hw 5

§ 2.2

2. (1) $f'(x)=2x$, $f'(0)=0$, $f'(x^2)=2x^2$, $f'(\sin(x))=2\sin x$

(2) $\frac{d}{dx}f(x^2)=f'(x^2)\cdot 2x=2x^2\cdot 2x=4x^3$

$\frac{d}{dx}f(\sin x)=f'(\sin x)\cdot\cos x=2\sin x\cos x=\sin(2x)$

(3) $[f(g(x))]'=f'(g(x))g'(x)$，两者不相同

3. (2) $y'=(\sec x)'=\left(\frac{1}{\cos x}\right)'=-\frac{1}{\cos^2 x}(-\sin x)=\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}=\tan x\cdot\sec x$

(4) $y'=3\cdot\sin^2(x)\cdot\cos x\cdot\cos 3x+\sin^3 x\cdot(-\sin 3x)\cdot 3$

$=3\sin^2 x(\cos x\cos 3x-\sin x\sin 3x)$

$=3\sin^2 x\cos(4x)$

(6) $y'=\frac{1}{3}\cdot 3\cdot\tan^2 x\cdot\sec^2 x-\sec^2 x+1$

$=\tan^2 x\cdot\sec^2 x-\tan^2 x$

$=\tan^2 x(\sec^2 x-1)$

$=\tan^4 x$

(8) $y'=5\cos^4\sqrt{1+x^2}\cdot(-\sin\sqrt{1+x^2})\cdot\frac{1}{2}\frac{2x}{\sqrt{1+x^2}}$

$=-5\cos^4\sqrt{1+x^2}\cdot\sin\sqrt{1+x^2}\cdot\frac{x}{\sqrt{1+x^2}}$

(10) $y'=\frac{1}{2a}\cdot\frac{x+a}{x-a}\cdot\frac{(x+a)\cdot 1-1\cdot(x-a)}{(x+a)^2}$ $(x\neq\pm a)$

$=\frac{1}{x^2-a^2}$

4. (2) $y'=\frac{1}{a}\cdot\frac{1}{1+\frac{x^2}{a^2}}\cdot\frac{1}{a}=\frac{1}{a^2+x^2}$

(4) $y'=\frac{1}{1+\frac{1}{x^2}}\cdot\left(-\frac{1}{x^2}\right)=\frac{-1}{x^2+1}$

(6) $y'=\frac{1}{2}\cdot\sqrt{x^2+a^2}+\frac{x}{2}\cdot\frac{1}{2}\cdot\frac{2x}{\sqrt{x^2+a^2}}+\frac{a^2}{2}\cdot\frac{a}{x+\sqrt{x^2+a^2}}\cdot\left(1+\frac{1}{2}\cdot\frac{2x}{\sqrt{x^2+a^2}}\right)\cdot\frac{1}{a}$

$=\frac{1}{2}\sqrt{x^2+a^2}+\frac{1}{2}\frac{x^2}{\sqrt{x^2+a^2}}+\frac{1}{2}\frac{a^2}{\sqrt{x^2+a^2}}$

$=\frac{1}{2}\sqrt{x^2+a^2}+\frac{1}{2}\sqrt{x^2+a^2}=\sqrt{x^2+a^2}$

(8) $y'=\frac{2}{\sqrt{a^2-b^2}}\frac{1}{\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)^2+1}\cdot\sqrt{\frac{a-b}{a+b}}\cdot\sec^2\frac{x}{2}\cdot\frac{1}{2}$ $(a>b\geq 0)$

$=\frac{1}{a+b}\cdot\frac{1}{\frac{a-b}{a+b}\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)}=\frac{1}{(a-b)\sin^2\left(\frac{x}{2}\right)+(a+b)\cos^2\left(\frac{x}{2}\right)}=\frac{1}{a+b\cos x}$

(10) $y' = \frac{1}{2} \cdot \frac{1+4x}{\sqrt{1+x+2x^2}}$

(12) $y' = \frac{1}{2} \cdot \frac{-2x}{\sqrt{a^2-x^2}} = \frac{-x}{\sqrt{a^2-x^2}}$

(14) $\ln|y| = \ln|x-1| + \frac{2}{3} \cdot \ln|3x+1| + \frac{1}{3}\ln|2-x|$

$$\frac{d\ln|y|}{dx} = \frac{y'(x)}{y(x)}$$

又 $\frac{d\ln|y|}{dx} = \frac{1}{x-1} + \frac{2}{3}\cdot\frac{1}{3x+1}\cdot 3 + \frac{1}{3}\frac{-1}{2-x}$

故 $y'(x) = y(x)\cdot\left(\frac{1}{x-1} + \frac{2}{3x+1} - \frac{1}{3}\frac{1}{2-x}\right)$

(16) $y'(x) = a^a \cdot x^{a-1} + \ln a \cdot a^{x^a}\cdot a x^{a-1} + \ln a \cdot a^x \cdot \ln a \cdot a^x$

$= a^a \cdot x^{a-1} + a\ln a \cdot a^{x^a}\cdot x^{a-1} + (\ln a)^2 \cdot a^x \cdot a^x$

5. $\theta(t) = \arctan\left(\frac{x(t)}{400}\right)$, $\theta'(t) = \frac{1}{1+\left(\frac{x(t)}{400}\right)^2}\cdot\frac{x'(t)}{400}$ ---(*)

$x(t)$ 满足方程 $\begin{cases} x(0)=0 \\ x'(0)=0 \\ x''(t)=8,\ t\geq 0 \end{cases}$

故 $x'(t) = 8t$

$x(t) = \frac{1}{2}8t^2 = 4t^2$

代入到(*)式

$$\theta'(t) = \frac{1}{1+\left(\frac{t^2}{100}\right)^2}\cdot\frac{8t}{400} = \frac{\frac{1}{50}8t}{1+\frac{t^4}{10000}},\quad t\geq 0$$

$t=10$ 代入 $\theta'(t)$，得 $\theta'(10) = 0.1$ (rad/s)

§2.3

2. 由 $x\to 0$ 时 $\alpha(x) = o(x^2)$，故 $\exists\,\eta(x)$，$\alpha(x) = \eta(x)\cdot x^2$ 且 $\lim_{x\to 0}\eta(x) = 0$

取 $\tilde{\eta}(x) = \eta(x)\cdot x$，则 $\alpha(x) = \tilde{\eta}(x)\cdot x^2 = (\eta(x)\cdot x)\,x = \tilde{\eta}(x)\cdot x$，

且 $\lim_{x\to 0}\tilde{\eta}(x) = \lim_{x\to 0}\eta(x)\cdot\lim_{x\to 0}x = 0$ 故 $\alpha(x) = o(x)$.

3. 由 $\alpha(x) = o(x)$，$\beta(x) = o(x)$ $(x\to 0)$

知 $\alpha(x) = \eta_1(x)\cdot x$，$\beta(x) = \eta_2(x)$，且 $\lim_{x\to 0}\eta_1(x) = \lim_{x\to 0}\eta_2(x) = 0$

令 $\eta(x) = \eta_1(x)+\eta_2(x)$，则 $\alpha(x)+\beta(x) = \eta_1(x)\cdot x + \eta_2(x)\cdot x = (\eta_1(x)+\eta_2(x))\cdot x = \tilde{\eta}(x)\cdot x$

且 $\lim_{x\to 0}\tilde{\eta}(x) = \lim_{x\to 0}(\eta_1(x)+\eta_2(x)) = \lim_{x\to 0}\eta_1(x) + \lim_{x\to 0}\eta_2(x) = 0$，故 $\alpha(x)+\beta(x) = o(x)$.

6. $\Delta x = 0.001$, $x_0 = 3$, $f'(x) = -\frac{2}{(x-1)^2}$, 增量 $\Delta y = f(x_0+\Delta x) - f(x_0) = \frac{2}{2.001} - 1 = \frac{-0.001}{2.001}$

微分为 $dy = f'(x)dx = f'(x_0)\Delta x = -\frac{1}{2}\cdot 0.001 = -\frac{0.001}{2}$

7. 令 $f(x) = x^{\frac{1}{5}}$, 由 $f(x_0+\Delta x) = f(x_0) + f'(x_0)\cdot\Delta x + o(\Delta x)$ $(\Delta x \to 0)$

取 $x_0 = 32$, $\Delta x = 0.16$, 由 $f'(x) = \frac{1}{5}\cdot x^{-\frac{4}{5}}$.

$\sqrt[5]{32.16} = f(32.16) = f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\cdot\Delta x = \sqrt[5]{32} + \frac{1}{5}\cdot 32^{-\frac{4}{5}}\cdot 0.16 = 2.002$

故 $\sqrt[5]{32.16}$ 的近似值为 2.002

9. 注：由于可以把(1)(2)中的y视为关于x的函数，因此可以用求微分的方法，也可以两边对x求导.

(1) 对方程两边关于x求导得

$2y\cdot y' - 2(y + x\cdot y') - 2x + 2 = 0$

$\Rightarrow y'(y-x) + 1 - x - y = 0$

$\Rightarrow y'(x) = \frac{1-x-y}{x-y}$

将 M(3,7) 代入其中得

$y'(3) = \frac{1-3-7}{3-7} = \frac{-9}{-4} = \frac{9}{4}$

(2) 对方程两边关于x求导

$e^{xy}(y + xy') - 5(2x\cdot y + x^2\cdot y') = 0$

$\Rightarrow y'(e^{xy}\cdot x - 5x^2) + e^{xy}\cdot y - 10x\cdot y = 0$

$\Rightarrow y'(x) = \frac{10xy - e^{xy}\cdot y}{e^{xy}\cdot x - 5x^2}$

将 $M(\frac{e^2}{10}, \frac{20}{e^2})$ 代入得

$y'(\frac{e^2}{10}) = \frac{10\cdot 2 - e^2\cdot\frac{20}{e^2}}{e^2\cdot\frac{e^2}{10} - 5(\frac{e^2}{10})^2} = 0$

10. (1) $dx = (2-2t)dt$, $dy = (3-3t^2)dt$

$y' = \frac{dy}{dx} = \frac{3-3t^2}{2-2t} = \frac{3}{2}\frac{1-t^2}{1-t} = \frac{3}{2}(1+t)$, $(t \neq 1)$

(2) $dx = (1\cdot \ln t + t\cdot\frac{1}{t})dt$, $dy = e^t dt$

$y' = \frac{dy}{dx} = \frac{e^t}{\ln t + 1}$ $(t > 0$ 且 $t \neq e^{-1})$

13) 由 $\arccos(x)' = \frac{-1}{\sqrt{1-x^2}}$, $|x|<1$

$\arcsin(x)' = \frac{1}{\sqrt{1-x^2}}$, $|x|<1$

知 $dx = \frac{1}{\sqrt{1-\frac{1}{1+t^2}}} \cdot (-\frac{1}{2}) \cdot (1+t^2)^{-\frac{3}{2}} \cdot 2t\,dt = \frac{\sqrt{1+t^2}}{\sqrt{1+t^2-1}} \cdot t \cdot (1+t^2)^{-\frac{3}{2}} = \frac{t}{|t|} \cdot \frac{1}{(1+t^2)}\,dt$

$dy = \frac{1}{\sqrt{1-\frac{t^2}{1+t^2}}} \cdot (\frac{1}{\sqrt{1+t^2}} + t \cdot (-\frac{1}{2}) \cdot (1+t^2)^{-\frac{3}{2}} \cdot 2t)\,dt$

$= \frac{\sqrt{1+t^2}}{\sqrt{1+t^2-t^2}} \cdot (\frac{1}{\sqrt{1+t^2}} - t^2 \cdot \frac{1}{(1+t^2)^{\frac{3}{2}}})\,dt$

$= \sqrt{1+t^2} \cdot (\frac{1+t^2-t^2}{(1+t^2)^{\frac{3}{2}}})\,dt$

$= \frac{1}{1+t^2}\,dt$

$y' = \frac{dy}{dx} = \frac{t}{|t|} = \begin{cases} 1, & t>0 \\ -1, & t<0. \end{cases}$

§ 2.4

3. $y' = \frac{(x+4)\cdot 1 - 1\cdot(x-3)}{(x+4)^2} = \frac{7}{(x+4)^2}$, $y'' = 7\cdot(-2)\cdot\frac{1}{(x+4)^3} = \frac{-14}{(x+4)^3}$

$2(y')^2 = 2\cdot\frac{7^2}{(x+4)^4}$, $(y-1)\cdot y'' = (\frac{x-3}{x+4} - 1)\cdot\frac{-14}{(x+4)^3} = \frac{-7}{x+4}\cdot\frac{-14}{(x+4)^3} = 2\frac{7^2}{(x+4)^4}$

故 $2(y')^2 = (y-1)\cdot y''$

4. $y(x)$ 是一个 6 次多项式，最高次项为 $-2^2\cdot 3^3\cdot x^6$ 即 $-108x^6$

$y^{(6)}(x)$ 仅与 $y(x)$ 的最高次项有关，因为低阶项在求 6 次导数后均为 0.

$y^{(6)}(x) = (-108x^6)^{(6)} = -108\cdot 6! = -77760$

从而被 $y^{(7)}(x) = 0$

7. $f'(x) = \frac{(-n)\cdot(-1)}{(1-x)^{n+1}} = \frac{n}{(1-x)^{n+1}}$, $f''(x) = \frac{n(n+1)}{(1-x)^{n+2}}$

利用数学归纳法可以证明 $f^{(k)}(x) = \frac{n(n+1)\cdots(n+k-1)}{(1-x)^{n+k}}$ …… (*)

$k=1$ 时结论成立，假设 k 时结论成立 ($k\geq 1$)，则

$f^{(k+1)}(x) = (f^{(k)}(x))'$

(利用假设) $= (\frac{n(n+1)\cdots(n+k-1)}{(1-x)^{n+k}})'$

$= \frac{n(n+1)\cdots(n+k-1)(n+k)}{(1-x)^{n+k+1}}$

故 $k+1$ 也使得结论成立，由数学归纳法，(*) 对任意 $k\geq 1$ 都成立.

因此 $f^{(k)}(0) = n(n+1)\cdots(n+k-1)$

8. 由 Leibniz 公式，令 $f(x)=x^2$, $g(x)=\ln(1+x)$

$$y^{(50)} = \sum_{i=0}^{50} \binom{50}{i} \cdot f^{(i)}(x)\, g^{(50-i)}(x)$$

由于 $f^{(i)}(x) \equiv 0$，当 $i \geq 3$

及数学归纳法可证 $g^{(k)}(x) = (-1)^{k-1}\cdot(k-1)!\,\frac{1}{(1+x)^k}$，$k \geq 1$（规定 $(-1)^0=1$, $0!=1$）

故 $y^{(50)}(x) = \binom{50}{0}\cdot f(x)\cdot g^{(50)}(x) + \binom{50}{1} f'(x) g^{(49)}(x) + \binom{50}{2} f''(x) g^{48}(x)$

$$= x^2\cdot(-1)\cdot 49!\,\frac{1}{(1+x)^{50}} + 50\cdot 2x\cdot(-1)^{48}\cdot 48!\,\frac{1}{(1+x)^{49}} + \frac{50\cdot 49}{2}\cdot 2\cdot(-1)\cdot 47!\,\frac{1}{(1+x)^{48}}$$

$$= -49!\,\frac{x^2}{(1+x)^{50}} + 100\cdot 48!\,\frac{x}{(1+x)^{49}} - 50\cdot 49\cdot 47!\,\frac{1}{(1+x)^{48}}$$

10. $y'(x) = e^{ax}\cdot a\cdot(C_1x+C_2) + e^{ax}(C_1) = e^{ax}(aC_1x+aC_2+C_1)$

$y''(x) = e^{ax}a(aC_1x+aC_2+C_1) + e^{ax}\cdot(aC_1)$

$= e^{ax}(a^2C_1x + a^2C_2 + 2aC_1)$

于是 $y''-2ay'+a^2y = e^{ax}(a^2C_1x + a^2C_2 + 2aC_1) - 2a\cdot e^{ax}(aC_1x+aC_2+C_1)$

$+ a^2 e^{ax}(C_1x+C_2)$

$= 0$

因为 $y=(C_1x+C_2)e^{ax}$ 是题目中方程的解。