  $x_0$ . It gives a trivialization  $mL \subset O_{X, x_0}$  for any integer  $m$ . Here  $mL$  means  $L^{\otimes m}$ , and we always write line bundles additively in this paper. In fact, for any local section  $s$  of  $mL$  at  $x_0$ , the quotient  $s_0^{-m}s$  is a rational function on  $X$  regular at  $x_0$  and we may identify  $s$  with  $s_0^{-m}s$ . Consider the power series expansion

$$s = \sum_{\alpha \in \mathbb{N}^d} a_\alpha t^\alpha.$$

Here we use the convention  $t^\alpha := t_1^{\alpha_1} \cdots t_d^{\alpha_d}$  for any  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ . The smallest  $\alpha \in \mathbb{N}^d$  with respect to the lexicographic order of  $\mathbb{Z}^d$  such that  $a_\alpha \neq 0$ , denoted by  $\nu_t(s)$ , is called the *valuation of  $s$* .

It follows that we have a map

$$\nu_t : H^0(X, mL) - \{0\} \rightarrow \mathbb{N}^d.$$

The *Okounkov body*  $\Delta_t(L)$  is defined to be the closure of

$$\Lambda_t(L) = \bigcup_{m \geq 1} \frac{1}{m} \nu_t(H^0(X, mL))$$

in  $\mathbb{R}^d$ . Here  $\nu_t(H^0(X, mL))$  stands for  $\nu_t(H^0(X, mL) - \{0\})$  throughout this paper.

As shown by [LM], the Okounkov body is convex and bounded and satisfies

$$\text{vol}(\Delta_t(L)) = \frac{1}{d!} \text{vol}(L).$$

Here we recall that the volume of  $L$  is defined by

$$\text{vol}(L) = \lim_{m \rightarrow \infty} \frac{\dim_K H^0(X, mL)}{m^d/d!}.$$

The existence of the above limit is an easy consequence of properties of the Okounkov body, and was implied by Fujita's approximation theorem in history.

To prepare for the arithmetic case, we introduce one more notation. For any  $\alpha \in \nu(H^0(X, L))$ , define

$$H^0(X, L)(\alpha) = \{s \in H^0(X, L) : s = t^\alpha + \text{higher order terms}\}.$$

In another word,  $H^0(X, L)(\alpha)$  consists of sections with valuation  $\alpha$  and leading coefficient equal to 1. Then  $H^0(X, L)(\alpha)$  gives a way to normalize the sections. Any  $s \in H^0(X, L)$  is uniquely written in the form  $\sum_{\alpha \in \nu(H^0(X, L))} b_\alpha s_\alpha$  with  $b_\alpha \in K$  and  $s_\alpha \in H^0(X, L)(\alpha)$ .

The valuation map  $\nu$  and the Okounkov body  $\Delta$  depend on the choice of  $x_0$  and  $t$ , but not on the choice of  $s_0$ . The set  $H^0(X, L)(\alpha)$  will be multiplied by  $s'_0(x_0)/s_0(x_0) \in K^\times$  if we change  $s_0$  to another  $s'_0$ .

## 1.2 Chebyshev transform

To define the Chebyshev transform, it is more convenient to work on adelic metrized line bundles in the sense of Zhang [Zh2]. We briefly recall the definition of adelic metrized line bundles. For more details, we refer to [Zh2] and §2.1 of the current paper.

Let  $X$  be a projective variety of dimension  $d$  over a number field  $K$ , and  $L$  be a line bundle over  $X$ . For any place  $v$  of  $K$ , by a  $v$ -adic metric on  $L$  we mean an assignment of a Galois invariant  $\mathbb{C}_v$ -norm to the fibre  $L_{\mathbb{C}_v}(x)$  at any point  $x \in X(\mathbb{C}_v)$ . Here  $\mathbb{C}_v$  denotes the completion of the algebraic closure of  $K_v$ . It induces the supremum norms on  $H^0(X_{\mathbb{C}_v}, L_{\mathbb{C}_v})$  given by

$$\|s\|_{v, \text{sup}} = \sup_{x \in X(\mathbb{C}_v)} \|s(x)\|_v.$$

An *adelic metric* on  $L$  is a collection  $\{\|\cdot\|_v\}_v$  of continuous  $v$ -adic metrics on  $L$  over all places  $v$  of  $K$  satisfies certain coherence condition. We usually write  $\bar{L} = (L, \{\|\cdot\|_v\}_v)$  and call it an *adelic metrized line bundle*.

Let  $\bar{L} = (L, \{\|\cdot\|_v\}_v)$  be an adelic metrized line bundle on  $X$  with  $L$  big. As in the algebraic case, take a rational regular point  $x_0 \in X(K)$  which exists by enlarging  $K$ , take a local coordinate  $t = (t_1, \dots, t_d)$  at  $x_0$ , and a base local section  $s_0$  that induces trivialization of  $L$ . The Okounkov body  $\Delta(L) = \Delta_t(L)$  is induced by the valuation map  $\nu = \nu_t$  on the global sections.

Fix a place  $v$ . The local Chebyshev transform  $c_v[\bar{L}]$  of  $\bar{L}$  will be a real-valued function on  $\Delta(L)$  constructed from  $\|\cdot\|_v$  by the method of [Ny]. For any  $\alpha \in \nu(H^0(X, L))$ , define

$$F_v[\bar{L}](\alpha) := \inf_{s \in H^0(X_{\mathbb{C}_v}, L_{\mathbb{C}_v})(\alpha)} \log \|s\|_{v, \text{sup}}.$$

Here

$$H^0(X_{\mathbb{C}_v}, L_{\mathbb{C}_v})(\alpha) = \{s \in H^0(X_{\mathbb{C}_v}, L_{\mathbb{C}_v}) : s = t^\alpha + \text{higher order terms}\}$$

is introduced above.

Let  $\{m_k\}_k$  be a sequence of positive integers tending to infinity, and let  $\alpha_k \in \frac{1}{m_k}\nu(H^0(X, m_k L))$  be a sequence convergent to some  $\alpha \in \Delta(L)$  under the Euclidean topology. Then  $c_v[\bar{L}](\alpha)$  is defined to be the limit

$$\lim_{k \rightarrow \infty} \frac{1}{m_k} F_v[m_k \bar{L}](m_k \alpha_k).$$

**Proposition 1.1.** *For any  $\alpha$  in the interior  $\Delta(L)^\circ$  of  $\Delta(L)$ , the above limit exists and does not depend on the sequence  $\{(m_k, \alpha_k)\}$ . Thus*

$$c_v[\bar{L}](\alpha) := \lim_{k \rightarrow \infty} \frac{1}{m_k} F_v[m_k \bar{L}](m_k \alpha_k).$$

is a well-defined function on  $\Delta(L)^\circ$ . Furthermore, it is convex and continuous.

Note that our definition of  $c_v$  differs from that in [Ny] by a factor 2. If  $v$  is archimedean, the result is exactly Theorem 1.1 and Theorem 1.2 of [Ny]. The non-archimedean case is very similar.

Go back to the globally metrized line bundle  $\bar{L}$ . We simply define

$$c[\bar{L}] := \sum_v c_v[\bar{L}].$$

The sum converges pointwise in  $\Delta(L)^\circ$ , which makes  $c[\bar{L}]$  a convex and continuous function on  $\Delta(L)^\circ$ . By the product formula, the sum  $c[\bar{L}]$  is independent of the choice of  $s_0$ .

### 1.3 Integration and volume

Let  $(X, \bar{L})$  be as above. The main result of this paper is as follows:

**Theorem 1.2.** *Assume that  $L$  is big. Then the following are true:*

(1) *The function  $c[\bar{L}]$  is integrable on  $\Delta(L)$ , and*

$$\int_{\Delta(L)} c[\bar{L}](\alpha) d\alpha \leq -\frac{1}{d!} \text{vol}_X(\bar{L}).$$

(2) *If the function  $\frac{1}{m} \sum_v F_v[m \bar{L}]$  is uniformly bounded for all  $m \geq 1$ , then*

$$\int_{\Delta(L)} c[\bar{L}](\alpha) d\alpha = -\frac{1}{d!} \text{vol}_X(\bar{L}).$$

Now we explain the definition of the adelic volume  $\text{vol}_\chi(\bar{L})$ . We first denote

$$\chi_{\text{sup}}(\bar{L}) = \log \frac{\text{vol}(B_{\text{sup}}(\bar{L}))}{\text{vol}(H^0(X, L)_{\mathbb{A}_K}/H^0(X, L))}.$$

Here  $\mathbb{A}_K = \prod_v K_v$  is the adèle ring of  $K$  and  $H^0(X, L)_{\mathbb{A}_K}$  is the adelization of  $H^0(X, L)$ . The unit ball

$$B_{\text{sup}}(\bar{L}) = \prod_v B_{v, \text{sup}}(\bar{L})$$

is a product of local unit balls

$$B_{v, \text{sup}}(\bar{L}) = \{s \in H^0(X, L)_{K_v} : \|s\|_{v, \text{sup}} \leq 1\}.$$

Note that both  $B_{\text{sup}}(\bar{L})$  and  $H^0(X, L)_{\mathbb{A}_K}/H^0(X, L)$  are compact, and the quotient of their volumes does not depend on the Haar measure on  $H^0(X, L)_{\mathbb{A}_K}$ . Now we can define

$$\text{vol}_\chi(\bar{L}) = \limsup_{m \rightarrow \infty} \frac{\chi_{\text{sup}}(X, m\bar{L})}{m^{d+1}/(d+1)!}.$$

Let  $\mathcal{X}$  be an arithmetic variety and  $\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$  be a hermitian line bundle over  $\mathcal{X}$ . Then an adelic metric  $\{\|\cdot\|_v\}_v$  is induced on the line bundle  $\mathcal{L}_K$  over  $\mathcal{X}_K$ . By the data  $(\mathcal{X}_K, \bar{\mathcal{L}}_K = (\mathcal{L}_K, \|\cdot\|_v))$ , we have the Chebyshev transform  $c[\bar{\mathcal{L}}_K]$  on  $\Delta(\mathcal{L}_K)$  depending on the choice of a flag. In this case  $\text{vol}_\chi(\bar{\mathcal{L}}_K)$  is equal to  $\text{vol}_\chi(\bar{\mathcal{L}})$  defined by

$$\text{vol}_\chi(\bar{\mathcal{L}}) = \limsup_{m \rightarrow \infty} \frac{\chi_{\text{sup}}(\mathcal{X}, m\bar{\mathcal{L}})}{m^{d+1}/(d+1)!}.$$

Here the usual arithmetic Euler characteristic is defined by

$$\chi_{\text{sup}}(\mathcal{X}, \bar{\mathcal{L}}) = \log \frac{\text{vol}(B_{\text{sup}}(\mathcal{X}, \bar{\mathcal{L}}))}{\text{vol}(H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}}/H^0(\mathcal{X}, \mathcal{L}))}.$$

with unit ball

$$B_{\text{sup}}(\mathcal{X}, \bar{\mathcal{L}}) = \{s \in H^0(\mathcal{X}, \mathcal{L})_{\mathbb{R}} : \|s\|_{\text{sup}} \leq 1\}$$

is the corresponding unit ball.

The ‘‘limsup’’ defining  $\text{vol}_\chi(\bar{\mathcal{L}})$  is proved to be a limit by Chen [Ch1] using Harder-Narasimhan filtrations. In the case that Theorem 1.2 (1) is true, we can recover Chen’s result by our treatment.

## 2 Chebyshev Transform

In this section, we prove Proposition 1.1 and Theorem 1.2. Proposition 1.1 will be proved immediately in §2.2. The proof of Theorem 1.2 is given in §2.4 after some more preparations in §2.3.

## 2.1 Conventions and preliminary results

### Lexicographic order

In this paper we denote  $\mathbb{N} = \{0, 1, 2, \dots\}$ . There is the usual lexicographic order in  $\mathbb{N}^d$ . Namely,  $(\alpha_1, \dots, \alpha_d) < (\alpha'_1, \dots, \alpha'_d)$  for two elements in  $\mathbb{N}^d$  if there is an  $i \in \{1, \dots, d\}$  such that  $(\alpha_1, \dots, \alpha_{i-1}) = (\alpha'_1, \dots, \alpha'_{i-1})$  and  $\alpha_i < \alpha'_i$ . The order is preserved by addition in the sense that  $\alpha + \beta \leq \alpha' + \beta'$  if  $\alpha \leq \alpha'$  and  $\beta \leq \beta'$ .

### Adelic metrics

Here we recall the notion of adelic metrized line bundle introduced by Zhang [Zh2].

Let  $X$  be a projective variety over a number field  $K$ , and  $L$  be a line bundle over  $X$ . For any place  $v$  of  $K$ , by a  $v$ -adic metric on  $L$  we mean the assignment of a Galois invariant  $\mathbb{C}_v$ -norm to the fibre  $L_{\mathbb{C}_v}(x)$  at any point  $x \in X(\mathbb{C}_v)$ . Here  $\mathbb{C}_v$  denotes the completion of the algebraic closure of  $K_v$ .

Let  $\mathcal{X}$  be an arithmetic variety and  $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$  be a hermitian line bundle over  $\mathcal{X}$ . Then the generic fibre  $\mathcal{L}_K$  is a line bundle on  $\mathcal{X}_K$ . We call  $(\mathcal{X}, \overline{\mathcal{L}})$  an integral model of the generic fibre  $(\mathcal{X}_K, \mathcal{L}_K)$ . For each place  $v$ , we will have a natural  $v$ -adic metric  $\|\cdot\|_v$  on  $\mathcal{L}_K$ . If  $v$  is archimedean, the metric is exactly the hermitian metric. If  $v$  is non-archimedean, a point  $x \in \mathcal{X}(\mathbb{C}_v)$  extends to  $\tilde{x} : \text{Spec}(O_{\mathbb{C}_v}) \rightarrow \mathcal{X}_{O_{\mathbb{C}_v}}$ . Then  $\tilde{x}^* \mathcal{L}_{O_{\mathbb{C}_v}}$  gives a lattice in  $\mathcal{L}_{\mathbb{C}_v}(x)$ , and thus induces a  $\mathbb{C}_v$ -norm on  $\mathcal{L}_{\mathbb{C}_v}(x)$ . It gives  $\|\cdot\|_v$ . We call a collection  $\{\|\cdot\|_v\}_v$  of  $v$ -adic metrics on  $\mathcal{L}_K$  over all places  $v$  of  $K$  obtained by this way an *algebraic adelic metric* on  $\mathcal{L}_K$ .

In general, let  $(X, L)$  be as above. A collection  $\{\|\cdot\|_v\}_v$  of  $v$ -adic metric on  $L$  over all places  $v$  of  $K$  is called an *adelic metric* if there is an algebraic adelic metric  $\{\|\cdot\|'_v\}_v$  on  $L$  satisfying the following coherence and continuity conditions:

- There exists a finite set  $S$  of places of  $K$  such that  $\|\cdot\|_v = \|\cdot\|'_v$  for all  $v \notin S$ ;
- The quotient  $\|\cdot\|_v / \|\cdot\|'_v$  is a continuous function on  $X(\mathbb{C}_v)$  for all places  $v$ .

We usually write  $\overline{L} = (L, \{\|\cdot\|_v\}_v)$  and call it an *adelic metrized line bundle*. It induces the supremum norms on  $H^0(X_{\mathbb{C}_v}, L_{\mathbb{C}_v})$  given by

$$\|s\|_{v, \text{sup}} = \sup_{x \in X(\mathbb{C}_v)} \|s(x)\|_v.$$

### Approximation of convex cone

We recall some consequences of a theorem in Khovanskii [Kh] which will be used later. Similar result are used in [LM] and [Ny].

Let  $\Gamma$  be a sub-semigroup of  $\mathbb{N}^{d+1}$ , and assume that  $\Gamma$  generates  $\mathbb{Z}^{d+1}$  as a group. Denote

$$\begin{aligned} \Sigma(\Gamma) &= \text{closed convex cone of } \Gamma \text{ in } \mathbb{R}^{d+1}; \\ \Delta(\Gamma) &= \Sigma(\Gamma) \cap (\mathbb{R}^d \times \{1\}); \\ \Gamma_m &= \Gamma \cap (\mathbb{N}^d \times \{m\}), \quad m \in \mathbb{N}; \\ \Lambda_m(\Gamma) &= \frac{1}{m} \Gamma_m \subset \Sigma(\Gamma), \quad m \in \mathbb{N}. \end{aligned}$$

We may also view  $\Delta(\Gamma)$  as a subset of  $\mathbb{R}^d$ , and  $\Lambda_m$  as a subset of  $\mathbb{Q}^d$ . Then  $\Delta(\Gamma)$  is a convex body of  $\mathbb{R}^d$  in the sense that it is convex and closed.

The following result is a rephrasal of Proposition 3 in §3 of Khovanskii [Kh]:

**Theorem 2.1** (Khovanskii). *If  $\Gamma$  is finitely generated as a semigroup, then there exists an element  $\gamma \in \Gamma$  such that  $(\gamma + \Sigma(\Gamma)) \cap \mathbb{N}^{d+1} \subset \Gamma$ .*

**Corollary 2.2.** *For any convex body  $D$  contained in  $\Delta(\Gamma)^\circ$ , there exists  $m_0 > 0$  such that*

$$D \cap \Lambda_m(\Gamma) = D \cap \frac{1}{m}\mathbb{N}^d$$

for all integers  $m > m_0$ .

*Proof.* It is immediately true if  $\Gamma$  is finitely generated by the above theorem. If  $\Gamma$  is arbitrary, we can find a finitely generated sub-semigroup  $\Gamma' \subset \Gamma$  such that  $\Delta(\Gamma')^\circ$  contains  $D$ . Then apply the theorem on  $\Gamma'$ .  $\square$

For applications to  $(X, \bar{L})$  as in the introduction, we will take

$$\Gamma = \bigcup_{m \geq 1} \nu(H^0(X, mL)) \times \{m\}.$$

The related notations are translated as:

$$\begin{aligned} \Delta(\Gamma) &= \Delta(L) \text{ Okounkov body;} \\ \Gamma_m &= \nu(H^0(X, mL)) \times \{m\}; \\ \Lambda_m(\Gamma) &= \Lambda_m(L) \times \{1\}. \end{aligned}$$

By [LM, Lemma 2.2], it generates the group  $\mathbb{Z}^{d+1}$  if  $L$  is big. Hence we can use the results above.

## 2.2 Basic properties

Resume the notations in the introduction. That is, let  $X$  be a projective variety over  $K$  of dimension  $d$ , and  $\bar{L} = (L, \{\|\cdot\|_v\}_v)$  be an adelic metrized line bundle on  $X$  with  $L$  big. As in the introduction, take a rational regular point  $x_0 \in X(K)$  which exists by enlarging  $K$ , take a local coordinate  $t = (t_1, \dots, t_d)$  at  $x_0$ , and a base local section  $s_0$  that induces a trivialization of  $L$ . Then we have a valuation map  $\nu = \nu_t$  on the global sections, and the Okounkov body  $\Delta(L) = \Delta_t(L)$  is the closure of

$$\Lambda(L) = \bigcup_{m \geq 1} \frac{1}{m} \nu(H^0(X, mL))$$

in  $\mathbb{R}^d$ . Denote

$$\Lambda_m(L) = \Lambda_{t,m}(L) = \frac{1}{m} \nu(H^0(X, mL)).$$

It is a finite subset of  $\Lambda(L)$ .

We first show Proposition 1.1 following the method of [Ny]. Let  $v$  be any place of  $K$ . For any positive integer  $m$  and  $\alpha \in \Lambda_m(L)$ , we would like to bound

$$F_v[m\bar{L}](m\alpha) = \inf_{s \in H^0(X_{\mathbb{C}_v}, mL_{\mathbb{C}_v})(m\alpha)} \{\log \|s\|_{v, \text{sup}}\},$$

where

$$H^0(X_{\mathbb{C}_v}, mL_{\mathbb{C}_v})(m\alpha) = \{s \in H^0(X_{\mathbb{C}_v}, mL_{\mathbb{C}_v}) : s = t^{m\alpha} + \text{higher order terms}\}.$$

Denote by  $|(m\alpha, m)|$  the sum of all components of  $m\alpha$  and  $m$ . The following result includes the non-archimedean case of [Ny, Lemma 5.4].

**Lemma 2.3.** *There exists a constant  $C$  independent of  $(m, \alpha)$  such that*

$$F_v[m\bar{L}](m\alpha) \geq C|(m\alpha, m)|.$$

*Proof.* The archimedean case is just Nyström's result. The non-archimedean is proved in the same way. Let  $v$  be non-archimedean. We will show that there is a constant  $C$  such that

$$\log \|s\|_{v, \text{sup}} \geq C|(m\alpha, m)|, \quad \forall s \in H^0(X, mL)(m\alpha).$$

The coordinate  $t = (t_1, \dots, t_d)$  maps points near  $x_0$  in  $X(\mathbb{C}_v)$  to  $\mathbb{C}_v^d$ . We can find an closed neighborhood  $U$  of  $x_0$  which is bijectively to the closed polydisc  $D$  of radius  $r > 0$  in  $\mathbb{C}_v^d$ . As in the archimedean case, we have

$$\|s(x)\|_v = h(x)^m |s_0^{-m} s|_v, \quad \forall x \in U.$$

Here  $h$  is some positively-valued continuous function on  $U$ . Let  $A > 0$  be a positive lower bound of  $h$ . It follows that

$$\|s\|_{v, \text{sup}} \geq \sup_{x \in U} \{h(x)^m |s_0^{-m} s|_v\} \geq A^m \sup_{x \in U} |s_0^{-m} s|_v.$$

View  $s_0^{-m} s$  as a convergent power series on  $D$ . It is of the form:

$$t^{m\alpha} + \text{higher order terms}.$$

Its maximal absolute value on  $D$  is equal to its Gauss norm. By definition, the Gauss norm is greater than or equal to  $r^{|m\alpha|}$ . It finishes the proof.  $\square$

With the above lemma, Proposition 1.1 is proved in the same way as [Ny, Proposition 5.6]. Now we want to have some bound on  $c_v[\bar{L}]$  by varying  $v$ . It suffices to bound the values of  $c_v[\bar{L}]$  on  $\Lambda(L)$ . For any  $\alpha \in \Lambda(L)$  in the interior of  $\Delta(L)$ , we have

$$c_v[\bar{L}](\alpha) = \lim_m \frac{1}{m} F_v[m\bar{L}](m\alpha).$$

Here the limit is only taken for positive integers  $m$  such that  $\alpha \in \Lambda_m(L)$ .



**Lemma 2.4.** *The following are true:*

(1a) *For all places  $v$ ,  $\frac{1}{m}F_v[m\bar{L}](m\alpha)$  has a lower bound independent of  $(m, \alpha)$ .*

(1b) *For all places  $v$ ,  $c_v[\bar{L}]$  is bounded from below.*

(2a) *For all but finitely many places  $v$ , we have  $F_v[m\bar{L}] \geq 0$  for all  $m \geq 1$ .*

(2b) *For all but finitely many places  $v$ , and  $c_v[\bar{L}] \geq 0$ .*

(3a) *For any  $\alpha \in \Lambda(L)$ , the value  $F_v[m\bar{L}](m\alpha) = 0$  for all but finitely many places  $v$  and for all  $m$  such that  $\alpha \in \Lambda_m(L)$ .*

(3b) *For any  $\alpha \in \Delta(L)^\circ$ , the value  $c_v[\bar{L}](\alpha) = 0$  for all but finitely many places  $v$ .*

*Proof.* It is easy to see that (1b) and (2b) are implied by (1a) and (2a) respectively. From (3a) to (3b) it requires a little argument.

By Lemma 2.3,

$$\frac{1}{m}F_v[m\bar{L}](m\alpha) > C|(\alpha, 1)|, \quad \alpha \in \Lambda_m(L).$$

The right-hand side is bounded since the Okounkov body is bounded. It proves (1a)

Now we prove (2a) and (3a). It will not impact the truth of the results if we change the adelic metric of  $\bar{L}$  at finitely many places. Thus we can assume that the adelic metric on  $\bar{L}$  is algebraic, i.e., it is induced by an integral model  $(\mathcal{X}, \bar{\mathcal{L}})$  of  $(X, L)$ .

We first show (2a). Let  $v$  be a non-archimedean place such that the fibre  $\mathcal{X}_{\mathbb{F}_v}$  of  $\mathcal{X}$  above  $v$  is irreducible and has multiplicity zero in  $\text{div}(s_0)$  and all  $\text{div}(t_i)$  as divisors on  $\mathcal{X}$ . It only excludes finitely many places. We claim that  $F_v[m\bar{L}] \geq 0$ . Otherwise, there is a section  $s \in H^0(X, mL)(m\alpha)$  with  $\|s\|_{v, \text{sup}} < 1$  for some  $\alpha \in \Lambda_m(L)$ . Then  $\text{div}(s)$ , as a divisor on  $\mathcal{X}$ , has positive multiplicity on  $\mathcal{X}_{\mathbb{F}_v}$ . This is impossible since

$$s = s_0^{m_1} t^{m_2} (1 + \text{higher order terms}).$$

Each factor on the right-hand side has multiplicity zero on  $\mathcal{X}_{\mathbb{F}_v}$ .

Now we consider (3a). By (2a), it suffices to show that  $F_v[m\bar{L}](m\alpha) \leq 0$  for all but finitely many places  $v$  and for all  $m \in M$ . Here  $M(\alpha)$  denote the set of positive integer  $m$  such that  $\alpha \in \Lambda_m(L)$ . It is a semigroup. Fix an  $m \in M(\alpha)$ , and pick any element  $s \in H^0(X, mL)(m\alpha)$ . Then  $\|s\|_{v, \text{sup}} = 1$  for almost all non-archimedean place  $v$ . For such a place  $v$ , by definition  $F_v[m\bar{L}](m\alpha) \leq 0$ . In another word, there is a finite set  $S(m)$  of places of  $K$  such that  $F_v[m\bar{L}](m\alpha) \leq 0$  for all  $v \notin S(m)$ . Observe that for  $m_1, m_2 \in M(\alpha)$ , we have  $m_1 + m_2 \in M(\alpha)$  and  $S(m_1 + m_2) \subset S(m_1) \cup S(m_2)$ . Since the semigroup  $M(\alpha)$  is finitely generated, it is easy to find a common  $S$  for all  $m \in M(\alpha)$ .

In the end, we consider (3b). By (3a), we see that the result is true if  $\alpha \in \Lambda(L)$ . For a general  $\alpha \in \Delta(\bar{L})^\circ$ , pick two elements  $\beta, \gamma \in \Lambda(\bar{L})$  such that the line segment between  $\beta$  and  $\gamma$  contains  $\alpha$ . We have showed  $c_v[\bar{L}](\beta) = c_v[\bar{L}](\gamma) = 0$  for almost all  $v$ . Since  $c_v[\bar{L}]$  is convex, we have  $c_v[\bar{L}](\alpha) \leq 0$  almost all  $v$ . By (1b), we conclude that  $c_v[\bar{L}](\alpha) = 0$  for almost all  $v$ .  $\square$

*Remark.* By the convexity of  $c_v[\bar{L}]$ , we actually know that for any closed convex polytope contained in  $\Delta(\bar{L})^\circ$ ,  $c_v[\bar{L}]$  vanishes identically on the polytope for almost all  $v$ .

Now we can introduce the global Chebyshev transform as the sum of the local ones. Denote

$$\begin{aligned} F[\bar{L}](\alpha) &:= \sum_v F_v[\bar{L}](\alpha), \quad \alpha \in \Lambda_1(L) \\ c[\bar{L}](\alpha) &:= \sum_v c_v[\bar{L}](\alpha), \quad \alpha \in \Delta(L)^\circ. \end{aligned}$$

By Lemma 2.4 (3a) (3b), both sums are finite and thus well-defined. Below is the global version of Proposition 1.1.

**Proposition 2.5.** *The following are true:*

- (1) *The function  $c[\bar{L}]$  is convex and continuous on  $\Delta(L)^\circ$ .*
- (2) *Let  $\{m_k\}_k$  be a sequence of positive integers divergent to infinity, and  $\alpha_k \in \Lambda_{m_k}(L)$  be a sequence convergent to some  $\alpha \in \Delta(L)^\circ$ . Then*

$$c[\bar{L}](\alpha) = \lim_{k \rightarrow \infty} \frac{1}{m_k} F[m_k \bar{L}](m_k \alpha_k).$$

*Proof.* These properties can be transferred from the local case since the definitions of  $F[\bar{L}]$  and  $c[\bar{L}]$  are essentially summations. For example, (1) is immediate.

Now we prove (2). We claim that there is a finite set  $S$  of places of  $F$  such that for any  $v \notin S$ ,

$$F_v[m_k \bar{L}](m_k \alpha_k) = 0, \quad c_v[\bar{L}](\alpha) = 0.$$

Once this is true, the result follows from the local case.

By Lemma 2.4 (2a) (3b), there is a finite set  $S$  of places of  $F$  such that for any  $v \notin S$ ,

$$F_v[m_k \bar{L}](m_k \alpha_k) \geq 0, \quad c_v[\bar{L}](\alpha) = 0.$$

It remains to show that the inverse direction of the inequality is true for some  $S$ .

As in §2.1, denote

$$\Gamma = \bigcup_{m \geq 1} \nu(H^0(X, mL)) \times \{m\}.$$

By definition  $(m_k \alpha_k, m_k)$  lies in the closed convex cone  $\Sigma(\Gamma)$ . Since  $\alpha_k \rightarrow \alpha$ , we can find a sub-semigroup  $\Gamma'$  generated by finitely many points  $(n_1 \beta_1, n_1), \dots, (n_r \beta_r, n_r)$  of  $\Gamma$  such that  $\Sigma(\Gamma')$  contains  $(m_k \alpha_k, m_k)$  for all  $k$ . By Theorem 2.1,  $(m_k \alpha_k, m_k) \in \Gamma'$  for  $k$  large enough.

By Lemma 2.4 (3a), there is a finite set  $S$  of places of  $F$  such that for any  $v \notin S$ ,

$$F_v[n_j \bar{L}](n_j \beta_j) = 0, \quad j = 1, \dots, r.$$

It implies that for any  $v \notin S$ ,

$$F_v[m_k \bar{L}](m_k \alpha_k) \leq 0, \quad \forall k \gg 0.$$

In fact, for sufficiently large  $k$  one can write

$$(m_k \alpha_k, m_k) = \sum_j a_j (n_j \beta_j, n_j), \quad a_j \in \mathbb{N}.$$

Then any choice of section  $s_j \in H^0(X_{\mathbb{C}_v}, n_j L_{\mathbb{C}_v})(n_j \beta_j)$  gives a section

$$\otimes_j s_j^{\otimes a_j} \in H^0(X_{\mathbb{C}_v}, m_k L_{\mathbb{C}_v})(m_k \alpha_k).$$

Then it is easy to have the bound. □

### 2.3 Euler characteristic and arithmetic degree

Let  $X, \bar{L}$  be as before. In this subsection we build relation between  $F[\bar{L}]$  and  $\chi_{\text{sup}}(\bar{L})$ . The supremum norm at archimedean place is not an inner product, so we first introduce the  $L^2$ -norm and compare these two norms.

#### Change of norm

Let  $v$  be an archimedean place. Fix a measure  $d\mu_v$  on  $X(\mathbb{C}_v)$ , which is assumed to be the push-forward measure of a positive smooth volume form on some resolution of singularity of  $X(\mathbb{C}_v)$ . It gives an  $L^2$ -norm by

$$\|s\|_{v, L^2}^2 = \int |s(z)|_v^2 d\mu_v, \quad s \in H^0(X, L)_{\mathbb{C}_v}.$$

Denote the associated bilinear pairing by  $\langle \cdot, \cdot \rangle_v$ .

**Lemma 2.6** (Gromov). *There exists a positive constant  $a$  such that*

$$am^{-d} \|s\|_{v, \text{sup}} \leq \|s\|_{v, L^2} \leq \|s\|_{v, \text{sup}}, \quad \forall m > 0, s \in H^0(X, mL)_{\mathbb{C}_v}.$$

For a proof, see [GS2] or [Yu1, Proposition 2.13]. Such a property is called Bernstein-Markov property by analysts.

Replacing the supremum norms by the  $L^2$ -norms at every archimedean place, we define the  $L^2$ -version  $(\chi_{L^2}(\bar{L}), F'_v[\bar{L}], c'_v[\bar{L}])$  of  $(\chi_{\text{sup}}(\bar{L}), F_v[\bar{L}], c_v[\bar{L}])$ . Their difference can be ignored by the above lemma.

First, the  $L^2$ -characteristic  $\chi_{L^2}$  is defined by

$$\chi_{L^2}(X, \bar{L}) = \log \frac{\text{vol}(B_{L^2}(\bar{L}))}{\text{vol}(H^0(X, L)_{\mathbb{A}_K} / H^0(X, L))}.$$

Here the unit ball

$$B_{L^2}(\bar{L}) = \prod_{v|\infty} B_{v,L^2}(\bar{L}) \times \prod_{v \nmid \infty} B_{v,\text{sup}}(\bar{L})$$

with

$$B_{v,L^2}(\bar{L}) = \{s \in H^0(X, L)_{K_v} : \|s\|_{v,L^2} \leq 1\}.$$

Second, for any archimedean  $v$ , we define

$$F'_v[m\bar{L}](m\alpha) := \inf_{s \in H^0(X_{\mathbb{C}_v}, mL_{\mathbb{C}_v})(m\alpha)} \{\log \|s\|_{v,L^2}\}.$$

For any  $\alpha \in \Delta(L)^\circ$ , choose a sequence  $\alpha_k \in \Delta_{m_k}(L)$  converging to a point  $\alpha$ , and define

$$c'_v[\bar{L}](\alpha) := \lim_{k \rightarrow \infty} \frac{1}{m_k} F'_v[m_k \bar{L}](m_k \alpha_k).$$

We have the following simple consequence of Lemma 2.6.

**Lemma 2.7.** *Let  $v$  be an archimedean place. The following are true:*

- (1)  $\chi_{L^2}(X, m\bar{L}) = \chi_{\text{sup}}(X, m\bar{L}) + O(m^d \log m)$ .
- (2)  $F'_v[m\bar{L}](m\alpha) = F'_v[m\bar{L}](\alpha) + O(\log m)$ .
- (3) For any  $\alpha \in \Delta(L)^\circ$ , the limit defining  $c'_v[\bar{L}](\alpha)$  exists and does not depend on the sequence  $\{\alpha_k\}$ . Furthermore,  $c'_v[\bar{L}] = c_v[\bar{L}]$ .

### Euler characteristic and arithmetic degree

Let  $s_1, s_2, \dots, s_N$  be a basis of  $H^0(X, L)$  over  $K$ , where  $N = \dim_K H^0(X, L)$ . Define

$$\deg H^0(X, \bar{L}) := - \sum_{v|\infty} \log \sqrt{\det(\langle s_i, s_j \rangle_v)_{i,j}} - \sum_{v \nmid \infty} \log \frac{\text{vol}(O_{K_v} s_1 + \dots + O_{K_v} s_N)}{\text{vol}(B_{v,\text{sup}}(\bar{L}))}.$$

**Proposition 2.8.** *The definition of  $\deg H^0(X, \bar{L})$  is independent of the basis  $s_1, s_2, \dots, s_N$ . Furthermore, there exist a positive constant  $a_K$  depending only on  $K$  such that*

$$|\chi_{L^2}(X, \bar{L}) - \deg H^0(X, \bar{L})| \leq a_K N \log N.$$

*Proof.* We first recall some basic algebraic number theory. One has a coset identity

$$\mathbb{A}_K/K \cong (K \otimes_{\mathbb{Q}} \mathbb{R}/O_K) \times \prod_{v \nmid \infty} O_{K_v}.$$

Here  $K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$  where  $r_1$  (resp.  $2r_2$ ) is the number of real (resp. imaginary) embeddings of  $K$  in  $\mathbb{C}$ . Endow  $K \otimes_{\mathbb{Q}} \mathbb{R}$  with the Lebesgue measure induced by the identity, then  $\text{vol}(K \otimes_{\mathbb{Q}} \mathbb{R}/O_K) = \sqrt{d_K}$ . Here  $d_K$  denotes the discriminant of  $K$  over  $\mathbb{Q}$ .

Go back to the current situation. By the product formula, it is easy to see that the definition is independent of the basis. Denote by  $M = O_K s_1 + \cdots + O_K s_N$  the free lattice in  $H^0(X, L)$  generated by the basis. We obtain

$$H^0(X, L)_{\mathbb{A}_K} / H^0(X, L) \cong (H^0(X, L) \otimes_{\mathbb{Q}} \mathbb{R} / M) \times \prod_{v \nmid \infty} M_v.$$

It follows that

$$\chi_{L^2}(X, \bar{L}) = \log \frac{\text{vol}(\prod_{v|\infty} B_{v, L^2}(\bar{L}))}{\text{vol}(H^0(X, L) \otimes_{\mathbb{Q}} \mathbb{R} / M)} + \sum_{v \nmid \infty} \log \frac{\text{vol}(B_{v, \text{sup}}(\bar{L}))}{\text{vol}(M_v)}.$$

Identify  $H^0(X, L)$  with  $K^N$  via the basis  $\{s_i\}$ . Endow  $H^0(X, L) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1 N} \times \mathbb{C}^{r_2 N}$  with the Lebesgue measure. We have  $\text{vol}(H^0(X, L) \otimes_{\mathbb{Q}} \mathbb{R} / M) = \sqrt{d_K^N}$ , and

$$\text{vol}(B_{v, L^2}(\bar{L})) = \begin{cases} \det(\langle s_i, s_j \rangle_v)^{-\frac{1}{2}} V(N) & \text{if } v \text{ is real;} \\ \det(\langle s_i, s_j \rangle_v) V(2N) & \text{if } v \text{ is imaginary.} \end{cases}$$

Here

$$V(N) = \pi^{\frac{N}{2}} / \Gamma(\frac{N}{2} + 1)$$

is the volume of the unit ball in the Euclidean space  $\mathbb{R}^N$ . It follows that

$$\chi_{L^2}(X, \bar{L}) = \deg H^0(X, \bar{L}) - \frac{1}{2} N \log d_K + r_1 \log V(N) + r_2 \log V(2N).$$

Then the result follows from Stirling's formula. □

*Remark.* The vector space  $H^0(X, L)$  over  $K$  is endowed with the  $L^2$ -norm at archimedean places and the supremum norms at non-archimedean places. Then it induces an adelic metric on the one-dimensional  $K$ -vector space  $\det H^0(X, L)$ , whose arithmetic degree is exactly  $\deg H^0(X, \bar{L})$  defined above. A more elegant expression is

$$\deg H^0(X, \bar{L}) = - \sum_v \log \|s_1 \wedge s_2 \wedge \cdots \wedge s_N\|_v.$$

The bound above is a Riemann-Roch type result.

The following is the fundamental identity we will use.

**Proposition 2.9.**

$$\deg H^0(X, \bar{L}) = - \sum_v \sum_{\alpha \in \Lambda_1(L)} F'_v[\bar{L}](\alpha).$$

*Proof.* For every  $\alpha \in \Lambda_1(L)$ , we pick an element  $s_\alpha$  of

$$H^0(X, L)(\alpha) = \{s \in H^0(X, L) : s = t^\alpha + \text{higher order terms}\}.$$

Then  $\{s_\alpha : \alpha \in \Lambda_1(L)\}$  forms a  $K$ -basis for  $H^0(X, L)$ . By definition,

$$\deg H^0(X, \bar{L}) = - \sum_{v|\infty} \log \sqrt{\det(\langle s_\alpha, s_\beta \rangle_v)_{\alpha, \beta \in \Lambda_1(L)}} - \sum_{v \nmid \infty} \log \frac{\text{vol}(\sum_{\alpha \in \Lambda_1(L)} O_{K_v} s_\alpha)}{\text{vol}(B_{v, \text{sup}}(\bar{L}))}.$$

We will show the match of the local terms at each place  $v$ .

Recall that for  $\alpha \in \Lambda_1(L)$ ,

$$F_v[\bar{L}](\alpha) = \inf_{s \in H^0(X_{\mathbb{C}_v}, L_{\mathbb{C}_v})(\alpha)} \{\log \|s\|_{v, \text{sup}}\}.$$

Let  $e_{v, \alpha} \in H^0(X_{\mathbb{C}_v}, L_{\mathbb{C}_v})(\alpha)$  be an element that takes the infimum. Then we have

$$\sum_{\alpha \in \Lambda_1(L)} F_v[\bar{L}](\alpha) = \log \prod_{\alpha \in \Lambda_1(L)} \|e_{v, \alpha}\|_{v, \text{sup}}$$

for non-archimedean  $v$ . One needs to replace the supremum norm by the  $L^2$ -norm in the above expression for archimedean  $v$ .

First assume that  $v$  is archimedean. Since both  $s_\alpha$  and  $e_{v, \alpha}$  are elements of  $H^0(X_{\mathbb{C}_v}, L_{\mathbb{C}_v})(\alpha)$ , the transition matrix between the basis  $\{e_{v, \alpha}\}_\alpha$  and  $\{s_\alpha\}_\alpha$  is upper-triangular with 1 on the diagonals. Thus it has determinant 1. It follows that

$$\det(\langle s_\alpha, s_\beta \rangle_v)_{\alpha, \beta \in \Lambda_1(L)} = \det(\langle e_{v, \alpha}, e_{v, \beta} \rangle_v)_{\alpha, \beta \in \Lambda_1(L)}.$$

A key property of  $\{e_{v, \alpha}\}_\alpha$  is that they form an orthogonal basis of  $H^0(X_{\mathbb{C}_v}, L_{\mathbb{C}_v})$ . Otherwise, assume that  $e_{v, \alpha}$  is not orthogonal to  $e_{v, \alpha'}$  for some  $\alpha < \alpha'$ . Then there will be an  $\epsilon \in K_v$  such that the norm of  $e_{v, \alpha} + \epsilon e_{v, \alpha'}$  is greater than  $e_{v, \alpha}$ . It contradicts to the choice of  $e_{v, \alpha}$ .

Hence, we simply have

$$\det(\langle s_\alpha, s_\beta \rangle_v)_{\alpha, \beta \in \Lambda_1(L)} = \prod_{\alpha \in \Lambda_1(L)} \|e_{v, \alpha}\|_{v, L^2}^2.$$

It gives the matching

$$\log \sqrt{\det(\langle s_\alpha, s_\beta \rangle_v)_{\alpha, \beta \in \Lambda_1(L)}} = \sum_{\alpha \in \Lambda_1(L)} F_v[\bar{L}](\alpha).$$

Next, assume that  $v$  is non-archimedean. The transition matrix between  $s_\alpha$  and  $e_{v, \alpha}$  still has determinant 1. It follows that we have

$$\text{vol}\left(\sum_{\alpha \in \Lambda_1(L)} O_{K_v} s_\alpha\right) = \text{vol}\left(\sum_{\alpha \in \Lambda_1(L)} O_{K_v} e_{v, \alpha}\right).$$

It remains to show that

$$\frac{\text{vol}(\sum_{\alpha \in \Lambda_1(L)} O_{K_v} e_{v,\alpha})}{\text{vol}(B_{v,\text{sup}}(\bar{L}))} = \prod_{\alpha \in \Lambda_1(L)} \|e_{v,\alpha}\|_{v,\text{sup}}.$$

We claim that

$$B_{v,\text{sup}}(\bar{L}) = \sum_{\alpha \in \Lambda_1(L)} O_{K_v} e_{v,\alpha}^\circ,$$

Here  $e_{v,\alpha}^\circ$  is any element in  $K_v e_{v,\alpha}$  with norm 1. It is easy to see that it implies what we want.

Now we prove the claim. Let  $s \in B_{v,\text{sup}}(\bar{L})$  be any element. We need to show that  $s$  belongs to  $\sum_{\alpha} O_{K_v} e_{v,\alpha}^\circ$ . We can uniquely write

$$s = \sum_{\alpha \in \Lambda_1(L)} a_{\alpha} e_{v,\alpha}, \quad a_{\alpha} \in K_v.$$

It suffices to show that  $|a_{\alpha}|_v \cdot \|e_{v,\alpha}\|_{v,\text{sup}} \leq 1$  for all  $\alpha$ .

Let  $\alpha_1 < \alpha_2 < \dots < \alpha_N$  be the elements of  $\Lambda_1(L)$  in the lexicographic order. If  $a_{\alpha_1} \neq 0$ , we have

$$1 \geq \|s\|_{v,\text{sup}} \geq |a_{\alpha_1}|_v \cdot \left\| e_{v,\alpha_1} + \sum_{\alpha \neq \alpha_1} a_{\alpha}^{-1} a_{\alpha} e_{v,\alpha} \right\|_{v,\text{sup}} \geq |a_{\alpha_1}|_v \cdot \|e_{v,\alpha_1}\|_{v,\text{sup}}.$$

Here the last inequality follows from the minimality of  $e_{v,\alpha_1}$ . It confirms the case  $\alpha = \alpha_1$ . Considering

$$s - a_{\alpha_1} e_{v,\alpha_1} = \sum_{\alpha \neq \alpha_1} a_{\alpha} e_{v,\alpha} \in B_{v,\text{sup}}(\bar{L}),$$

we can show that  $|a_{\alpha_2}|_v \cdot \|e_{v,\alpha_2}\|_{v,\text{sup}} \leq 1$ . Inductively, we will have the result for all  $\alpha$ .  $\square$

## 2.4 Proof of the main result

Now we are ready to prove Theorem 1.2. Note that if we change the metric  $\|\cdot\|_v$  to  $e^{-a}\|\cdot\|_v$  for some constant  $a \in \mathbb{R}$ , then  $F_v[\bar{L}]$  will increase by  $a$ . By this fact it is easy to see that the truth of Theorem 1.2 does not change. By Lemma 2.4, we can assume that all  $F_v[\bar{L}] \geq 0$  for all  $v$ . Then everything involved in the summation and integrals below are non-negative.

### The first part

We first prove Theorem 1.2 (1). By Proposition 2.8 and Proposition 2.9,

$$\chi_{L^2}(X, m\bar{L}) = - \sum_v \sum_{\alpha \in \Lambda_m(L)} F'_v[m\bar{L}](m\alpha) + O(m^d \log m).$$

By Lemma 2.7, we have the supremum version

$$\chi_{\text{sup}}(X, m\bar{L}) = - \sum_v \sum_{\alpha \in \Lambda_m(L)} F_v[m\bar{L}](m\alpha) + O(m^d \log m).$$

By Lemma 2.4 (3a), there are only finitely many nonzero terms in the double summation, so we can change the order of the summation. It yields

$$\chi_{\text{sup}}(X, m\bar{L}) = - \sum_{\alpha \in \Lambda_m(L)} F[m\bar{L}](m\alpha) + O(m^d \log m).$$

Here we recall that

$$F[m\bar{L}](m\alpha) = \sum_v F_v[m\bar{L}](m\alpha).$$

Rewrite it as

$$\frac{\chi_{\text{sup}}(X, m\bar{L})}{m^{d+1}} = -\frac{1}{m^d} \sum_{\alpha \in \Lambda_m(L)} \frac{1}{m} F[m\bar{L}](m\alpha) + O\left(\frac{1}{m} \log m\right). \quad (1)$$

Observe that

$$\frac{1}{m} F[m\bar{L}](m\alpha) \geq c[\bar{L}](\alpha).$$

In fact, for any  $\alpha \in \Lambda_m(L)$ , the sequence  $\frac{1}{mk} F_v[mk\bar{L}](mk\alpha)$  decreases to  $c_v[\bar{L}](\alpha)$  as  $k \rightarrow \infty$ . Hence the limit is always smaller. It follows that (1) gives

$$-\frac{\chi_{\text{sup}}(X, m\bar{L})}{m^{d+1}} \geq \frac{1}{m^d} \sum_{\alpha \in \Lambda_m(L)} c[\bar{L}](\alpha) + O\left(\frac{1}{m} \log m\right).$$

By Corollary 2.2, for any convex body  $D$  contained in  $\Delta(L)^\circ$ ,

$$-\frac{\chi_{\text{sup}}(X, m\bar{L})}{m^{d+1}} \geq \frac{1}{m^d} \sum_{\alpha \in D \cap \frac{1}{m}\mathbb{N}^d} c[\bar{L}](\alpha) + O\left(\frac{1}{m} \log m\right).$$

The right-hand side is a Riemann sum for  $c[\bar{L}]$ , except for some exceptions on the boundary of  $D$  which can be ignored. Taking limit, we obtain

$$-\frac{1}{d!} \text{vol}_\chi(\bar{L}) \geq \int_D c[\bar{L}](\alpha) d\alpha.$$

Let  $D \rightarrow \Delta(L)$ . We obtain

$$-\frac{1}{d!} \text{vol}_\chi(\bar{L}) \geq \int_{\Delta(L)} c[\bar{L}](\alpha) d\alpha.$$

Then we see that  $c[\bar{L}]$  and all  $c_v[\bar{L}]$  are integrable by the following lemma.



## The second part

Assume that  $\frac{1}{m}F[m\bar{L}]$  is uniformly bounded. Then our regularization method is the similar to Nyström's proof.

Recall that in (1) we have

$$\frac{\chi_{\text{sup}}(X, m\bar{L})}{m^{d+1}} = -\frac{1}{m^d} \sum_{\alpha \in \Lambda_m(L)} \frac{1}{m} F[m\bar{L}](m\alpha) + O\left(\frac{1}{m} \log m\right).$$

Define a step function  $\tilde{c}_m : \mathbb{R}^n \rightarrow \mathbb{R}$  as follows. Define

$$\tilde{c}_m(\alpha + \beta) = \frac{1}{m} F[m\bar{L}](m\alpha), \quad \forall \alpha \in \Lambda_m(L), \beta \in [0, 1/m)^d.$$

Define  $\tilde{c}_m(x)$  to be zero if  $x \in \mathbb{R}^n$  can not be written in the form  $\alpha + \beta$  describe above form. Then the summation on the right-hand side is exactly equal to the integral of  $\tilde{c}_m$ . We get

$$\frac{\chi_{\text{sup}}(X, m\bar{L})}{m^{d+1}} = - \int_{\mathbb{R}^n} \tilde{c}_m(x) dx + O\left(\frac{1}{m} \log m\right). \quad (2)$$

By Proposition 2.5 and Corollary 2.2, we know that  $\tilde{c}_m(x)$  converges to  $c[\bar{L}](x)1_{\Delta(L)^{\circ}}(x)$  almost everywhere. Note that  $\tilde{c}_m(x)$  is uniformly bounded and uniformly supported on a bounded domain. We can use dominant convergence theorem to conclude that

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} \tilde{c}_m(x) dx = \int_{\Delta(L)} c[\bar{L}](x) dx.$$

Take limit in (2). It proves the result.

We conjecture that  $\frac{1}{m}F[m\bar{L}]$  is uniformly bounded for all big  $L$ . It is easy to see that the result does not depend on the adelic metric on  $L$ . In fact, if  $\{\|\cdot\|_v\}_v$  and  $\{\|\cdot\|'_v\}_v$  are two metrics of  $L$ , then we can find a positive constant  $b_v \geq 1$  for each place  $v$  such that  $b_v^{-1}\|\cdot\|_v \leq \|\cdot\|'_v \leq b_v\|\cdot\|_v$ . Furthermore, we can take  $b_v = 1$  for almost all  $v$ . Then

$$-\sum_v \log b_v \leq \sum_v \frac{1}{m} F_v[\|\cdot\|'_v](m\alpha, m) - \sum_v \frac{1}{m} F_v[\|\cdot\|_v](m\alpha, m) \leq \sum_v \log b_v.$$

Then one of the sums is bounded if and only if the other one is bounded.

The following are some simple examples for which the uniform bound is easy to obtain.

*Example. (1)* The standard valuation in projective space. More precisely,  $X = \mathbb{P}^d$  with homogeneous coordinate  $(Z_0, \dots, Z_d)$ , base point  $x_0 = (1, 0, \dots, 0)$ , and local coordinate  $t = (Z_1/Z_0, \dots, Z_d/Z_0)$ . The line bundle  $\bar{L} = (O(1), \{\|\cdot\|_v\}_v)$  with arbitrary adelic metric, and the trivialization is given by the base section  $s_0 = Z_0$ . Then it is easy to verify that

$$\frac{1}{m} F[m\bar{L}](m\alpha) \leq \max_{0 \leq i \leq d} \sum_v \log \|Z_i\|_{v, \text{sup}}.$$

Note that the summation on the right-hand side is always a finite sum by the definition of adelic metric.

(2) Assume that the value semi-group

$$\Gamma = \bigcup_{m \geq 1} \nu(H^0(X, mL)) \times \{m\}$$

is finitely generated. As in the proof of Proposition 2.5, we have

$$\frac{1}{m} F[m\bar{L}](m\alpha) \leq \max_{1 \leq j \leq r} \frac{1}{n_j} F[n_j\bar{L}](n_j\beta_j)$$

where  $\{(n_j\beta_j, n_j) : j = 1, \dots, r\}$  is a set of generator of  $\Gamma$ . It happens if  $X$  is a toric variety and  $x_0, \text{div}(s_0), \text{div}(t_1), \dots, \text{div}(t_d)$  are invariant under the torus action.

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