

6.2.

1. (2).

$$\frac{\sin(x^3+y^4)}{x^2+y^2} = \frac{\sin(x^3+y^4)}{x^3+y^4} \cdot \frac{x^3+y^4}{x^2+y^2}$$

$$0 \le \left|\frac{x^3}{x^2+y^2}\right| = \left|\frac{x^2}{x^2+y^2} \cdot x\right| \le |x|$$

$$0 \le \left|\frac{y^4}{x^2+y^2}\right| = \left|\frac{y^2}{x^2+y^2} \cdot y^2\right| \le |y|^2$$

$$= \lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^4)}{x^2+y^2} = 0$$

(4) $\sqrt[3]{y-1} = z \qquad z \to 0$

$$\lim_{(x,z)\to(0,0)} \frac{x^3+z^3}{x^2+z^2} = 0 \qquad (\text{由 (2)})$$

$$\left|\frac{x^3}{x^2+z^2}\right| = \left|\frac{x^2}{x^2+z^2} \cdot x\right| \le |x|$$

(6) $$\lim_{(x,y,z)\to(1,-2,0)} \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln 5$$

2.

(1) $y = kx$

$$\frac{x^4 - k^2x^2}{x^4 + k^2x^2} = \frac{x^2 - k^2}{x^2 + k^2}$$

$k \neq 0$. $\lim \frac{x^4 - y^2}{x^4 + y^2} = -1$

$k = 0$ $\lim \frac{x^4 - y^2}{x^4 + y^2} = 1$

极限不存在.

(2)

$$\lim_{\substack{y=x \\ x \to 0}} f(x,y) = 0$$

$$\lim_{\substack{y=2x \\ x \to 0}} f(x,y) = \lim_{x \to 0} \frac{-3x}{x} = -3 \neq 0$$

极限不存在

3. (2) 令 $x^2 + y^2 = r^2$ $(r \to 0)$

原式 $= \frac{1 - \cos(r^2)}{r^2} \cdot \frac{r^4}{r^4 \sin^2\theta \cos^2\theta}$

$$= \frac{1 - \cos(r^2)}{r^4} \frac{r^4}{r^6 \sin^2\theta \cos^2\theta} \sim \frac{1}{2r^2 \sin^2\theta \cos^2\theta} \to \infty$$

(4) $$\left|\frac{x^k y^{n-k}}{(x^2+y^2)^{\frac{n}{2}}}\right| \le \frac{\rho^k \cdot \rho^{n-k}}{\rho^n} = \rho \to 0$$

(所有项表达式为 $x^k y^{n-k}$, $0 \le k \le n$, 且有限项和).

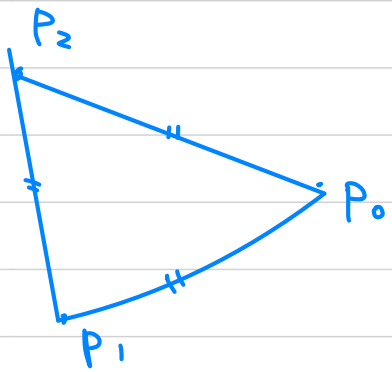
16.3

2.

解 $f(x,y) = d(P, P_0)$ $P(x,y)$

$= \sqrt{(x-x_0)^2 + (y-y_0)^2}$ $P_0(x_0, y_0)$

$|f(P_1, P_0) - f(P_2, P_0)|$



$= |d(P_1, P_0) - d(P_2, P_0)|$

$\le d(P_1, P_2)$

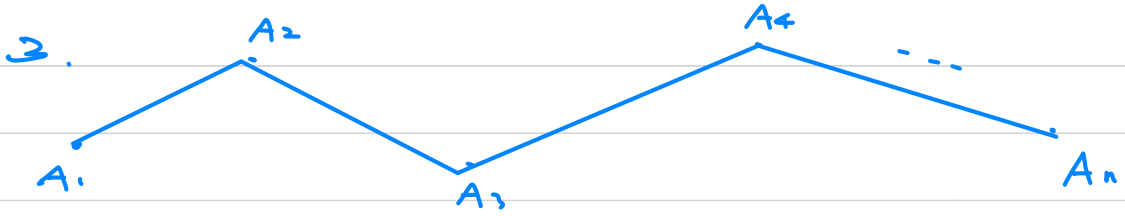
$\Rightarrow$ $f(x,y)$ 是连续函数.

且 $\overline{D}$ 是有界闭区域

$\Rightarrow$ $\exists\ S \in \overline{D}$ s.t.

$$\min f(x,y) = d(S, P_0)$$

3.



连结 A_i 从 A_1 到 A_n 的折线段可以容易表示. $A_i = (x_i, y_i)$

$$\begin{cases} x = a(t) \\ y = b(t) \end{cases} \quad \text{满足} \quad \begin{matrix} a(t_i) = x_i \\ b(t_i) = y_i \end{matrix} \qquad \alpha \le t \le \beta.$$

$$1 \le i \le n.$$

令 $g(t) = f(a(t), b(t))$

由一元函数介值定理, 至少

$$g(s) = \frac{1}{n} \cdot \sum_{i=1}^{n} g(t_i)$$

即

$$f(a(s), b(s)) = \frac{1}{n} \sum_{i=1}^{n} f(x_i, y_i)$$

5.

由于 $z = f(x,y)$ 在单位圆内连续

$\forall (x_0, y_0) \in D$,

$\forall \varepsilon > 0$, $\exists \delta > 0$,

$$|x - x_0| + |y - y_0| < \delta$$

$$|f(x,y) - f(x_0, y_0)| < \varepsilon$$

$$g(t_0) = f(x(t_0), y(t_0)) = f(x_0, y_0)$$

$$|g(t) - g(t_0)| = |f(x(t), y(t)) - f(x(t_0), y(t_0)|$$

由于 $x(t), y(t)$ 关于 t 连续.

$\exists \tilde{\delta}$. $\quad |t - t_0| < \tilde{\delta}$

$$|x(t) - x(t_0)| < \frac{\delta}{2}$$

$$|y(t) - y(t_0)| < \frac{\delta}{2}$$

再由上述 $f(x, y)$ 连续性可得.

$$|g(t) - g(t_0)| < \varepsilon.$$

6. $\dfrac{1}{1-(x^2+y^2)}$

6.4.

(2) $z_x = \dfrac{1}{\sqrt{x^2+y^2}} - \dfrac{x^2}{(\sqrt{x^2+y^2})^3} = \dfrac{y^2}{(\sqrt{x^2+y^2})^3}$

$$z_y = \frac{xy}{-(\sqrt{x^2+y^2})^3}$$

(4). $z_x = \frac{-y^2}{(x-y)^2}$

$z_y = \frac{x^2}{(x-y)^2}$

(6) $z_x = e^{-xy} + x(-y)e^{-xy}$

$= (1-xy)e^{-xy}$

$z_y = -x^2 e^{-xy}$

(8). $u_x = y^z \cdot z x^{z-1}$

$u_y = x^z z \cdot y^{z-1}$

$u_z = (xy)^z \cdot \ln xy$

3.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{|x|+|y|}$$

$$= \lim \frac{x}{|x|+|y|} \cdot x + \lim \frac{y}{|x|+|y|} \cdot y = 0$$

$= f(0,0)$

故 $f(x,y)$ 连续.

$$f_x(0,0) = \lim_{\substack{x\to 0 \\ y=0}} \frac{f(x,y)-f(0,0)}{x}$$

$$= \lim_{\substack{x\to 0 \\ y=0}} \frac{x^2y^2}{(|x|+y|)x}$$

$$= \lim_{x\to 0} \frac{x^2}{|x|\cdot x} = \lim_{x\to 0} \frac{x}{|x|}$$ 不存在

$x\to 0^+$, $f_x(0,0)=1$

$x\to 0^-$, $f_x(0,0)=-1$

6.

$$u_x = 3e^{-3y}\sin 3x$$

$$u_{xx} = -3e^{-3y}\cdot 3\cos 3x = -9e^{-3y}\cos 3x$$

$$u_y = -3e^{-3y}\cos 3x$$

$$u_{yy} = 9e^{-3y}\cos 3x.$$

$$\Rightarrow \quad u_{xx} + u_{yy} = 0$$

7. $u_x = v_y$

$u_y = -v_x$

$$\begin{cases} u_{xx} = v_{yx} & v_{yy} = u_{xy} \\ u_{yy} = -v_{xy} & -v_{xx} = u_{yx} \end{cases}$$

(u, v 有连续二阶偏导数 $\Rightarrow u_{xy} = u_{yx}$, $v_{xy} = v_{yx}$)

$\Rightarrow$ $du = 0$, $dv = 0$

9.

$$z_x = -\sin y + \frac{1}{1-xy}$$

$$\Rightarrow z = -x\sin y - \frac{1}{y}\ln|1-xy| + g(y)$$

$$z(0,y) = g(y) = 2\sin y + y^2$$

$$\Rightarrow z(x,y) = (2-x)\sin y + y^2 - \frac{1}{y}\ln|1-xy|$$

12.

$$z_x = x - \frac{y}{x^2+y^2}$$

$$\frac{y}{x^2+y^2} = \frac{y}{y^2}\cdot\frac{1}{1+(\frac{x}{y})^2} = \frac{1}{y}\cdot\frac{1}{1+(\frac{x}{y})^2}$$

$$\Rightarrow z = \frac{x^2}{2} - \arctan\frac{x}{y} + g(y)$$

$$z_y = -\frac{-\frac{x}{y^2}}{1+(\frac{x}{y})^2} + g'(y)$$

$$= \frac{x}{x^2+y^2} + g'(y) = y$$

$$\Rightarrow g(y) = \frac{y^2}{2} + C$$

$$\Rightarrow \quad z = \frac{x^2+y^2}{2} - \arctan\frac{x}{y} + C$$

14.

$$\lim_{(x,y)\to(0,0)} \sqrt{|xy|} = 0$$

$$f_x(0,0) = \lim_{\substack{y=0 \\ x\to 0}} \frac{f(x,0)-f(0,0)}{x} = 0$$

同理 $f_y(0,0) = 0$.

若 $f(x,y)$ 在 $(0,0)$ 处可微

则应有 $\begin{cases} x\to 0 \\ y\to 0 \end{cases}$

$$f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y = o(\sqrt{x^2+y^2})$$

$$\| \quad \sqrt{|xy|}.$$

$$\frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}} \qquad \text{令 } y = kx$$

$$\| \quad \frac{\sqrt{|k|}}{\sqrt{1+|k|^2}} \qquad \text{随 } k \text{ 变化且可能不等.}$$

故 $f(x,y)$ 不可微于 $(0,0)$

16.

(1)

$$f_x(0,y) = \lim_{x\to 0} \frac{f(x,y) - f(0,y)}{x}$$

$$= \lim_{x\to 0} \frac{(x^2-y^2)xy}{(x^2+y^2)x}$$

$$= \lim_{x\to 0} \frac{x^2-y^2}{x^2+y^2} \cdot y$$

$$= -y.$$

(2)

$$f_x(0,0) = \lim_{\substack{y=0 \\ x\to 0}} \frac{f(x,0) - f(0,0)}{x}$$

$$= 0$$

(3).

$$f_{xy}(0,0) = \lim_{\substack{x=0 \\ y\to 0}} \frac{f_x(0,y) - f_x(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{-y}{y} = -1$$

(4).

$$f_y(x,0) = \lim_{y\to 0} \frac{f(x,y)-f(x,0)}{y}$$

$$= \lim_{y\to 0} \frac{x^2y^2}{x^2+y^2}\cdot\frac{xy}{y} = x$$

$$f_y(0,0) = \lim_{y\to 0} \frac{f(x,0)-f(0,0)}{y} = 0$$

$$f_{yx}(0,0) = \lim_{\substack{y=0\\ x\to 0}} \frac{f_y(0,y)-f_y(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x}{x} = 1$$

6.5

1. (2)

$$z_x = z_u \cdot u_x + z_v v_x$$

$$= \frac{2u}{v}\cdot ye^x + -\frac{u^2}{v^2}\cdot \ln y$$

$$= \frac{(2x-1)ye^{2x}}{x^2\ln y}.$$

$$z_y = z_u \cdot u_y + z_v \cdot v_y$$

$$= \frac{2u}{v} \cdot e^x + -\frac{u^2}{v^2} \frac{x}{y} \cdot$$

$$= \frac{y e^{2x} (2\ln y - 1)}{x \ln^2 y} .$$

(4)

$$z_x = f_1' y + f_2' \frac{1}{y}$$

$$z_y = f_1' x + f_2' \cdot \frac{x}{-y^2}$$

4.

$$z_x = n x^{n-1} f\left(\frac{y}{x^2}\right) + x^n f'\left(\frac{y}{x^2}\right) \frac{-2y}{x^3}$$

$$z_y = x^n f'\left(\frac{y}{x^2}\right) \frac{1}{x^2}$$

$$x \cdot z_x = n x^n f + x^{n+1} f'\left(\frac{y}{x^2}\right) \frac{-2y}{x^3}$$

$$2y \cdot z_y = 2y x^n f' \frac{1}{x^2}$$

$$x z_x + 2y z_y = n x^n f = n z$$

9

$$f(tx, ty, tz) = t^n f(x, y, z)$$

两端同时对 t 求导.

$$x f_x(tx, ty, tz) + y f_y(tx, ty, tz) + z f_z(tx, ty, tz)$$

$$\| $$

$$n t^{n-1} f(x, y, z)$$

令 $t=1$ 则得证.

10.

令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$f(r\cos\theta, r\sin\theta) = F(r, \theta)$$

$$F_r = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r}$$

$$= f_x \cos\theta + f_y \sin\theta$$

$$= \frac{1}{r}(f_x r\cos\theta + f_y r\sin\theta)$$

$$= \frac{1}{r}(x f_x + y f_y) = 0$$

故证毕