hwk.

习题 3.3

2.

使用多项式的除法知

$3x^4+x^2+1 = (x^2+x-6)(3x^2-3x+22) - 40x+133$

$I = \int \frac{3x^4+x^2+1}{x^2+x-6}dx = \int \left((3x^2-3x+22) + \frac{-40x+133}{x^2+x-6}\right)dx$

$= x^3 - \frac{3}{2}x^2 + 22x + \int \frac{-40x+133}{x^2+x-6}dx$

设 $\frac{-40x+133}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2}$，代入 $x=0, 1$.

解得 $A = -\frac{253}{5}$, $B = \frac{53}{5}$

$I = x^3 - \frac{3}{2}x^2 + 22x - \frac{253}{5}\int \frac{dx}{x+3} + \frac{53}{5}\int \frac{dx}{x-2}$

$= x^3 - \frac{3}{2}x^2 + 22x - \frac{253}{5}\ln|x+3| + \frac{53}{5}\ln|x-2| + C$

4. 令 $t = x-1$

$I = \int \frac{dx}{(x+1)^2(x-2)} = \int \frac{dt}{t^2(t-1)}$

$= -\int \frac{\frac{1}{t}\,d(\frac{1}{t})}{1-\frac{1}{t}} \xlongequal{z=\frac{1}{t}} -\int \frac{z\,dz}{1-z} = \int (1+\frac{1}{z-1})dz$

$= z + \ln|z-1| + C \xlongequal{z=\frac{1}{x-1}} \frac{1}{x-1} + \ln\left|\frac{x-2}{x-1}\right| + C$

6.

$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

$\Rightarrow 1 = x^2(A+B) + x(-A+B+C) + C+A$

$\begin{cases} A+B=0 \\ -A+B+C=0 \\ A+C=1 \end{cases} \Rightarrow \begin{cases} A=1/3 \\ B=-1/3 \\ C=2/3 \end{cases}$

$I = \int \frac{1}{x^3+1}dx = \int \frac{1}{3}\frac{dx}{x+1} + \int \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}dx$

$-\frac{1}{3}\int \frac{x-2}{x^2-x+1}dx = -\frac{1}{3}\int \frac{x-\frac{1}{2}}{x^2-x+1}dx - \frac{1}{3}\int \frac{-\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}dx$

$= -\frac{1}{3}\cdot\frac{1}{2}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}$

$= -\frac{1}{6}\cdot\ln|x^2-x+1| + \frac{1}{2}\int \frac{\frac{1}{\frac{\sqrt{3}}{2}}dx}{(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}})^2+1}$

$= -\frac{1}{6}\ln|x^2-x+1| + \frac{2}{3}\int \frac{d(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}})\frac{\sqrt{3}}{2}}{(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}})^2+1}$

$= -\frac{1}{6}\ln\left|x^2-x+1\right| + \frac{1}{\sqrt{3}}\cdot\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$

$I = \frac{1}{3}\cdot\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C.$

8.

$x^3+x^2+2 = x(x^2+2) + x^2-2x+2$

$= x(x^2+2) + x^2+2-2x$

$I = \int \frac{x^3+x^2+2}{(x^2+2)^2}dx = \int \frac{x}{x^2+2}dx + \int \frac{dx}{x^2+2} - \int \frac{2x}{(x^2+2)^2}dx$

$= \frac{1}{2}\cdot\ln(x^2+2) + \int \frac{\frac{1}{2}\,d(\frac{x}{\sqrt{2}})\sqrt{2}}{(\frac{x}{\sqrt{2}})^2+1} - \int \frac{dx^2}{(x^2+2)^2}$

$= \frac{1}{2}\ln(x^2+2) + \frac{\sqrt{2}}{2}\arctan(\frac{x}{\sqrt{2}}) + \frac{1}{x^2+2} + C$

10.

$I = \int \frac{\cos x\,dx}{\sin^2 x + \sin x - 6} \xlongequal{z=\sin x} \int \frac{dz}{(z+3)(z-2)}$

$= \frac{1}{5}\int \frac{1}{z-2}dz - \frac{1}{5}\int \frac{1}{z+3}dz$

$= \frac{1}{5}\ln|z-2| - \frac{1}{5}\ln|z+3| + C$

$= \frac{1}{5}\cdot\ln\left|\frac{\sin x-2}{\sin x+3}\right| + C$

12.

$\frac{1}{(x+2)(x^2-x+2)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+2}$

$\Rightarrow 1 = (A+B)x^2 + (2B-2A+C)x + 2A+2C$

$\begin{cases} A+B=0 \\ 2B-2A+C=0 \\ 2A+2C=1 \end{cases} \Rightarrow \begin{cases} B=-\frac{1}{10} \\ A=\frac{1}{10} \\ C=\frac{2}{5} \end{cases}$

$$I=\int\frac{dx}{(x+2)(x^2-2x+2)}=\int\frac{\frac{1}{10}}{x+2}dx+\int\frac{-\frac{1}{10}x+\frac{2}{5}}{x^2-2x+2}dx$$

$$\int\frac{-\frac{1}{10}x+\frac{2}{5}}{x^2-2x+2}dx=-\frac{1}{10}\int\frac{x-4}{x^2-2x+2}dx$$

$$=-\frac{1}{10}\int\frac{\frac{1}{2}(2x-2)-3}{x^2-2x+2}dx$$

$$=-\frac{1}{10}\frac{1}{2}\cdot\int\frac{2x-2}{x^2-2x+2}dx+\frac{3}{10}\int\frac{dx}{x^2-2x+2}$$

$$=-\frac{1}{20}\cdot\ln(x^2-2x+2)+\frac{3}{10}\int\frac{dx}{(x-1)^2+1}$$

$$=-\frac{1}{20}\ln(x^2-2x+2)+\frac{3}{10}\arctan(x-1)+C$$

$$I=\frac{1}{10}\cdot\ln|x+2|-\frac{1}{20}\ln(x^2-2x+2)+\frac{3}{10}\arctan(x-1)+C.$$

14. 设 $t=\tan(\frac{x}{2})$, $\frac{x}{2}=\arctan(t)$

$$dx=\frac{2}{1+t^2},\quad \sin(x)=\frac{2t}{1+t^2}$$

$$\cos(x)=\frac{1-t^2}{1+t^2},$$

$$I=\int\frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt$$

$$=\int\frac{2}{1+t^2+2t+1-t^2}dt$$

$$=\int\frac{dt}{t+1}=\ln|t+1|=\ln|\tan(\frac{x}{2})+1|+C$$

16.

$$I=\int\sec^4x\,dx=\int(\tan^2x+1)\,d\tan x$$

$$=\frac{1}{3}\tan^3x+\tan x+C$$

18. ~~$\int\frac{d\tan x}{\tan x+1}$~~

$$I=\int\frac{\cos^3x\,dx}{\sin x+\cos x}\cdot\overset{u=\tan x}{=}\int\frac{(\tan^2x+1)^{-1}}{\tan x+1}dx$$

$$=\int\frac{1}{(u^2+1)(u+1)}\frac{du}{1+u^2}$$

$$=\int$$

$$\frac{1}{(1+u^2)^2(1+u)}=\frac{1}{1+u^2}\left(\frac{1}{(1+u^2)(1+u)}\right)$$

$$=\frac{1}{1+u^2}\cdot\left(\frac{1}{1+u}+\frac{1-u}{1+u^2}\right)\frac{1}{2}$$

$$=\frac{1}{2}\frac{1}{(1+u^2)(1+u)}+\frac{1}{2}\frac{1-u}{(1+u^2)^2}$$

$$=\frac{1}{4}\frac{1}{1+u}+\frac{1}{4}\frac{1-u}{1+u^2}-\frac{1}{2}\frac{u}{(1+u^2)^2}+\frac{1}{2}\frac{1}{(1+u^2)^2}$$

$$=\frac{1}{4}\frac{1}{1+u}+\frac{1}{4}\frac{1}{1+u^2}-\frac{1}{4}\frac{u}{1+u^2}-\frac{1}{2}\frac{u}{(1+u^2)^2}+\frac{1}{2}\frac{1}{(1+u^2)^2}$$

$$I=\frac{1}{4}\cdot\ln|1+u|+\frac{1}{4}\arctan u-\frac{1}{8}\ln(1+u^2)+\frac{1}{4}\frac{1}{1+u^2}+$$

$$\frac{1}{4}\arctan u+\frac{1}{8}\cdot\frac{2u}{1+u^2}+C$$

$$=\frac{1}{4}\cdot\ln|1+\tan x|+\frac{1}{4}\frac{\tan^2x}{1+\tan^2x}+$$

$$+\frac{1}{2}x-\frac{1}{8}\ln(1+\tan^2x)+\frac{1}{4}\frac{1}{1+\tan x}+\frac{1}{4}\frac{\tan x}{1+\tan^2x}+C$$

$$=\frac{1}{4}\ln|1+\tan x|+\frac{x}{2}+\frac{1}{4}\ln|\cos x|+\frac{1}{4}\cdot\cos^2x+\frac{1}{4}\tan x\cos^2x$$

$$+C$$

20.

$$I=\int\sin^6x\,dx=\int\left(\frac{1-\cos(2x)}{2}\right)^3dx$$

$$=\frac{1}{8}\int(1-3\cdot\cos(2x)+3\cdot\cos^2(2x)-\cos^3(2x))\,dx$$

$$=\frac{1}{8}x-\frac{3}{8}\int\cos(2x)dx+\frac{3}{8}\int\cos^2(2x)\,dx$$

$$-\frac{1}{8}\int\cos^3(2x)dx.$$

$$\int\cos(2x)dx=\frac{\sin(2x)}{2}+C$$

$$\int\cos^2(2x)\,dx=\int\frac{1+\cos(4x)}{2}dx$$

$$=\frac{1}{2}x+\frac{1}{2}\cdot\frac{\sin(4x)}{4}+C$$

$$\int\cos^3(2x)dx=\int(1-\sin^2(2x))\,d\sin(2x)\frac{1}{2}$$

$$=\frac{1}{2}\cdot\sin(2x)-\frac{1}{2}\cdot\frac{1}{3}\cdot\sin^3(2x)+C$$

$$I=\frac{1}{8}x-\frac{3}{8}\frac{\sin(2x)}{2}+\frac{3}{8}\left(\frac{1}{2}x+\frac{\sin4x}{8}\right)$$

$$-\frac{1}{8}\cdot\left(\frac{1}{2}\sin(2x)-\frac{\sin^3(2x)}{6}\right)+C$$

$$I=\frac{5}{16}x-\frac{1}{4}\sin(2x)+\frac{1}{48}\sin^3(2x)+\frac{3}{64}\sin(4x)+C$$

22.

$I=\int \frac{dx}{\sin x+2\cos x}$ 令 $t=\tan(\frac{x}{2})$

$=\int \frac{1}{\frac{2t}{1+t^2}+2\cdot\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$

$=\int \frac{2}{2t+2-2t^2}\,dt$

$=\int \frac{dt}{-t^2+t+1}=-\int\frac{dt}{t^2-t-1}$

$=-\int\frac{dt}{(t-\frac{1}{2})^2-(\frac{\sqrt5}{2})^2}$

$=-\int\frac{\frac{1}{2\cdot\frac{\sqrt5}{2}}d(\frac{t-\frac12}{\frac{\sqrt5}{2}})}{(\frac{t-\frac12}{\frac{\sqrt5}{2}})^2-1}$

$=\frac{\sqrt5}{5}\times\left(-\int\frac{d(\frac{t-\frac12}{\frac{\sqrt5}{2}})}{(\frac{t-\frac12}{\frac{\sqrt5}{2}})^2-1}\right)+\frac{\sqrt5}{5}\times\int\frac{d(\frac{t-\frac12}{\frac{\sqrt5}{2}})}{(\frac{t-\frac12}{\frac{\sqrt5}{2}})^2+1}$

$=\frac{\sqrt5}{5}\cdot(-1)\cdot\ln\left|\frac{t-\frac12}{\frac{\sqrt5}{2}}-1\right|+\frac{\sqrt5}{5}\cdot\ln\left|\frac{t-\frac12}{\frac{\sqrt5}{2}}+1\right|+C$

$=\frac{\sqrt5}{5}\cdot\ln\left|\frac{t-\frac12+\frac{\sqrt5}{2}}{t-\frac12-\frac{\sqrt5}{2}}\right|+C$

24.

$I=\int\frac{dx}{\sin^4x}=\int\frac{1}{\sin^2x}(\frac{1}{\sin^2x}-1+1)\,dx$

$=-\int(\cot^2(x)+1)\,d\cot(x)=-\frac13\cot^3x-\cot x+C$

26. 令 $u=\sqrt[6]{x-1}$

$I=\int\frac{1-\sqrt{x-1}}{1+\sqrt[3]{x-1}}\cdot dx=\int\frac{1-u^3}{1+u^2}\cdot 6u^5\,du$

$=6\cdot\int\frac{u^5-u^8}{1+u^2}\,du$

由多项式除法知

$u^5-u^8=(-u^6+u^4+u^3-u^2-u-1)(1+u^2)+u+1$

$I=6\int\frac{u^5-u^8}{1+u^2}\,du$

$=6\int(-u^6+u^4+u^3-u^2-u-1)\,du+6\int\frac{u}{1+u^2}\,du+6\int\frac{du}{1+u^2}$

$=-\frac67u^7+\frac65u^5+\frac32u^4-2u^3-3u^2-6u$

$+3\ln(1+u^2)+6\arctan(u)+C$

28.

$I=\int(x+1)^{-\frac23}(x-1)^{-\frac43}\,dx$

令 $t=\sqrt[3]{\frac{x+1}{x-1}}$

$I=\int(x+1)^{-1}(x-1)^{-1}\cdot\sqrt[3]{\frac{x+1}{x-1}}\,dx$

$=\int\frac{t^3+1}{2t^3}\cdot\frac{t^3+1}{2}\cdot t\cdot(-6)\cdot\frac{t^2}{(t^3-1)^2}\,dt$

$=-\frac32\int dt=-\frac32t+C$

$=-\frac32\sqrt[3]{\frac{x+1}{x-1}}+C$

30. 令 $u=(1+x^{\frac13})^{\frac12}$

$I=\int\frac{(u^2-1)^3}{u}\cdot6(u^2-1)^2u\,du$

$=6\int(u^2-1)^5\,du$

$=6\int(u^{10}-5u^8+10u^6-10u^4+5u^2-1)\,du$

$=6\cdot\frac{1}{11}u^{11}-6\cdot5\cdot\frac19u^9+6\cdot10\cdot\frac17u^7-6\cdot10\cdot\frac15u^5$

$+6\cdot5\cdot\frac13u^3-6u+C$

32.

$I=\int\frac{2x+1+2}{\sqrt{x^2+x}}\,dx$

$=2\cdot\sqrt{x^2+x}+\int\frac{2\,dx}{\sqrt{x^2+x+1}}$

$=2\sqrt{x^2+x}+2\int\frac{dx}{\sqrt{(x+\frac12)^2+(\frac{\sqrt3}{2})^2}}$

$= 2\sqrt{x^2+x} + 2\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + C$

34.

$I = \int \sqrt{5-2x+x^2}\,dx = \int \sqrt{(x-1)^2+2^2}\,dx$

$= \frac{x-1}{2}\sqrt{x^2-2x+5} + 2\cdot\ln\left(x-1+\sqrt{x^2-2x+5}\right) + C$

习题3.4

4. $I = \int_0^{\pi} x\sin x\,dx = -\int_0^{\pi} x\,d\cos x$

$= -x\cdot\cos x\Big|_0^{\pi} + \int_0^{\pi} 1\cdot\cos x\,dx$

$= -(\pi\cdot(-1)-0) + \sin x\Big|_0^{\pi}$

$= \pi$

8. $I = \int_0^3 x^3\sqrt{1-x^2}\,dx = \frac{1}{2}\int_0^3 \sqrt{1-x^2}\,dx^2$

$\overset{z=x^2}{=} \frac{1}{2}\cdot\int_0^9 \sqrt{1-z}\,dz = -\frac{1}{2}\cdot\frac{2}{3}(1-z)^{\frac{3}{2}}\cdot\frac{3}{4}\Big|_0^9$

$= -\frac{3}{8}\left((-8)^{\frac{3}{2}}-1\right) = -\frac{3}{8}\cdot(16-1) = -\frac{45}{8}$

12.

$I_{11} = \int_0^{\frac{\pi}{2}} \sin^{11}x\,dx$

$I_{11} = \frac{10}{11}I_9 = \frac{10}{11}\cdot\frac{8}{9}I_7 = \frac{10}{11}\cdot\frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1$

16.

$I = \int_0^1 \arcsin x\,dx = x\arcsin x\Big|_0^1 - \int_0^1 x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$

$= (1\cdot\frac{1}{2}\pi - 0) - \frac{1}{2}\int_0^1 \frac{dx^2}{\sqrt{1-x^2}}$

$= \frac{1}{2}\pi + \frac{1}{2}\cdot\sqrt{1-x^2}\cdot 2\Big|_0^1$

$= \frac{1}{2}\pi + (0-1) = \frac{1}{2}\pi - 1$

22. $x' = x - \frac{1}{2}\pi$

$I = \int_0^{\pi} x f(\sin x)\,dx$

$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (x'+\frac{1}{2}\pi) f(\sin(x'-\frac{1}{2}\pi))\,dx'$

$= \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (x'+\frac{1}{2}\pi) f(\cos(x'))\,dx'$

$= \frac{1}{2}\pi\cdot\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(\cos(x'))\,dx'$

$= \pi\cdot\int_0^{\frac{1}{2}\pi} f(\cos x')\,dx'$

$t = \frac{1}{2}\pi - x'$

$= \pi\cdot\int_0^{\frac{1}{2}\pi} f(\sin x)\,dx$

23.

$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\frac{1}{2}\pi}\frac{\sin(x)}{1+\cos^2 x}\,dx$

$= -\int_0^{\frac{1}{2}\pi}\frac{d\cos(x)}{1+\cos^2 x}\cdot\pi$

$= -\arctan(\cos x)\Big|_0^{\frac{1}{2}\pi}$

$= -1\cdot(0-\frac{1}{4}\pi)$

$= \frac{1}{4}\pi$

24.

(1) $F(x+T) = \frac{x+T}{T}\int_0^T f(t)\,dt - \int_0^{x+T} f(t)\,dt$

$= \frac{x}{T}\int_0^T f(t)\,dt - \int_0^x f(t)\,dt$

$= F(x)$, $\forall x\in\mathbb{R}$

故 $F(x)$ 以 T 为周期

(2) 不妨设

$\frac{1}{x}\int_0^x f(t)\,dt - \frac{1}{T}\int_0^T f(t)\,dt = -\frac{F(x)}{x}$

由于 $F(x)$ 在 $\mathbb{R}$ 上连续，且为周期函数，

故 $\exists B>0$，$|F(x)|<B$，$\forall x\in\mathbb{R}$

$$\Rightarrow \lim_{X\to+\infty} -\frac{F(x)}{X}=0$$

$$\Rightarrow \lim_{X\to+\infty} \frac{1}{X}\int_0^X f(t)\,dt-\frac{1}{T}\int_0^T f(t)\,dt=0$$

$$\Rightarrow \lim_{X\to+\infty} \frac{1}{X}\int_0^X f(t)\,dt=\frac{1}{T}\int_0^T f(t)\,dt$$

25.

证：设 $f(x_0)>0$（$f(x_0)<0$ 时类似讨论）

若 对 $\forall x\in(x_0,x_0+T)$，有 $f(x)>0$，

则 $\int_0^T f(x)\,dx=\int_{x_0}^{x_0+T} f(x)\,dx>0$ 矛盾.

从而 $\exists \xi\in(x_0,x_0+T)$，$f(\xi)<0$.

由 $f(x)$ 连续性，及 $f(x_0+T)=f(x_0)>0$，$f(\xi)<0$

及介值定理，$\exists \eta_1,\eta_2$，$\eta_1\in(x_0,\xi)$，$\eta_2\in(\xi,x_0+T)$

使得 $f(\eta_1)=f(\eta_2)=0$.

□

26.

$$I=\int_0^{2m\pi}\frac{dx}{\sin^4x+\cos^4x}=m\cdot\int_0^{2\pi}\frac{dx}{\sin^4x+\cos^4x}$$

$$=m\int_0^{2\pi}\frac{dx}{(\sin^2x+\cos^2x)^2-2\sin^2x\cos^2x}=m\cdot\int_0^{2\pi}\frac{dx}{1-\frac{1}{2}\cdot\sin^2(2x)}$$

$$=4m\int_0^{\frac{\pi}{2}}\frac{dx}{1-\frac{1}{2}\sin^2(2x)}$$

$$=4m\int_0^{\frac{\pi}{2}}\frac{2\,dx}{\cos^2(2x)+1}$$

$$=8m\int_0^{\frac{\pi}{2}}\frac{d\tan(2x)\cdot\frac{1}{2}}{1+\sec^2 2x}$$

$$=4m\int_0^{\frac{\pi}{2}}\frac{d\tan(2x)}{\tan^2(2x)+2}\quad\xrightarrow{z=\tan(2x)}\quad 4m\int$$

$$=4m\left(\int_0^{\frac{\pi}{4}}\frac{d\tan(2x)}{\tan^2(2x)+2}+\int_{\frac{\pi}{4}}^{\frac{1}{2}\pi}\frac{d\tan(2x)}{\tan^2(2x)+2}\right)$$

$$=4m\left(\int_0^{\frac{1}{4}\pi}\frac{d\tan(2x)}{\tan^2(2x)+2}+\int_{-\frac{1}{4}\pi}^{0}\frac{d\tan(2x')}{\tan^2(2x')+2}\right)\quad x'=x-\frac{1}{2}\pi$$

$z=\tan(2x)$

$$I=4m\left(\int_0^{+\infty}\frac{dz}{z^2+2}+\int_{-\infty}^{0}\frac{dz}{z^2+2}\right)$$

$$=4m\left(\int_{-\infty}^{+\infty}\frac{dz}{z^2+2}\right)$$

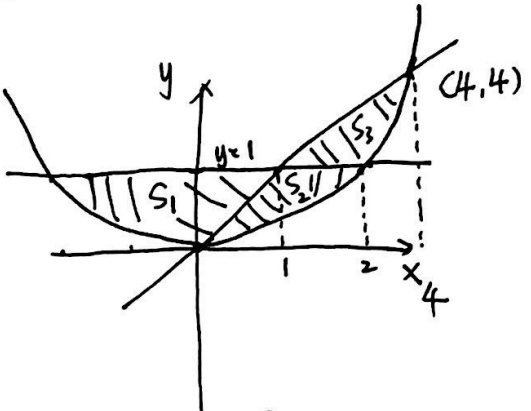
$$=4m\cdot\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\frac{z}{\sqrt{2}}\cdot\sqrt{2}}{\left(\frac{z}{\sqrt{2}}\right)^2+1}$$

$$=2\sqrt{2}m\,\arctan\left(\frac{z}{\sqrt{2}}\right)\Big|_{-\infty}^{+\infty}$$

$$=2\sqrt{2}m\cdot\pi$$

习题3.5.

2.



阴影共三部分，S_1，S_2，S_3.

$$S_1=\int_0^1 y\,dy+\int_0^1\sqrt{4y}\,dy$$

$$=\frac{1}{2}y^2\Big|_0^1+\sqrt{4}\cdot\frac{2}{3}\cdot y^{\frac{3}{2}}\Big|_0^1$$

$$=\frac{1}{2}+\frac{4}{3}=\frac{11}{6}$$

$$S_2=\int_0^1\sqrt{4y}\,dy-\int_0^1 y\,dy$$

$$=\frac{4}{3}-\frac{1}{2}=\frac{5}{6}$$

$$S_3=(S_2+S_3)-S_2$$

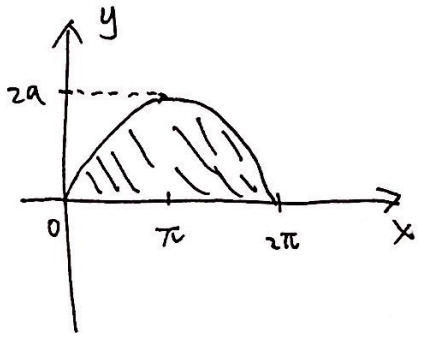
$$=\int_0^4 x\,dx-\int_0^4\frac{x^2}{4}dx-S_2$$

$$=\frac{1}{2}x^2\Big|_0^4-\frac{1}{4}\cdot\frac{1}{3}x^3\Big|_0^4-S_2$$

$$=\frac{1}{2}\cdot 16-\frac{64}{12}-\frac{5}{6}$$

$$=\frac{11}{6}$$

4.



注意到 $x(t)=a(t-\sin t)$，$t>0$

是单增的且 $x(0)=0$，及 $y(t)=a(1-\cos t)$

关于 $t=\pi$ 的对称性得上图.

$$S=\int_0^{2\pi} y(t)\,dx(t)$$

$$=\int_0^{2\pi} a(1-\cos t)\cdot a(1-\cos t)\,dt$$

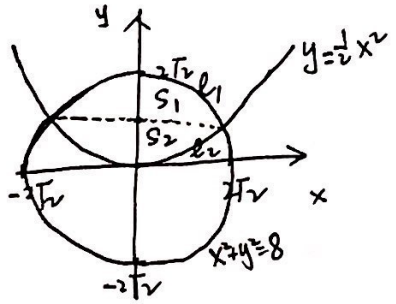
$$=a^2\cdot\int_0^{2\pi}(1-2\cos(t)+\cos^2 t)\,dt$$

$$=a^2\cdot 2\pi-2a^2\cdot\sin(t)\Big|_0^{2\pi}+a^2\int_0^{2\pi}\frac{1+\cos(2t)}{2}\,dt$$

$$=2\pi a^2-0+a^2\cdot\frac{1}{2}\cdot 2\pi+0$$

$$=3\pi a^2$$

6.



联立方程得.曲线相交于 $(2,2)$，$(-2,2)$

设 S_1 为 $x=0$，$y=2$，$x^2+y^2=8$，围成的在第一象限的图形的面积.

设 S_2 为 $y=2$，$x=0$，$y=\frac{1}{2}x^2$ 围成的在第一象限的图形的面积.

设 S_1 对应的曲线为 l_1，S_2 对应曲线为 l_2

则 l_1：$x=\sqrt{8-y^2}$，$2\le y\le 2\sqrt{2}$

l_2：$x=\sqrt{2}\sqrt{y}$，$0\le y\le 2$

$$S_1=\int_2^{2\sqrt{2}}\sqrt{8-y^2}\,dy$$

$$y=2\sqrt{2}\sin(t)\ =\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}2\sqrt{2}\cdot\cos t\cdot 2\sqrt{2}\cos t\,dt$$

$$=8\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\cos^2(t)\,dt$$

$$=8\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\frac{1+\cos(2t)}{2}\,dt$$

$$=8\cdot\frac{1}{2}\cdot\frac{1}{4}\pi+8\cdot\frac{1}{2}\cdot\frac{\sin(2t)}{2}\Big|_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}$$

$$=\pi+2\cdot(0-1)$$

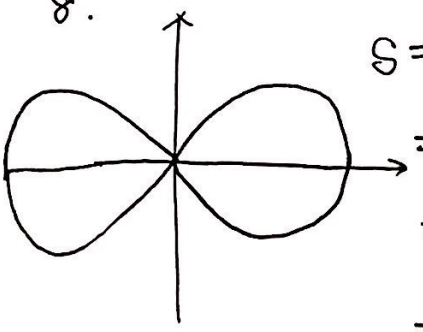
$$=\pi-2$$

$$S_2=\int_0^2\sqrt{2}\sqrt{y}\,dy=\sqrt{2}\cdot\frac{2}{3}y^{\frac{3}{2}}\Big|_0^2=\frac{8}{3}$$

因此上部分面积为 $2(S_1+S_2)=2\pi+\frac{4}{3}$

下部分面积为 $\pi(2\sqrt{2})^2-(2\pi+\frac{4}{3})=6\pi-\frac{4}{3}$

8.



$$S=4\cdot\int_0^{\frac{1}{4}\pi}\frac{1}{2}r^2(\varphi)\,d\varphi$$

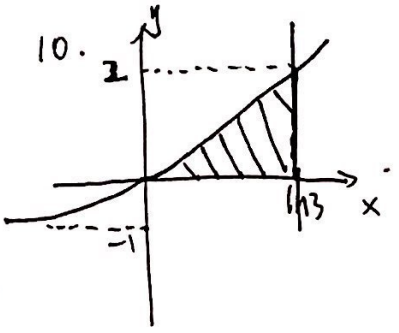
$$=2\cdot\int_0^{\frac{1}{4}\pi}a^2\cdot\cos(2\varphi)\,d\varphi$$

$$=2a^2\cdot\frac{\sin(2\varphi)}{2}\Big|_0^{\frac{1}{4}\pi}$$

$$=a^2\cdot(1-0)$$

$$=a^2$$

10.



$$V=\int_0^{\ln 3}\pi y^2\,dx$$

$$=\pi\int_0^{\ln 3}(e^x-1)^2\,dx$$

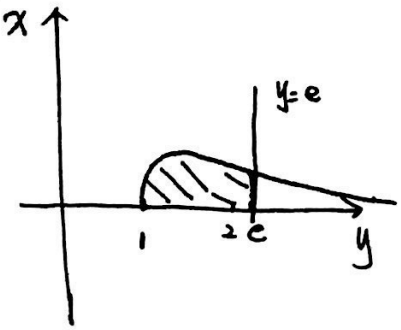
$$=\pi\int_0^{\ln 3}(e^{2x}-2e^x+1)\,dx$$

$$=\pi\,\frac{e^{2x}}{2}\Big|_0^{\ln 3}-2\pi e^x\Big|_0^{\ln 3}+\pi\cdot\ln 3$$

$$=\frac{9}{2}\pi-\frac{1}{2}\pi-2\pi(3-1)+\pi\cdot\ln 3$$

$$=\pi\cdot\ln 3$$

12. 翻转x,y轴不会改变所求体积.



$x'(y)=\sqrt{8}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\ln y}}\cdot\frac{1}{y}\cdot\frac{1}{y}+\sqrt{8}\cdot\sqrt{\ln y}\cdot(-\frac{1}{y^2})$

$=\frac{\sqrt{8}}{y^2}\cdot\frac{1}{\sqrt{\ln y}}(\frac{1}{2}-\ln y)$

令 $x'(y)>0$, $y\in(1,\sqrt{e})$

$x'(y)<0$, $y\in(\sqrt{e},+\infty)$

故 $x(y)$ 在 $(1,\sqrt{e})$ 单增, 在 $(\sqrt{e},+\infty)$ 单减

$x(1)=\lim_{y\to 1^+}\sqrt{8}\frac{\sqrt{\ln y}}{y}=0$

又 $\lim_{y\to+\infty}\sqrt{8}\frac{\sqrt{\ln y}}{y}=0$,

故可绘制上图.

$V=\int_1^e \pi x^2\,dy$

$=\int_1^e \pi\cdot 8\cdot\frac{\ln y}{y^2}\,dy$

$=-\int_1^e \pi\cdot 8\cdot\ln y\cdot d(\frac{1}{y})$

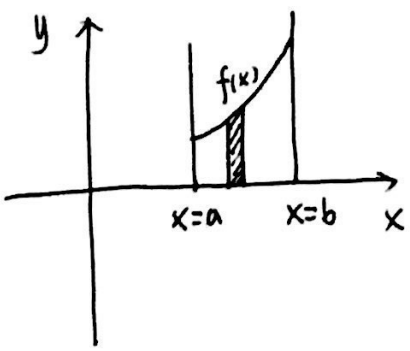
$=-\pi 8\cdot\ln y\cdot\frac{1}{y}\Big|_1^e+\int_1^e \pi\cdot 8\cdot\frac{1}{y}\cdot\frac{1}{y}\,dy$

$=-8\pi\cdot(\frac{1}{e}-0)+8\pi\cdot(-\frac{1}{y})\Big|_1^e$

$=-8\pi\cdot\frac{1}{e}+8\pi(-\frac{1}{e}+1)$

$=8\pi[1-\frac{2}{e}]$

13.



将 $[a,b]$ 进行划分

$T_n: x_0=a<x_1<x_2<\cdots<x_n=b$

$\Delta x_i=x_i-x_{i-1}$, $i=1,\cdots,n$.

在每个区间段 $[x_{i-1},x_i]$ 上, 用高为 $f(x_{i-1})$ 宽为 Δx_i 的矩形绕 y 轴旋转所成圆筒代表面体积 S_n

$y=f(x)$ 在 $[x_{i-1},x_i]$ 与 $x=a, x=b$ 围成图形绕 y 轴旋转所成体积. $S_n=\pi(x_{i-1}+\Delta x_i)^2 f(x_{i-1})-\pi(x_{i-1})^2 f(x_{i-1})$

则总体积近似为

$V\approx\sum_{i=1}^n S_n=\sum_{i=1}^n \pi\cdot 2x\cdot\Delta x_i f(x_{i-1})+\pi(\Delta x_i)^2 f(x_{i-1})$

由于 $\lim_{n\to\infty}\sum_{i=1}^n \pi(\Delta x_i)^2\cdot f(x_{i-1})$

$\le\lim_{n\to\infty}\sum_{i=1}^n \pi\Delta x_i f(x_{i-1})\cdot\max\{\Delta x_i\}$

$=0$ (当 $\lambda(T_n)=\max_{i=1,\cdots n}\{\Delta x_i\}\to 0$ 时)

故 $V\approx\lim_{n\to\infty}\sum_{i=1}^n \pi 2x\Delta x_i f(x_{i-1})$

$=\int_a^b 2\pi\cdot x f(x)\,dx$

16.

$S=\int_1^3\sqrt{1+(y'(x))^2}\,dx$

$=\int_1^3\sqrt{1+(\frac{1}{2}x^2-\frac{1}{2}\frac{1}{x^2})^2}\,dx$

$=\int_1^3\sqrt{1+\frac{1}{4}(x^4+\frac{1}{x^4}-2)}\,dx$

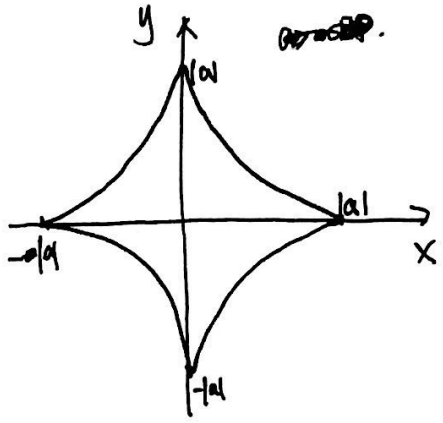
$=\int_1^3\frac{1}{2}\cdot(x^2+\frac{1}{x^2})\,dx$

$=\frac{1}{2}\cdot\frac{1}{3}\cdot x^3\Big|_1^3-\frac{1}{2}\cdot\frac{1}{x}\Big|_1^3$

$=\frac{1}{6}3^3-\frac{1}{6}-\frac{1}{2}(\frac{1}{3}-1)$

$=\frac{26}{6}+\frac{3}{6}=\frac{14}{3}$

18.



在 $t\in[0,\frac{1}{2}\pi]$，星形线可表示为方程

$$(\frac{x}{a})^{\frac{2}{3}}+(\frac{y}{a})^{\frac{2}{3}}=1$$

故可绘制上图.

$$l=4\cdot\int_0^{\frac{1}{2}\pi}\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\cdot dt$$

$$=4\int_0^{\frac{1}{2}\pi}\sqrt{[a\cdot3\cdot\cos^2t\cdot(-\sin t)]^2+(a3\cdot\sin^2t\cos t)^2}\,dt$$

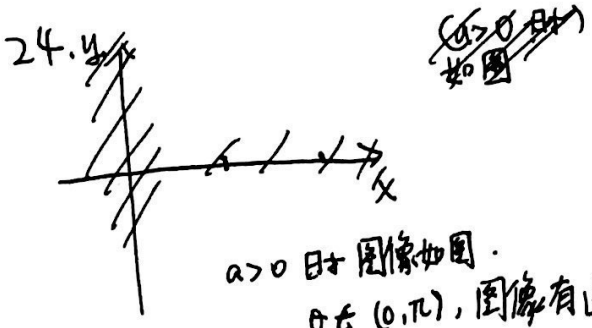
$$=12|a|\cdot\int_0^{\frac{1}{2}\pi}\sqrt{\cos^4t\sin^2t+\sin^4t\cos^2t}\,dt$$

$$=12|a|\cdot\int_0^{\frac{1}{2}\pi}\sin(t)\cdot\cos(t)\,dt$$

$$=6|a|\int_0^{\frac{1}{2}\pi}\sin(2t)\,dt=6|a|\cdot\frac{-\cos(2t)}{2}\Big|_0^{\frac{1}{2}\pi}$$

$$=6|a|\cdot(-\frac{-1}{2}-(-\frac{1}{2}))=6|a|$$

24.



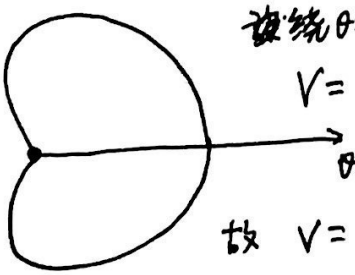
~~($a>0$ 时)~~ ~~如图~~

$a>0$ 时图像如图.

$\theta\in(0,\pi)$，图像有函数表示 $y=y(x)$

曲线 $\theta=0$ 旋转体体积为

$$V=\pi\cdot\int y(x)\sqrt{1+(y'(x))^2}\,dx$$



故 $V=\pi\int y(x)\sqrt{1+(\frac{dy}{dx})^2}\,dx$

$$=\pi\int y(x)\sqrt{(dx)^2+(dy)^2}$$

$$=\pi\int y(x)\,ds(x)$$

$$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{(dr)^2+(r\,d\theta)^2}$$

$$V=2\pi\int y(x)\sqrt{(dr)^2+(r\,d\theta)^2}$$

$$=2\pi\int_0^{\pi}r(\theta)\cdot\sin\theta\cdot\sqrt{(\frac{dr}{d\theta})^2+r^2}\,d\theta$$

$$=2\pi\int_0^{\pi}|a|(1+\cos\theta)\cdot\sin\theta\cdot\sqrt{a^2(-\sin\theta)^2+a^2(1+\cos\theta)^2}\,d\theta$$

$$=2\sqrt{2}\pi|a|^2\int_0^{\pi}(1+\cos\theta)\sin\theta\cdot\sqrt{1+\cos\theta}\,d\theta$$

$$=2\sqrt{2}\pi a^2\cdot\int_0^{\pi}(1+\cos\theta)^{\frac{3}{2}}\,d\cos\theta$$

$$=-2\sqrt{2}\pi a^2\cdot\frac{2}{5}\cdot(1+\cos\theta)^{\frac{5}{2}}\Big|_0^{\pi}$$

$$=-2\sqrt{2}\pi a^2\frac{2}{5}\cdot(0-2^{\frac{5}{2}})$$

$$=\frac{32}{5}\pi a^2$$