Research Statement

Xinyi Yuan

My research interest lies in Diophantine geometry, Arakelov geometry, algebraic dynamics, Shimura varieties and automorphic forms. In particular, I focus on interdisciplinary results among these areas. My works are mainly 3-fold:

(1) Arakelov geometry and its application to algebraic dynamics and Diophantine geometry. My inspiration of Arakelov geometry also comes from results of algebraic geometry and complex analysis.

(2) Gross–Zagier type formulas which relate derivatives of L-functions to heights of special cycles on Shimura varieties. Such formulas lie in the intersection of Arakelov geometry, Shimura variety and automorphic forms.

(3) Arithmetic problems over related to the BSD conjecture over global function fields and the Tate conjecture over finite fields. For this subject, there are many approaches beyond the Gross–Zagier and Kolyvagin treatment.

1 Arakelov Geometry

Most of my works in Arakelov geometry are related to volumes of Hermitian line bundles and version of the arithmetic Hodge index theorem.

1.1 Volumes of arithmetic line bundles

Let $\mathcal{X}$ be an arithmetic variety over $\mathbb{Z}$ of absolute dimension n, and $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ a hermitian line bundle on $\mathcal{X}$. Define

$$h^0(\overline{\mathcal{L}}) = \log \#\{s \in H^0(\mathcal{X}, \mathcal{L}) : \|s\|_{\sup} \leq 1\}.$$

The volume of $\overline{\mathcal{L}}$ is defined by

$$\mathrm{vol}(\overline{\mathcal{L}}) = \limsup_{N\to\infty} \frac{h^0(\overline{\mathcal{L}}^{\otimes N})}{N^n/n!}$$

As in the geometric case, the "lim sup" is actually a limit. It was first proved by Chen [Ch1], and also implied my construction of Okounkov bodies in [Yu2].

Motivation to study the volumes of arithmetic line bundles

The volume of a hermitian line bundle, as a birational invariant, is a fundamental object in Arakelov geometry. For expositions of volumes in the classical case, we refer to [La].

Lower bound of the volume in the arithmetic case can be used to control heights of algebraic points, and thus deduce results in algebraic dynamics and Diophantine geometry. This procedure is crucial in Vojta's proof of the Mordell conjecture in [Vo1] and Faltings's proof of Lang's conjecture in [Fa2]. It is expected that the theory of volumes will lay the foundation in attacking Diophantine problems such as Vojta's conjecture (cf. [Vo2]).

Arithmetic version of Siu's Inequality

The arithmetic Hilbert–Samuel formula due to Gillet–Soulé [GS] and Bismut–Vasserot asserts that, if $\overline{\mathcal{L}}$ is ample in the sense of Zhang [Zh1], then $\mathrm{vol}(\overline{\mathcal{L}}) = \hat{c}_1(\overline{\mathcal{L}})^n$. The following is the arithmetic analogue of a theorem of Siu in algebraic geometry.

Theorem 1. *([Yu1]) If $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ are two ample hermitian line bundles over $\mathcal{X}$, then*

$$\mathrm{vol}(\overline{\mathcal{L}} \otimes \overline{\mathcal{M}}^{\otimes(-1)}) \geq \hat{c}_1(\overline{\mathcal{L}})^n - n \cdot \hat{c}_1(\overline{\mathcal{L}})^{n-1}\hat{c}_1(\overline{\mathcal{M}}).$$

It can be viewed as an extended version of the arithmetic Hilbert–Samuel formula. The extension is very meaningful in that it can be applied to line bundles without any positivity condition, since $\overline{\mathcal{L}} \otimes \overline{\mathcal{M}}^{\otimes(-1)}$ can be any line bundle. Many results and techniques in complex geometry and Arakelov theory, including the arithmetic Hilbert-Samuel formula, depend heavily on some positivity assumptions. As an application, an equidistribution result over algebraic dynamics is deduced. See Theorem 5.

Okounkov body for arithmetic line bundles

Extending an idea of Okounkov [Ok1, Ok2], Lazarsfeld and Mustaţă [LM] explored a systematic way to construct a convex body (called Okounkov body) from a line bundle on projective varieties. The volume of the Okounkov body gives the volume of the line bundle. A lot of important results in algebraic geometry are recovered by the construction.

An arithmetic analogue is introduced and treated in [Yu2, Yu3], where Okounkov bodies are constructed from the arithmetic object $(\mathcal{X}, \overline{\mathcal{L}})$. The volumes of these Okounkov bodies approximate $\mathrm{vol}(\overline{\mathcal{L}})$, and basic properties of the volume function is proved in [Yu2, Yu3]. For example, I obtain the arithmetic Fujita approximation theorem that the volume of a big hermitian line bundle can be approximated by the volumes of certain ample hermitian line bundles. The arithmetic Fujita approximation theorem is also proved by Chen [Ch2] independently.

Effective upper bound of the volume

The main theorem of [YtZ1] is as follows.

Theorem 2. *([YtZ1]) Let $\mathcal{X}$ be an arithmetic surface over $\mathbb{Z}$, and $\overline{D}$ be a nef arithmetic divisor on $\mathcal{X}$. Then*

$$\widehat{h}^0(\overline{D}) \leq \frac{1}{2}\overline{D}^2 + 4d\log(3d).$$

Here $d = \deg D_{\mathbb{Q}}$.

It is an effective version of the arithmetic Hilbert–Samuel formula. We have generalizations of the result to function fields and high dimensions in [YtZ2, YtZ3].

1.2 Hodge index theorem for adelic line bundles

This subsection is to describe the main results of [YsZ1], which generalizes the arithmetic Hodge index theorem of Faltings [Fa1], Hriljac [Hr] and Moriwaki [Mo] from integral Hermitian line bundles to adelically metrized line bundles over number fields. The work of [YsZ2] further generalizes the theorem further to finitely generated fields.

Global Hodge index theorem

The main result of [YsZ1] is the following Hodge index theorem.

Theorem 3. *([YsZ1]) Let $\pi : X \to \mathrm{Spec}\bar{\mathbb{Q}}$ be a normal and integral projective variety of dimension $n \geq 1$. Let $\overline{M}$ be an integrable adelic $\mathbb{Q}$-line bundle on X, and $\overline{L}_1, \cdots, \overline{L}_{n-1}$ be $n-1$ nef adelic $\mathbb{Q}$-line bundles on X. Assume $M \cdot L_1 \cdots L_{n-1} = 0$ and that each L_i is big. Then*

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0.$$

Moreover, if $\overline{L}_i$ is arithmetically positive and $\overline{M}$ is $\overline{L}_i$-bounded for each i, then the equality holds if and only if

$$\overline{M} \in \pi^*\widehat{\mathrm{Pic}}(\bar{\mathbb{Q}})_{\mathbb{Q}}.$$

The theorem has an immediate consequence in algebraic dynamics as we shall see next section. The paper also proves a local version of the theorem, which is essentially the following non-archimedean Calabi–Yau Theorem.

Non-archimedean Calabi–Yau Theorem

The prestigious Calabi–Yau theorem is as follows. *Let L be an ample line bundle on a complex projective manifold X of dimension n. Let ω be a positive and smooth (n,n)-form on X with $\int_X \omega = \deg(L)$. Then there exists a positive hermitian metric h on L such that $c_1(L,h)^n = \omega$. Moreover, the metric h is unique up to scalar multiples.*

The following is the uniqueness part of a non-archimedean analogue of the Calabi–Yau theorem.

Theorem 4. *([YsZ1]) Let X be a projective variety of dimension n over a complete non-archimedean valued field. Let L be an ample line bundle over X, and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two semipositive metrics on L. Then:*

$$c_1(L, \|\cdot\|_1)^n = c_1(L, \|\cdot\|_2)^n \Longleftrightarrow \frac{\|\cdot\|_1}{\|\cdot\|_2} \text{ constant.}$$

Here $c_1(L, \|\cdot\|)^n$ denotes the measure associated to $(L, \|\cdot\|)$, introduced by Chambert-Loir [CL], on the Berkovich analytic space X^{an} of X constructed by Berkovich [Be].

2 Algebraic dynamical systems

Let K be a field. A polarized algebraic dynamical system over K is a triple (X, f, L), where X is a projective variety over K, $f : X \to X$ is an endomorphism, and L is an ample line bundle on X such that $f^*L \cong L^{\otimes q}$ for some $q > 1$. The most popular examples are abelian varieties and projective spaces.

Equidistribution Theorem

Assume that K is a number field. There is a canonical height function $\hat{h}_f : X(\overline{K}) \to \mathbb{R}_{\geq 0}$ introduced by Call–Silverman [CS]. Similar to Neron–Tate height on abelian varieties, the canonical height is obtained by Tate's limiting argument from any Weil height function in the class of L. Furthermore, $\hat{h}_f(x) = 0$ if and only if x is preperiodic in the sense that the orbit $\{x, f(x), f^2(x), \cdots\}$ is finite.

Theorem 5 ([Yu1])**.** *Let $\{x_m\}_{m=1}^{\infty}$ be a generic sequence of points in $X(\overline{K})$ with $\hat{h}_f(x_m) \to 0$. Then the Galois orbit of the sequence is equidistributed on the analytic space $X_{K_v}^{\mathrm{an}}$ at any place v of K.*

The previously known cases were mainly the abelian variety case by Szpiro–Ullmo–Zhang [SUZ], the torus case by Bilu [Bi], and the non-archimedean analogue of these two cases by Chambert-Loir [CL]. The proof in [SUZ] uses the arithmetic Hilbert–Samuel formula, which does not work here since in our general case "negative" line bundles appears in the proof. Theorem 2 overcomes this difficulty. Another feature of the theorem is that it gives a uniform proof including all the previous cases.

Rigidity of the Set of Preperiodic Points

Let (X, f, L) be a polarized algebraic dynamical system over K of characteristic zero. The set $\mathrm{Prep}(f)$ of preperiodic points in $X(\overline{K})$ determines many aspects of the dynamics. For example, its accumulation points form the Julia set over the complex numbers.

Theorem 6 ([YsZ1, YsZ2])**.** *Let (X, f, L) and (X, g, L) be two polarized algebraic dynamical systems, sharing the same underlying space X and the same polarization L, over any field K of characteristic zero. Then the following are equivalent:*

(a) $\mathrm{Prep}(f) = \mathrm{Prep}(g)$*;*

(b) $g\mathrm{Prep}(f) = \mathrm{Prep}(f)$*;*

(c) $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *is Zariski dense in* X*.*

The theorem works for any field of characteristic zero including $\mathbb{C}$, while the proof is arithmetic. We reduce the problem to number fields, and then use Theorem 5 and Theorem 4. In the number field case, we actually prove that the conditions of the theorem are equivalent to $\hat{h}_f = \hat{h}_g$. Hence the arithmetic of f and g are essentially the same.

In the case $X = \mathbb{P}^1$, the theorem is obtained independently by Baker–DeMarco [BD].

3 Special Value Formulas

Here we introduce the Gross–Zagier formula proved in [YZZ] and the averaged Colmez conjecture proved in [YsZ3].

Gross–Zagier Formula under any Ramification

Gross and Zagier proved their original formula on certain modular curves in [GZ], and S. Zhang extended it to quite general cases over totally real fields in [Zh2, Zh3, Zh4]. There are many difficulties in the computation of automorphic forms and arithmetic intersection on Shimura curves, so these proofs only work under a lot of unramified conditions.

In [YZZ], we prove the Gross–Zagier formula in the most general setting, as expected by Gross [Gr]. More precisely, we compute the central derivative of the Rankin–Selberg L-function $L(s, \pi, \chi)$ with root number -1 in the following case:

- $\pi = \otimes_v \pi_v$ is cuspidal automorphic representation of GL_2 over a totally real number field F, discrete of weight 2 at all archimedean places;
- $\chi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ is a character of finite order for a totally imaginary quadratic extension E over F. Assume that $\chi|_{\mathbb{A}^\times}$ is reciprocal to the central character of π.

This general formula automatically simplifies many important results on Heegner points depending on the original formulas of Gross–Zagier and S. Zhang. For example, we can simply sharpen the indefinite case of Mazur's conjecture proved in the work of Cornut–Vatsal [CV].

Our new treatment simplifies the previous proofs in many respects. It follows the framework of the central value formula by Waldspurger [Wa], uses Kudla's generating function with coefficients given by special cycles as a substitute of the theta function, and takes a more representation-theoretic point of view. The identity at unramified places follows from explicit computation, and computation of the identity at bad places is avoided by an argument using the multiplicity one result of local linear functionals by Tunnell [Tu] and Saito [Sa].

Averaged Colmez conjecture

Let E be a CM field of degree $[E:\mathbb{Q}]=2g$, with the maximal totally real subfield F. Let $\Phi\subset \mathrm{Hom}(E,\mathbb{C})$ be a CM-type, i.e., a subset such that $\Phi\cap\bar\Phi=\emptyset$ and $\Phi\cup\bar\Phi=\mathrm{Hom}(E,\mathbb{C})$. Let A_Φ be a CM abelian variety over $\mathbb{C}$ of CM type (O_E,Φ). By the theory of complex multiplication, A_Φ is defined over a number field K and has a smooth and projective integral model over O_K. Denote by $h(A_\Phi)$ the Faltings height of A_Φ, which is the arithmetic degree of the Hodge bundle of A_Φ.

The averaged Colmez conjecture proved by [YsZ3] is the following theorem.

Theorem 7. *([YsZ3]) Let E/F be a CM extension, $\eta=\eta_{E/F}$ be the corresponding quadratic character, and d_F (resp. $d_{E/F}$) be the absolute discriminant of F (resp. the norm of the relative discriminant of E/F). Then*

$$\frac{1}{2^g}\sum_{\Phi}h(\Phi)=-\frac{1}{2}\frac{L'(0,\eta)}{L(0,\eta)}-\frac{1}{4}\log(d_{E/F}d_F),$$

where Φ runs through the set of all CM types of E.

The theorem is independently proved by [AGHM]. By the recent work of Jacob Tsimerman [Ts] et al, the theorem implies the Andre–Oort conjecture: *Let X be a Shimura variety of abelian type over $\mathbb{C}$. Let $Y\subset X$ be a closed subvariety which contains a Zariski dense subset of special points of X. Then Y is a special subvariety.*

4 Arithmetic over function fields

This section describes my recent work [Yu4]. It proves some results related to the BSD conjecture of abelian varieties over global function fields, or equivalently the Tate conjecture of projective and smooth surfaces over finite fields.

Positivity of Hodge bundle

Let S be a projective and smooth curve over a field k, and $K=k(S)$ be the function field of S. Let A be an abelian variety over K, and $\mathcal{A}$ be the Neron model of A over S. The Hodge bundle of A over $\mathcal{S}$ is defined to be the locally free $\mathcal{O}_S$-module

$$\overline{\Omega}_A=\overline{\Omega}_{\mathcal{A}/S}=e^*\Omega^1_{\mathcal{A}/S},$$

where $\Omega^1_{\mathcal{A}/S}$ denotes the relative differential sheaf, and $e:S\to\mathcal{A}$ denotes the identity section of $\mathcal{A}$. The main theorem of [Yu4] is as follows.

Theorem 8 ([Yu4])**.** *Denote $K=k(t)$ and $S=\mathbb{P}^1_k$ for a finite field k. Let A be an abelian variety over K. Then A is isogenous to $B\times_K C_K$, where C is an abelian variety over k, and B is an abelian variety over K with an ample Hodge bundle over S.*

In the following, we introduce two other consequences of the theorem.

Partial finiteness of Tate–Shafarevich group

Let A be an abelian variety over a global function field K of characteristic p. The Tate–Shafarevich group $Ш(A)$ is conjectured to be finite, and the finiteness of $Ш(A)[p^\infty]$ is actually equivalently to the BSD conjecture for A by the works of Tate et al. Denote by $F^n : A \to A^{(p^n)}$ the relative p^n-Frobenius morphism over K. Define

$$Ш(A)[F^n] = \ker(Ш(F^n) : Ш(A) \to Ш(A^{(p^n)}))$$

and

$$Ш(A)[F^\infty] = \bigcup_{n\geq 1} Ш(A)[F^n].$$

Both are subgroups of $Ш(A)$. Note that $F^n : A \to A^{(p^n)}$ is a factor of the multiplication $[p^n] : A \to A$, so $Ш(A)[F^\infty]$ is a subgroup of $Ш(A)[p^\infty]$.

Theorem 9 ([Yu4]). *Denote $K = k(t)$ for a finite field k. Let A be an abelian variety over K with everywhere semi-abelian reduction over $S = \mathbb{P}^1_k$. Then $Ш(A)[F^\infty]$ is finite.*

Variation of Tate conjectures

One version of the prestigious Tate conjecture for divisors is as follows. *Let X be a projective and smooth variety over a finite field k of characteristic p. Then for any prime $\ell \neq p$, the cycle class map*

$$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^2(X_{\bar{k}}, \mathbb{Q}_\ell(1))^{\mathrm{Gal}(\bar{k}/k)}$$

is surjective.

Theorem 10 ([Yu4]). *If the Tate conjecture holds for all projective and smooth surfaces X over finite fields with $H^1(X, \mathcal{O}_X) = 0$, then the Tate conjecture holds for all projective and smooth surfaces over finite fields.*

The proof uses Theorem 8, and also makes crucial use of the equivalence between the BSD conjecture and the Tate conjecture.

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