习题 2.5

2. $\int (1+\sqrt{x})^2 \, dx$

$= \int (1 + 2\sqrt{x} + x) \, dx$

$= x + \frac{4}{3} x^{\frac{3}{2}} + \frac{1}{2} x^2 + C$

4. $\int \tan^2 t \, dt$

$= \int \frac{\sin^2 t}{\cos^2 t} \, dt$

$= \int \frac{1 - \cos^2 t}{\cos^2 t} \, dt$

$= \int \sec^2 t - 1 \, dt$

$= \tan t - t + C$

6. $\int \frac{x^2 + 3}{1 + x^2} \, dx$

$= \int \frac{x^2 + 1 + 2}{1 + x^2} \, dx$

$= \int 1 + \frac{2}{1 + x^2} \, dx$

$= x + 2 \arctan x + C$

8. $\int (1 + \cos^2 x) \sec^2 x \, dx$

$= \int (\sec^2 x + 1) \, dx$

$= \tan x + x + C$

10. $\int (\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}) \, dx$

$= \ln |x| - \frac{2}{x} - \frac{3}{2x^2} + C$

12. $\int (2 \operatorname{ch} x - \operatorname{sh} x) \, dx$

$= \int (e^x + e^{-x} - \frac{e^x - e^{-x}}{2}) \, dx$

$= \int (\frac{e^x}{2} + \frac{3}{2} e^{-x}) \, dx$

$= \frac{e^x}{2} - \frac{3}{2} e^{-x} + C$

14. $\int \frac{1}{\sin^2 x \cos^2 x} \, dx$

$= \int (\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}) \, dx$

$= -\cot x + \tan x + C$

16. $\int \frac{1}{x^2 (1 + x^2)} \, dx$

$= \int \frac{1}{x^2} - \frac{1}{1 + x^2} \, dx$

$= -\frac{1}{x} - \arctan x + C$

18. $x f'(x) + f(x) = [x f(x)]'$

因此 $[x f(x)]' = x^3 + 1$

$x f(x) = \frac{1}{4} x^4 + x + C$

$f(x) = \frac{1}{4} x^3 + 1 + \frac{C}{x}, \quad x \neq 0$

习题 2.6

3. $\sum_{i=1}^{n} f(\xi_i) \Delta x_i = \sum_{i=1}^{n} (\frac{i-1}{n})^2 \cdot \frac{1}{n}$

$= \sum_{i=1}^{n} \frac{(i-1)^2}{n^3}$

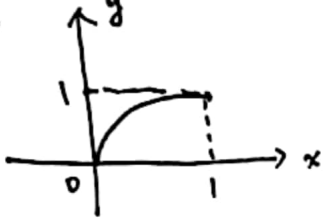
$= \frac{\frac{1}{6}(n-1) \cdot n \cdot (2n-1)}{n^3}$

$\lim_{n \to +\infty} \sum_{i=1}^{n} \frac{(i-1)^2}{n^3} = \lim_{n \to +\infty} \frac{\frac{1}{3} n^3 - 3n^2 + n}{n^3}$

$= \frac{1}{3}$

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4.



考虑 $y=\sqrt{x}$ 的反函数 $x=y^2$ 在 $[0,1]$ 上积分

$$\int_0^1 y^2\,dy=\frac{1}{3}$$

由积分的几何意义知，$\int_0^1 \sqrt{x}\,dx=1-\frac{1}{3}=\frac{2}{3}$

5. (1) $\int_0^{\frac{\pi}{2}}(1+\sin x)dx>\int_0^{\frac{\pi}{2}}1\,dx=\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}}(1+\sin x)dx<\int_0^{\frac{\pi}{2}}(1+1)\,dx=\pi$$

(2) 函数 $y=2+x-x^2$ 在 $[0,1]$ 上有最大值 $\frac{9}{4}$，最小值 2

因此

$$I=\int_0^1\sqrt{x}\,dx<\int_0^1\sqrt{2+x-x^2}\,dx<\int_0^1\sqrt{\frac{9}{4}}\,dx=\frac{3}{2}$$

7.

$\Rightarrow$：若 $y=f(x)$ 在 $[a,b]$ 区间上可积

则对任意分割 $T: x_0<x_1<\cdots<x_n$

对任意 $\varepsilon>0$，存在 $\delta>0$，对任意 $\xi_i\in[x_{i-1},x_i]$，都有

$$\left|\sum_{i=1}^n f(\xi_i)\Delta x_i-I\right|<\varepsilon$$

只要 $\lambda(T)<\delta$. 其中 I 为某一常数

分别取 $\xi_i=x_{i,m}$ 与 $\xi_i=x_{i,M}$

其中 $x_{i,m}$，$x_{i,M}$ 为 $f(x)$ 在 $[x_{i-1},x_i]$ 上最小值和最大值点..

则对 $\forall\varepsilon>0$，$\exists\delta>0$，只要 $\lambda(T)<\delta$，

都有 $\left|\sum_{i=1}^n m_i\Delta x_i-I\right|<\varepsilon$

$$\left|\sum_{i=1}^n M_i\Delta x_i-I\right|<\varepsilon$$

即 $\lim\limits_{\lambda(T)\to0}\sum\limits_{i=1}^n m_i\Delta x_i=\lim\limits_{\lambda(T)\to0}\sum\limits_{i=1}^n M_i\Delta x_i=I$

$\Leftarrow$：对任意分割 T，任取 $\xi_i\in[x_{i-1},x_i]$

则有 $\sum\limits_{i=1}^n m_i\Delta x_i\le\sum\limits_{i=1}^n f(\xi_i)\Delta x_i\le\sum\limits_{i=1}^n M_i\Delta x_i$

设 $\lim\limits_{\lambda(T)\to0}\sum\limits_{i=1}^n m_i\Delta x_i=\lim\limits_{\lambda(T)\to0}\sum\limits_{i=1}^n M_i\Delta x_i=I$

I 为某一常数.

则对 $\forall\varepsilon>0$，$\exists\delta>0$，当 $\lambda(T)<\delta$ 时，有

$$\left|\sum_{i=1}^n m_i\Delta x_i-I\right|<\varepsilon\quad\left|\sum_{i=1}^n M_i\Delta x_i-I\right|<\varepsilon$$

因此

$$-\varepsilon<\sum_{i=1}^n m_i\Delta x_i-I<\sum_{i=1}^n f(\xi_i)\Delta x_i-I<\sum_{i=1}^n M_i\Delta x_i-I<\varepsilon$$

$$\Rightarrow\left|\sum_{i=1}^n f(\xi_i)\Delta x_i-I\right|<\varepsilon$$

因此 $\lim\limits_{\lambda(T)\to0}f(\xi_i)\Delta x_i=I$

即 $f(x)$ 在 $[a,b]$ 上可积.

习题 2.7

1. (1) $F'(x)=\frac{2x}{1+x^2}$

(2) $G'(x)=\sin(1+x^2)^2\cdot 2x$

(3) $H'(x)=-x^2\cos x$

(4) $L'(x)=e^{-x^4}\cdot 2x-e^{-x^2}$

2. 考察极限

$$\lim_{x\to a^+}\frac{F_0(x)-F_0(a)}{x-a}=\lim_{x\to a^+}\frac{\int_a^x f(t)dt-0}{x-a}$$

由积分中值定理，$\exists c\in[a,x]$

$$\int_a^x f(t)\,dt=f(c)(x-a)$$

因此上述极限 $=\lim\limits_{x\to a^+}f(c)$

当 $x\to a^+$ 时，$c\to a$，且 $f(x)$ 在 a 处连续

因此 $\lim\limits_{x\to a^+}f(c)=f(a)$，即 $F_0'(a+0)=f(a)$

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5. $|F(x_1)-F(x_2)| = \left|\int_a^{x_1} f(t)\,dt - \int_a^{x_2} f(t)\,dt\right|$

$= \left|\int_a^{x_1} f(t)\,dt + \int_{x_1}^{x_2} f(t)\,dt - \int_{x_1}^{x_2} f(t)\,dt - \int_a^{x_2} f(t)\,dt\right|$

$= \left|\int_{x_1}^{x_2} f(t)\,dt\right|$. 不妨设 $x_1 < x_2$

$\le \int_{x_1}^{x_2} |f(t)|\,dt$

$\le L(x_2 - x_1)$

若 $x_1 \ge x_2$，有 $\left|\int_{x_1}^{x_2} f(t)\,dt\right| \le \int_{x_2}^{x_1} |f(t)|\,dt$

$\le L(x_1 - x_2)$

因此 $|F(x_1) - F(x_2)| \le L|x_1 - x_2|$

6. $G'(x) = e^x \int_0^x \sin z\,dz$

$G''(x) = e^x \int_0^x \sin z\,dz + e^x \sin x$

$= e^x(1 - \cos x + \sin x)$

习题 2.8

1. (1) $\int_0^1 x^3\,dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}$

(3) $\int_0^{2\pi} \sin x\,dx = -\cos x\Big|_0^{2\pi} = 0$

(5) $\int_0^{\pi} (2\sin x + x^3)\,dx = -2\cos x + \frac{1}{4}x^4\Big|_0^{\pi}$

$= 4 + \frac{\pi^4}{4}$

3. (1) $\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{n}\sin\frac{k}{n} = \int_0^1 \sin x\,dx$

$= 1 - \cos 1$

(3) $\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{n+k}$

$= \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} \cdot \frac{1}{n}$

$= \int_0^1 \frac{1}{1+x}\,dx$

$= \ln 2$

4. (1) $\int_{-1}^{1} |x|\,dx = \int_{-1}^{0} -x\,dx + \int_0^1 x\,dx$

$= 1$

(3) $\int_0^1 x\left|\frac{1}{2} - x\right|\,dx = \int_0^{\frac{1}{2}} x\left(\frac{1}{2} - x\right)\,dx + \int_{\frac{1}{2}}^{1} x\left(x - \frac{1}{2}\right)\,dx$

$= \frac{1}{8}$

(4) $\int_0^2 (x - [x])\,dx$

$= \int_0^1 x\,dx + \int_1^2 (x-1)\,dx$

$= 1$

5. $F(x) = \int_a^x F'(t)\,dt + F(a)$

由积分中值定理，$\exists\, c \in [a, x]$，使

$\int_a^x F'(t)\,dt = F'(c)(x-a)$

取 $x = b$，即得

$F(b) - F(a) = F'(c)(b-a)$

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