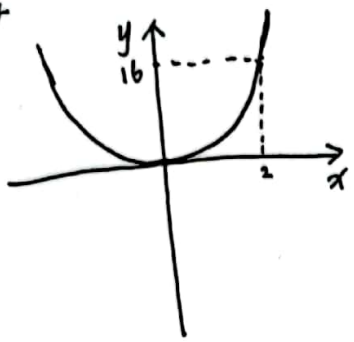
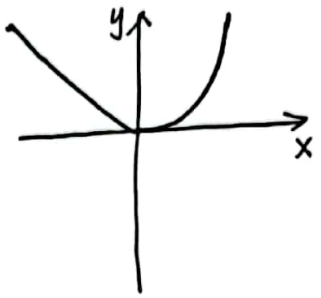
习题 1.2

9. (1) $y = x^4$

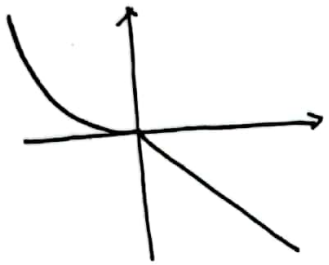
$-\infty < x < +\infty$



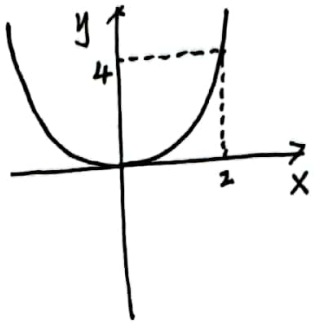
(2) $y = \begin{cases} x^2, & x \geq 0 \\ -x, & x < 0 \end{cases}$



(3) $y = \begin{cases} -x, & x > 0 \\ x^2, & x \leq 0 \end{cases}$



(4) $y = x^2$, $-\infty < x < +\infty$



10.

(1) $y = \frac{x}{2} - \frac{2}{x} \Rightarrow x^2 - 2xy - 4 = 0 \Rightarrow x = \frac{2y \pm \sqrt{4y^2 - 4\times(-4)}}{2} = y \pm \sqrt{y^2+4}$

由 $\sqrt{y^2+4} > |y|$ 及 $x \in (0, +\infty)$，$x = y + \sqrt{y^2+4}$，因此 $y = x + \sqrt{x^2+4}$ 为(1)的反函数，其定义域为 $(-\infty, +\infty)$

(2) $y = \operatorname{sh} x$ $(-\infty < x < +\infty)$ 的值域是 $(-\infty, +\infty)$，故其反函数定义域为 $(-\infty, +\infty)$.

由 $y = \operatorname{sh} x = \frac{e^x - e^{-x}}{2}$ 可解得 $e^x = y \pm \sqrt{y^2+1}$，由 $e^x > 0$ 及 $\sqrt{y^2+1} > |y|$ 知 $e^x = y + \sqrt{y^2+1}$，即 $x = \ln(y + \sqrt{y^2+1})$ 从而 $y = \ln(x + \sqrt{x^2+1})$ $(x \in (-\infty, +\infty))$ 为(2)的反函数.

(3) $y = \operatorname{ch} x$ $(0 \leq x < +\infty)$ 的值域为 $[1, +\infty)$，故其反函数定义域为 $[1, +\infty)$.

由 $y = \operatorname{ch} x = \frac{e^x + e^{-x}}{2}$ 可解得 $e^x = y \pm \sqrt{y^2-1}$ 由 $|y| \geq \sqrt{y^2-1}$ $(y \geq 1)$ 及 $x \in [0, +\infty)$ 知 $e^x = y + \sqrt{y^2-1}$，即 $x = \ln(y + \sqrt{y^2-1})$，从而 $y = \ln(x + \sqrt{x^2-1})$ $(x \in [1, +\infty))$ 为(3)的反函数.

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13. $y=\sqrt{1+x}-\sqrt{x}=\frac{(\sqrt{1+x}-\sqrt{x})(\sqrt{1+x}+\sqrt{x})}{\sqrt{1+x}+\sqrt{x}}=\frac{1}{\sqrt{1+x}+\sqrt{x}}<\frac{1}{\sqrt{2}+1}$ $(x>1)$ ②

15. "$\Rightarrow$" 设 $f: X\to Y$ 是有界函数，则 f 存在上界 N，下界 M，使得对任何 $x\in X$，$M\le f(x)\le N$，取 $C:=\max\{|M|,|N|\}$，则有

$$|f(x)|\le C,\ \forall x\in X$$

"$\Leftarrow$" 设 $|f(x)|\le C,\ \forall x\in X$，则

$$-C\le f(x)\le C,\ \forall x\in X$$

因为 $-C, C$ 分别为 f 的下界和上界，故 f 为有界函数.

16. 由(15) f, g 有界从而有 C_1, C_2 使得

$$|f(x)|\le C_1,\ \forall x\in X$$
$$|g(x)|\le C_2,\ \forall x\in X$$

故 $|(f\cdot g)(x)|=|f(x)\cdot g(x)|\le |f(x)|\cdot|g(x)|\le C_1\cdot C_2,\ \forall x\in X$

再利用(15)知 $f\cdot g$ 为有界函数.

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3. 由习题2知 $\lim_{n\to\infty}|a_n| = |l|$

(1) 取 $\varepsilon=1$，则存在 $N\in\mathbb{N}$，$n>N$ 时，有

$$|\,|a_n|-|l|\,| < \varepsilon = 1 \Rightarrow |a_n|-|l|<1$$

即 $|a_n| < |l|+1$

(2) 取 $M := \max\{|a_1|, |a_2|, \cdots, |a_N|, |l|+1\}$

则有 $|a_n| \le M$，$n=1,2,3,\cdots$

4.

(1) $\dfrac{3n+1}{2n-3} - \dfrac{3}{2} = \dfrac{2(3n+1)-3\cdot(2n-3)}{(2n-3)\cdot 2} = \dfrac{11}{4n-6}$

对任意 $\varepsilon>0$，取 $N=\left[\frac{3}{2}+\frac{11}{4\varepsilon}\right]$，则 $n>N$ 时，

$n \ge \left[\frac{3}{2}+\frac{11}{4\varepsilon}\right]+1 \ge \frac{3}{2}+\frac{11}{4\varepsilon} \Rightarrow 4n-6 \ge \frac{11}{\varepsilon} \Rightarrow \frac{11}{4n-6} \le \varepsilon$

故 $\lim_{n\to\infty}\frac{3n+1}{2n-3} = \frac{3}{2}$

(2) 令 $\left|\dfrac{\sqrt[3]{n^2}\sin(n)}{n+1}\right| = \left|\dfrac{\sin(n)}{n^{\frac{1}{3}}+n^{-\frac{2}{3}}}\right| < \dfrac{1}{n^{\frac{1}{3}}}$

对任意 $\varepsilon>0$，取 $M=\left\lceil\frac{1}{\varepsilon^3}\right\rceil$，则 $n>M$ 时，有

$n > \frac{1}{\varepsilon^3} \Rightarrow \left|\frac{1}{n^{\frac{1}{3}}}\right| < \varepsilon$

故 $\lim_{n\to\infty}\dfrac{\sqrt[3]{n^2}\sin(n)}{n+1} = 0$

(3) 令 $\frac{1}{|q|} = 1+\varepsilon$，则由 $|q|<1$ 知 $\varepsilon>0$，且

$$|n^2 q^n| = n^2|q|^n = n^2/(1+\varepsilon)^n = \frac{n^2}{(1+\varepsilon)^n}$$

考虑到 $(1+\varepsilon)^n = 1+n\cdot\varepsilon+\frac{n(n+1)}{2}\varepsilon^2+\frac{n(n+1)(n-2)}{6}\varepsilon^3+\cdots > \frac{n(n+1)(n-2)}{6}\varepsilon^3 \quad (n>3)$

故 $|n^2q^n| < \dfrac{n^2\,6}{n(n+1)(n-2)\varepsilon^3} < \dfrac{n\,6}{(n-2)^2\varepsilon^3} = \dfrac{6}{(n-4+\frac{4}{n})\varepsilon^3} \le \dfrac{6}{(n-3)\varepsilon^3} \quad (n\ge 4)$

取对任意 $\eta>0$，取 $N := \left\lceil\frac{6}{\varepsilon^3\eta}\right\rceil+3$，则 $n>N$ 时有

$n > \left\lceil\frac{6}{\varepsilon^3\eta}\right\rceil+3 \Rightarrow n > \frac{6}{\varepsilon^3\eta}+3 \Rightarrow \frac{6}{(n-3)\varepsilon^3} < \eta \Rightarrow |n^2q^n| < \frac{6}{(n-3)\varepsilon^3} < \eta$

故 $\lim_{n\to\infty} n^2\cdot q^n = 0$.

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(4) $\left|\frac{n!}{n^n}\right| = \left|\frac{n\times(n-1)\times(n-2)\times\cdots\times 2\times 1}{n^n}\right| < \frac{1}{n}$

对任意 $\varepsilon>0$，取 $N:=\lceil\frac{1}{\varepsilon}\rceil$，则当 $n>N$ 时，

$$n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon$$

故 $\lim\limits_{n\to\infty}\frac{n!}{n^n}=0$

(5) $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(n+1)(n)} = \frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n+1}-\frac{1}{n} = 1-\frac{1}{n}$

对任意 $\varepsilon>0$，取 $N:=\lceil\frac{1}{\varepsilon}\rceil$，则当 $n>N$ 时，$n>\frac{1}{\varepsilon}$，从而

$$\left|\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(n+1)n}-1\right| = \frac{1}{n} < \frac{1}{1/\varepsilon} = \varepsilon$$

故 $\lim\limits_{n\to\infty}\left(\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{(n+1)n}\right)=1$

(6) $\frac{1}{(n+1)^{\frac{3}{2}}}+\cdots+\frac{1}{(2n)^{\frac{3}{2}}} < \frac{n}{(n+1)^{\frac{3}{2}}} < \frac{n}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{1}{2}}}$

取 对任意 $\varepsilon>0$，取 $N:=\lceil\frac{1}{\varepsilon^2}\rceil$，则当 $n>N$ 时，$n>\frac{1}{\varepsilon^2}$，从而

$$\left|\frac{1}{(n+1)^{\frac{3}{2}}}+\cdots+\frac{1}{(2n)^{\frac{3}{2}}}-0\right| < \frac{1}{n^{\frac{1}{2}}} < \varepsilon$$

故 $\lim\limits_{n\to\infty}\left(\frac{1}{(n+1)^{\frac{3}{2}}}+\cdots+\frac{1}{(2n)^{\frac{3}{2}}}\right)=0$

5. 对任意 $\varepsilon>0$，

由 $\lim\limits_{n\to\infty}a_n=0$，存在 M_1，$n>M_1$ 时，有 $|a_n|<\varepsilon/M$.

从而 $|a_nb_n| \le |a_n|\cdot|b_n| < |a_n|\cdot M < \varepsilon$

故 $\lim\limits_{n\to\infty}a_nb_n=0$

7. (1) $\lim\limits_{n\to\infty}\sqrt{n+1}-\sqrt{n} = \lim\limits_{n\to\infty}\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

(2) $\lim\limits_{n\to\infty}\frac{n^3+3n^2-100}{4n^3-n+2} = \lim\limits_{n\to\infty}\frac{1+3\frac{1}{n}-\frac{100}{n^3}}{4-\frac{1}{n^2}+\frac{2}{n^3}} = \frac{1}{4}$

(3) $\lim\limits_{n\to\infty}\frac{(2n+\frac{10}{n})^4}{n^4+n^3} = \lim\limits_{n\to\infty}\frac{(2n+10/n)^4}{1+\frac{1}{n}} = \frac{\lim\limits_{n\to\infty}(2+10/n)^4}{\lim\limits_{n\to\infty}(1+\frac{1}{n})} = 由\ \frac{(\lim\limits_{n\to\infty}(2+\frac{10}{n}))^4}{1} = 2^4$

(4) $\lim\limits_{n\to\infty}\left[(1+\frac{1}{n})^n\right]^{-2} = \left[\lim\limits_{n\to\infty}(1+\frac{1}{n})^n\right]^{-2} = e^{-2}$

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(5) $\lim_{n\to\infty}(1-\frac{1}{n})^n = \lim_{n\to\infty}(\frac{n-1}{n})^n = \lim_{n\to\infty}(\frac{n}{n-1})^{-n} \overset{n-1=n}{=} \lim_{n\to\infty}(1+\frac{1}{n-1})^{-(n-1)-1}$

$= \lim_{n\to\infty}[(1+\frac{1}{n-1})^{n-1}]^{-1}\cdot(1+\frac{1}{n-1})^{-1} = \lim_{n\to\infty}[(1+\frac{1}{n-1})^{n-1}]^{-1}\cdot\lim_{n\to\infty}(1+\frac{1}{n-1})^{-1}$

$\overset{n'=n-1}{=} \lim_{n'\to\infty}[(1+\frac{1}{n'})^{n'}]^{-1}\cdot 1 = [\lim_{n'\to\infty}(1+\frac{1}{n'})^{n'}]^{-1} = e^{-1}$

(6) 首先，由(5)知 $\lim_{n\to\infty}(1-\frac{1}{n})^{M_1 n} = \lim_{n\to\infty}[(1-\frac{1}{n})^n]^{M_1} = [\lim_{n\to\infty}(1-\frac{1}{n})^n]^{M_1} = e^{-M_1}$ ($M_1>0$)

~~故 对任意 $\varepsilon>0$，存在 M_2，当 $n>M_2$ 时 $(1-\frac{1}{n})^{M_2 n} < 2e^{M_1}$~~

对任意 $\varepsilon>0$，取 $M_1 := \lceil \ln(\frac{\varepsilon}{2}) \rceil$，则 $e^{-M_1} < \frac{\varepsilon}{2}$.

由 $\lim_{n\to\infty}(1-\frac{1}{n})^{M_1 n} = e^{-M_1}$ 知 存在 M_2，$n>M_2$ 时 $(1-\frac{1}{n})^{M_1 n} < 2e^{-M_1}$

取 $N = \max\{M_1, M_2\}$，则当 $n>N$ 时

$$(1-\frac{1}{n})^{n^2} < (1-\frac{1}{n})^{M_1 n} < 2e^{-M_1} < \varepsilon$$

从而 $\lim_{n\to\infty}(1-\frac{1}{n})^{n^2} = 0$

(7) $\lim_{n\to\infty}(1-\frac{1}{n^2})^n = \lim_{n\to\infty}(1-\frac{1}{n})^n(1+\frac{1}{n})^n = \lim_{n\to\infty}(1-\frac{1}{n})^n\cdot\lim_{n\to\infty}(1+\frac{1}{n})^n = e^{-1}\cdot e = 1$

注意：在第13)(4)(5)(6)题的计算中，使用了

$$\lim_{n\to\infty}(2+\frac{1}{n})^4 = [\lim_{n\to\infty}(2+\frac{1}{n})]^4$$

$$\lim_{n\to\infty}[(1+\frac{1}{n})^n]^{-1} = [\lim_{n\to\infty}(1+\frac{1}{n})^n]^{-1}$$

$$\lim_{n\to\infty}[(1-\frac{1}{n})^n]^M = [\lim_{n\to\infty}(1-\frac{1}{n})^n]^M$$

这实际是使用了初等函数的连续性，即

若 $\lim_{n\to\infty}x_n = a$，则 $\lim_{n\to\infty}x_n^4 = (\lim_{n\to\infty}x_n)^4$

$$\lim_{n\to\infty}x_n^{-1} = (\lim_{n\to\infty}x_n)^{-1}$$

$$\lim_{n\to\infty}x_n^M = (\lim_{n\to\infty}x_n)^M$$

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9.

根据提示，仅需证明 对任意固定 $k\in\mathbb{N}$

$$e \geq 1+1+\frac{1}{2!}+\cdots+\frac{1}{k!} \quad \cdots\cdots\cdots (*)$$

考虑 $(1+\frac{1}{n})^{n+k} = 1+(n+k)\frac{1}{n}+\frac{(n+k)(n+k-1)}{2!}\cdot\frac{1}{n^2}+\cdots+\frac{(n+k)\cdots(n+1)}{k!}\frac{1}{n^k}+\cdots$

$$\Rightarrow \quad (1+\frac{1}{n})^{n+k} > 1+1+\frac{1}{2!}+\cdots+\frac{1}{k!}$$

上式两边同时令 $n\to\infty$，并利用 §3.2 定理3 (课本)

$$\lim_{n\to\infty}(1+\frac{1}{n})^{n+k} \geq 1+1+\frac{1}{2!}+\cdots+\frac{1}{k!}$$

$$\Rightarrow \quad e=\lim_{n\to\infty}(1+\frac{1}{n})^{n}\cdot(1+\frac{1}{n})^{k} \geq 1+1+\frac{1}{2!}+\cdots+\frac{1}{k!}$$

从而 (*) 式得证.

由 (*) 式 $e \geq \lim_{n\to\infty}(1+\frac{1}{1}+\frac{1}{2!}+\cdots+\frac{1}{k!})$

结合已知 $e \leq \lim_{n\to\infty}(1+\frac{1}{1}+\frac{1}{2!}+\cdots+\frac{1}{n!})$

$$\Rightarrow \quad e=\lim_{n\to\infty}(1+\frac{1}{1}+\frac{1}{2!}+\cdots+\frac{1}{n!})$$

10. 首先 $|x_n| \leq k|x_{n-1}| \leq k^2|x_{n-2}| \leq \cdots \leq k^n|x_1|$

不妨设 $x_1\neq 0$ ($x_1=0$ 时易证)

对任意 $\varepsilon>0$，取 $N:=\lceil \ln(\frac{\varepsilon}{|x_1|})/\ln k \rceil+1$，当 $n>N$ 时

$$n > \ln(\tfrac{\varepsilon}{|x_1|})/\ln k+1 \Rightarrow \quad k^n|x_1|<\varepsilon$$

$$\Rightarrow |x_n| \leq k^n|x_1| < \varepsilon$$

故 $\lim_{n\to\infty} x_n = 0$.

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