习题4.3

1. (2) $\frac{1}{2}\ln\frac{1-x}{1+x} = \frac{1}{2}\left(\ln(1-x) - \ln(1+x)\right)$.

$= \frac{1}{2}\left(-x + \frac{x^2}{2} - \frac{x^3}{3} + \cdots - \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)\right)$

$= \frac{1}{2}\cdot\left(2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots\right)$

$= \sum_{k=0}^{n} \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})$

(4) $\frac{x^3+2x-1}{x-1}$

$= -\left(x^2+2x-1\right)\cdot\left(1 + x + x^2 + \cdots + x^n + o(x^n)\right)$

$= -\left(x^2 + x^3 + \cdots + x^{n+2} + o(x^{n+2})\right)$

$-\left(2x + 2x^2 + \cdots + 2x^{n+1} + o(x^{n+1})\right)$

$+\left(1 + x + x^2 + \cdots + x^n + o(x^n)\right)$

$= 1 - x - 2x^2 - 2x^3 - \cdots - 2x^n + o(x^n)$

2.

(1) $e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)\right)\left(x - \frac{x^3}{6} + o(x^4)\right)$

$= x + x^2 + \frac{x^3}{3} + o(x^4)$.

(2) $\sqrt{1-2x+x^3} - \sqrt{1-3x+x^2}$

$= \left(1 + (x^3-2x)\right)^{\frac{1}{2}} - \left(1 + (x^2-3x)\right)^{\frac{1}{2}}$

$= 1 + \frac{1}{2}(x^3-2x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(x^3-2x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{6}(x^3-2x)^3 + o\left((x^3-2x)^3\right)$

$- \left(1 + \frac{1}{2}(x^2-3x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(x^2-3x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{6}(x^2-3x)^3 + o\left((x^2-3x)^3\right)\right)$

$= \frac{1}{2}x + \frac{1}{8}x^2 + \frac{15}{16}x^3 + o(x^3)$.

4.

(2) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

$= \lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)} = \lim_{x\to 0}\frac{e^x-1-x}{x^2} = \lim_{x\to 0}\frac{\frac{x^2}{2}+o(x^2)}{x^2} = \frac{1}{2}$

5. $f(x) = \sin(a+x) - (\sin a + x\cos a)$

将 $f(x)$ 在 0 点展开.

$$f(x) = f(0) + f'(0)x + \frac{f''(\xi)}{2}x^2$$

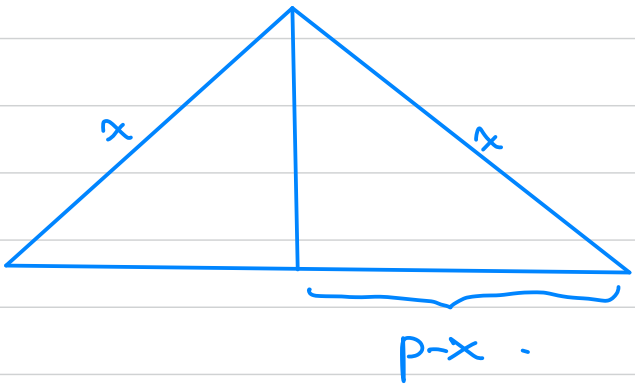
$$0 < \xi < x$$

$$f''(x) = -\sin(\xi + x) \qquad f'(0) = f(0) = 0$$

$$\Rightarrow \quad |f(x)| \leq \frac{1}{2}|x|^2 .$$

第七题.

3.



所绕旋转体为同轴.

$$V(x) = 2 \cdot \frac{1}{3}\cdot\pi\left(x^2 - (p-x)^2\right)(p-x)$$

$$= \frac{2\pi}{3}\, p(2x-p)(p-x)$$

method 1. 令 $V'(x) = 0 \Rightarrow x = \frac{3}{4}p$

method 2:

$$V(x) = \frac{\pi p}{3}(2x-p)(2p-2x)$$

$$\leq \frac{\pi p}{3}\cdot\left(\frac{2x-p+2p-2x}{2}\right)^2$$

if and only if

$$2x-p = 2p-2x \Rightarrow x = \frac{3}{4}p.$$

此时周长 $\frac{3p}{4}$, 底边 $\frac{p}{2}$

4.

$$f'(x) = 3x^2+2lx+k$$

$$\Rightarrow \begin{cases} f'(-1)=0 \\ f(-1)=2 \end{cases} \Rightarrow \begin{cases} l=0 \\ k=-3 \end{cases}$$

$$\Rightarrow f(x) = x^3-3x$$

$$f'(x) = 3x^2-3, \quad f''(x) = 6x.$$

极值点有 $x=1$ 或 $x=-1$.

$f''(1) = 6 > 0 \Rightarrow f(1)$ 是极小值.

$f''(-1) = -6 < 0 \Rightarrow$ $f(-1)$ 是极大值.

$$\begin{cases} f(0) = 0 \\ f(1) = -2 \quad \text{最小值} \\ f(3) = 18 \quad \text{最大值} \end{cases}$$

5.

$$r = \frac{a}{\cos\varphi}$$

$$\Rightarrow J = K \cdot \frac{\sin\varphi}{\left(\frac{a}{\cos\varphi}\right)^2}$$

$$= \frac{K}{a^2} \cdot \sin\varphi \cos^2\varphi$$

method 1: $\because J'(\varphi) = 0 \Rightarrow \tan\varphi = \frac{1}{\sqrt{2}}$

method 2

$$\sin\varphi\cos^2\varphi = \sqrt{(\sin\varphi\cos\varphi)^2}$$

注需要凑 $\sin^2\varphi \cos^2\varphi \cos^2\varphi$

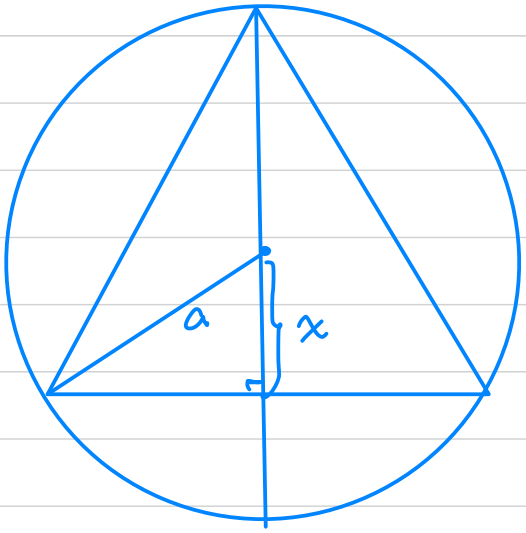
$$= \frac{1}{2} \cdot 2\sin^2\varphi \cos^2\varphi \cos^2\varphi$$

$$\leq \frac{1}{2}\left(\frac{2\sin^2\varphi + \cos^2\varphi + \cos^2\varphi}{3}\right)^3$$

if and only if

$$2\sin^2\varphi = \cos^2\varphi \quad \text{或} \quad \tan\varphi = \frac{1}{\sqrt{2}}$$

7.



$$V(x) = \frac{1}{3}\pi \cdot (a^2 - x^2) \cdot (x+a)$$

method 1: $V'(x) = 0 \Rightarrow x = \frac{a}{3}$

method 2:

$$V(x) = \frac{\pi}{3} \cdot \frac{1}{2} \cdot 2(a-x)(x+a)(x+a)$$

$$\leq \frac{\pi}{3} \cdot \frac{1}{2} \left(\frac{2a-2x+x+a+x+a}{3} \right)^3$$

if and only if

$$2a-2x = x+a \Rightarrow x = \frac{a}{3}$$

高应是 $\frac{4a}{3}$. 底面半径为 $\frac{2\sqrt{2}}{3}a$.

9.

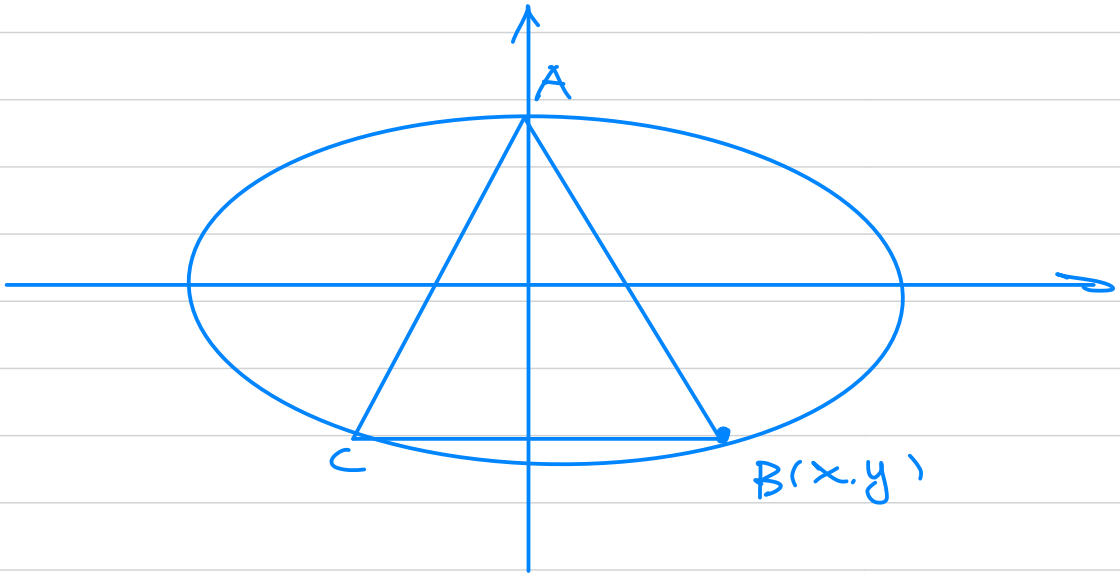
$$d_{y}^{2} = \left(\frac{y^2}{4} - 18\right)^2 + y^2$$

$$\because \left(d_{y}^{2}\right)' = 0 \Rightarrow y^2 = 64$$

$$\Rightarrow y = \pm 8. \quad x = 16$$

$(16, 8)$ $(16, -8)$ (该抛物线的对称性)

11.



$$S_{\triangle ABC} = \frac{1}{2} \cdot 2|x| \cdot (|y| + b)$$

~~由于对称性不妨假设~~ $x \cdot y > 0$ (第一象限).

$$S_{\triangle ABC} = x(y+b) = a\sqrt{1 - \frac{y^2}{b^2}}(y+b).$$

$$= \frac{a}{b} \cdot \sqrt{(b-y)(b+y)(b+y)(b+y)}$$

$$= \frac{1}{\sqrt{3}} \frac{a}{b} \sqrt{(2b-2y)(b+y)(b+y)(b+y)}$$

$$\leq \frac{a}{\sqrt{3}b} \cdot \sqrt{\left(\frac{2b-2y+b+y+b+y+b+y}{4}\right)^4}$$

if and only if. $2b-2y = b+y$

$$y = \frac{b}{2}$$

$$\max S_{\triangle ABC} = \frac{a}{\sqrt{3}b} \cdot \left(\frac{6b}{4}\right)^2 = \frac{36ab^2}{4 \cdot 4\sqrt{3}b} = \frac{3\sqrt{3}}{4}ab$$