

1.6.1.

$$P(x) = a_n x^{2n+1} + a_{n-1} x^{2n} + \cdots + a_0$$

$$= x^{2n+1}\left(a_n + \frac{a_n}{x} + \cdots + \frac{a_0}{x^{2n+1}}\right)$$

$$\lim_{x\to\infty}\left(a_n + \frac{a_n}{x} + \cdots + \frac{a_n}{x^{2n+1}}\right) = a_n$$

不妨设 $a_n > 0$

由此知 $P(x)$ 在 $|x| >> 1$ 时，$P(x)$ 与 x^{2n+1} 同符号

$P(x) \to +\infty$，$x \to +\infty$，$P(x) \to -\infty$，$x \to -\infty$

由于 $P(x)$ 是连续函数，由介值定理可得 $\exists\, \xi \in (-\infty, +\infty)$

$$P(\xi) = 0$$

1.6.2.

令 $g(x) = x - \varepsilon \sin x$

$$x - \varepsilon \le g(x) \le x + \varepsilon$$

由此知 $-\infty < g(x) < \infty$，由连续函数介值定理

$\forall\, y_0 \in \mathbb{R}$，$g(x) = y_0$ 必有解.

下证 $g(x)$ 是单射.

$\forall\, x_2 \ne x_1$ $\quad g(x_2) - g(x_1) = x_2 - x_1 - \varepsilon(\sin x_2 - \sin x_1)$

(不妨设 $x_2 > x_1$)

$$> x_2 - x_1 - \varepsilon|x_2 - x_1| > 0$$

$\Rightarrow$ $g(x)$是单增 故解必唯一.

1.6 3.

$$f(x_1) < \frac{m_1 \cdot f(x_1) + m_2 f(x_2)}{m_1 + m_2} < f(x_2)$$

由连续函数介值定理. $\exists\ \xi \in (a,b)$

$$f(\xi) = \frac{m_1 \cdot f(x_1) + m_2 f(x_2)}{m_1 + m_2}$$

1.6. 4.

令 $g(x) = f(x) - x$

$g(0) = f(0) - 0 \geq 0$

$g(1) = f(1) - 1 \leq 0$

由连续函数介值定理. $\exists\ t \in [0,1]$

$g(t) = 0 = f(t) - t.$

i.e. $f(t) = t.$

2.1.1.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

$\Rightarrow$ 代入 $x=0$, $x=\frac{1}{2}$

$\Rightarrow$ $f'(0)=0$, $f'(\frac{1}{2})=1$

在 $x=x_0$ 处切线斜率为 $f'(x_0)$

故知在 $(0,0)$ 处切线与 x 轴平行,

在 $(\frac{1}{2}, \frac{1}{4})$ 处切线与 x 轴夹角为 $\frac{\pi}{4}$ $(\tan\frac{\pi}{4}=1)$.

2.1.3.

(1) $y' = 2^x \ln 2$. $\Rightarrow$ $y'(0) = \ln 2$

$$y - y(0) = y'(0)(x-0)$$

$$y = x\ln 2 + 1$$

(2) $y' = 2x$ $y'(3) = 6$

$$y - y(3) = y'(3)(x-3) \Rightarrow y = 6x - 7$$

2.1.5.

$y' = 2x+2$

$2x+2=4 \Rightarrow x=1$

切线：$y-y(1)=y'(1)(x-1)$

$y=4x+2$

法线：$y-y(1)=-\frac{1}{y'(1)}(x-1)$

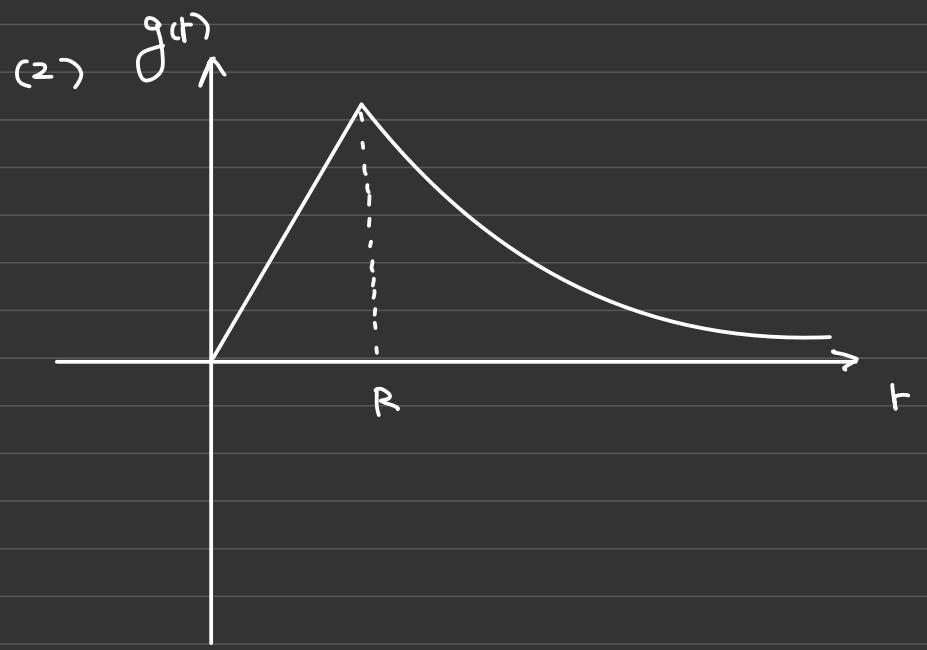
$y=-\frac{x}{4}+\frac{25}{4}$

2.1.6.

(1) 是

$$\lim_{r\to R^-} g(r) = \lim_{r\to R^+} g(r) = \frac{GM}{R^2}$$

(2)



(3) $r \neq R$ 时可导

$r = R$ 时不可导.

在 $r=R$ 处左导数 $g'_-(R) = \frac{GM}{R^2}$

右导数 $g'_+(R) = \frac{-2GM}{R^3}$

2.1.8.

(2) $y' = 5(6x^2-2) + (5x+3)\cdot 12x$

$= 90x^2 + 36x - 10$

(4) $y' = \frac{(9+2x)(5x+6) - 5(9+x^2)}{(5x+6)^2} = \frac{5x^2+12x+54}{(5x+6)^2}$

(6) $y' = \frac{6x^2}{-(x^3-1)^2}$

(8) $y' = 10^x + x10^x \cdot \ln 10$

(10) $y' = e^x(\sin x + \cos x)$

2.1.9.

$$P(x) = (x-x_0)^k g(x) \quad (g(x_0) \neq 0)$$

$$P'(x) = k(x-x_0)^{k-1} g(x) + (x-x_0)^k g'(x)$$

$$= (x-x_0)^{k-1} \underbrace{(k g(x) + (x-x_0) g'(x))}$$

代入 $x = x_0$

$$k g(x_0) + (x_0 - x_0) g'(x_0) = k g(x_0) \neq 0$$

$\Rightarrow$ x_0 是 $P'(x)$ 的 $k-1$ 重根.

2.1.10.

$$f(-x) = f(x)$$

两边对 x 求导

$$-f'(-x) = f'(x)$$

代入 $x = 0$

$$\Rightarrow \quad 2f'(0) = 0 \Rightarrow f'(0) = 0$$

2.1.11.

$$f(x_0+\Delta x) - f(x_0-\Delta x) = f(x_0+\Delta x) - f(x_0) + f(x_0) - f(x-\Delta x).$$

$$\Rightarrow \lim \frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x} = \lim \frac{f(x_0+\Delta x)-f(x_0)}{2\Delta x}$$

$$+ \lim \frac{f(x_0)-f(x_0-\Delta x)}{2\Delta x}$$

$$= \frac{1}{2}f'(x_0) + \frac{1}{2}f'(x_0) = f'(x_0)$$

2.1.13.

$$f_-'(0) = \lim_{x\to 0^-} \frac{\frac{x}{1+e^{\frac{1}{x}}}}{x-0} = \lim_{x\to 0^-} \frac{1}{1+e^{\frac{1}{x}}} = 1$$

$$f_+'(0) = \lim_{x\to 0^+} \frac{\frac{x}{1+e^{\frac{1}{x}}}}{x-0} = \lim_{x\to 0^+} \frac{1}{1+e^{\frac{1}{x}}} = 0$$

2.1.14.

$$f_-'(a) = \lim_{x\to a^-} \frac{|x-a|\varphi(x)}{x-a} = \lim_{x\to a^-} \frac{(a-x)\varphi(x)}{x-a} = -\lim_{x\to a^-} \varphi(x) = -\varphi(a)$$

$$f_+'(a) = \lim_{x\to a^+} \frac{|x-a|\varphi(x)}{x-a} = \lim_{x\to a^-} \frac{(x-a)\varphi(x)}{x-a} = \lim_{x\to a^-} \varphi(x) = \varphi(a)$$

故 $f_-'(a) \neq f_+'(a)$ $(\varphi(a)\neq 0)$

不可导这点...