Lecture 25

Plan: 1. 分部积分法
2. 有理函数

由 $(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$

定理 6.23 设 $u(x), v(x)$ 可导，若 $\int u(x)v'(x)dx$ 存在，则 $\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$.

证略

注：(1) 也写成 $\int u\,dv = uv - \int v\,du$.

(2) v' 常选 e^{ax}, $\sin x$, $\cos x$, x^n, $\sinh x$, $\cosh x$ 等。

例 $I = \int x^3 \ln x\,dx$

$u = \ln x$, $dv = x^3$

$du = \frac{1}{x}dx$ $v = \frac{1}{4}x^4$

$I = \frac{1}{4}x^4\ln x - \int \frac{1}{4}x^3 dx$

$= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$

例 $I = \int \arctan x\,dx$

$u = \arctan x$ $dv = dx$

$du = \frac{1}{1+x^2}$ $v = x$

$I = x\arctan x - \int \frac{x}{1+x^2}dx$

$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$

例. $I = \int \sqrt{x^2-1}\,dx$

[法I]: $I = x\sqrt{x^2-1} - \int x\,d\sqrt{x^2-1}$

$= x\sqrt{x^2-1} - \int \frac{x^2}{\sqrt{x^2-1}}dx$

$= x\sqrt{x^2-1} - \int \frac{x^2-1+1}{\sqrt{x^2-1}}dx$

$= x\sqrt{x^2-1} - \int\sqrt{x^2-1}\,dx - \int\frac{1}{\sqrt{x^2-1}}dx$

$= x\sqrt{x^2-1} - I - \int\frac{1}{\sqrt{x^2-1}}dx$

$\int\frac{1}{\sqrt{x^2-1}}dx \overset{x=\cosh t}{=\!=} \int 1\,dt = t + C$

$= \ln|x+\sqrt{x^2-1}| + C$

$\Rightarrow I = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln|x+\sqrt{x^2-1}| + C$

[法II] $I \overset{x=\cosh t}{=\!=} \int \sinh^2 t\,dt$

$= \int \frac{\cosh 2t - 1}{2}dt$

$= \frac{1}{4}\sinh 2t - \frac{1}{2}t + C$

$= \frac{1}{2}\sinh t\cosh t - \frac{1}{2}t + C$

$= \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\ln|x+\sqrt{x^2-1}| + C$

例. $I = \int e^{ax}\cos bx\,dx$

$J = \int e^{ax}\sin bx\,dx$

[法I] $I = \frac{1}{b}\int e^{ax}d(\sin bx)$

$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}J$.

$J = -\frac{1}{b}\int e^{ax}d(\cos bx)$

$= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}I$.

$$\Rightarrow I=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C$$

$$J=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$$

[法II] $I+iJ=\int e^{ax}e^{ibx}dx$

例. $I=\int\frac{\sin x}{a\cos x+b\sin x}dx=\int\frac{-d\cos x}{a\cos x+b\sin x}$

$$J=\int\frac{\cos x}{a\cos x+b\sin x}dx=\int\frac{\pm d\sin x}{a\cos x+b\sin x}$$

$$aJ+bI=\int\frac{a\cos x+b\sin x}{a\cos x+b\sin x}dx=x+C$$

$$-aI+bJ=\int\frac{d(a\cos x+b\sin x)}{a\cos x+b\sin x}$$

$$=\ln|a\cos x+b\sin x|+C$$

$$\Rightarrow I=\frac{1}{a^2+b^2}(bx-a\ln|a\cos x+b\sin x|)+C.$$

$$J=\frac{1}{a^2+b^2}(ax+b\ln|a\cos x+b\sin x|)+C$$

例 $I_n=\int\frac{dx}{(x^2+a^2)^n}$

$$I_n=\frac{x}{(x^2+a^2)^n}-\int x\,d\frac{1}{(x^2+a^2)^n}$$

$$=\frac{x}{(x^2+a^2)^n}+2n\int\frac{x^2}{(x^2+a^2)^{n+1}}dx$$

$$=\frac{x}{(x^2+a^2)^n}+2nI_n-2na^2I_{n+1}$$

$$\Rightarrow I_{n+1}=\frac{x}{2na^2(x^2+a^2)^n}+\frac{2n-1}{2na^2}I_n$$

特别地 $I_1=\int\frac{dx}{x^2+a^2}$

$$=\int\frac{1}{a^2}\frac{dx}{(\frac{x}{a})^2+1}$$

$$=\frac{1}{a}\arctan x+C$$

$$I_2=\frac{x}{2a^2(x^2+a^2)}+\frac{1}{2a^3}\arctan x+C\cdots$$

☆ 有理函数

$$R(x)=\frac{P(x)}{Q(x)}$$

$P(x)$、$Q(x)$ 为互素多项式。

策略:

假分式 → 真分式 ⟶ 最简真分式

P次数 > Q次数　　P次数 ≤ Q次数

只考虑: $(p^2-4q<0)$

$$\frac{A}{x-a},\quad\frac{A}{(x-a)^n}\ (n>1)$$

$$\frac{Ax+B}{x^2+px+q},\quad\frac{Ax+B}{(x^2+px+q)^n}\ (n>1)$$

(1) $\int\frac{A}{x-a}dx=A\ln|x-a|+C$

(2) $n>1$

$$\int\frac{A}{(x-a)^n}dx=A\int(x-a)^{-n}d(x-a)$$

$$=-\frac{A}{n-1}(x-a)^{1-n}+C$$

(3) $x^2+px+q=(x+\frac{p}{2})^2+q-\frac{p^2}{4}$

$z=x+\frac{p}{2}\quad dx=dz$

由 $p^2-4q<0$，记 $a^2=q-\frac{p^2}{4}$

$$\int\frac{Ax+B}{x^2+px+q}dx=\int\frac{Ax+B}{(x+\frac{p}{2})^2+(q-\frac{p^2}{4})}dx$$

$$=\int\frac{A(z-\frac{p}{2})+B}{z^2+a^2}dz$$

$$=\int\frac{Az}{z^2+a^2}dz+\int\frac{B-A\frac{p}{2}}{z^2+a^2}dz$$

$$=\frac{1}{2}A\ln z^2+a^2+(B-\frac{Ap}{2})\frac{1}{a}\arctan\frac{z}{a}+C$$

$$=\frac{1}{2}A\ln(x^2+px+q)$$

$$+\frac{2B-Ap}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}}+C$$

(4) $n>1$

$$\int \frac{Ax+B}{(x^2+px+q)^n}dx = \int \frac{Az}{(z^2+a^2)^n}dz + (B-\frac{Ap}{2})\int \frac{1}{(z^2+a^2)^n}dz$$

递推求解

Steps: 1. 因式分解（分母）

2. 待定系数（最简真分式）

3. 分别求解

例 $I=\int \frac{x\,dx}{(x^2+1)(x-1)}$

设 $\frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$

$$= \frac{(Ax+B)(x-1)+C(x^2+1)}{(x^2+1)(x-1)}$$

$$= \frac{(A+C)x^2+(B-A)x-B+C}{(x^2+1)(x-1)}$$

$\Rightarrow A+C=0 \quad -A+B=1 \quad -B+C=0$

$\Rightarrow A=-\frac{1}{2},\ B=\frac{1}{2},\ C=\frac{1}{2}$

$$I = -\frac{1}{2}\int \frac{x-1}{x^2+1}dx + \frac{1}{2}\int \frac{dx}{x-1}$$

$$= -\frac{1}{4}\ln(x^2+1) + \frac{1}{2}\arctan x + \frac{1}{2}\ln|x-1| + C$$

例 $I=\int \frac{x^3+1}{x^4-3x^3+3x^2-x}dx$

$x^4-3x^3+3x^2-x = x(x-1)^3$

设 $\frac{x^3+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)}$

$\Rightarrow A=-1,\ B=2\quad C=1\quad D=2$

$$I = -\int \frac{dx}{x} + 2\int \frac{dx}{(x-1)^3} + \int \frac{dx}{(x-1)^2} + \int \frac{2dx}{x-1}$$

$$= -\ln|x| - \frac{1}{(x-1)^2} - \frac{1}{(x-1)} + 2\ln|x-1| + C$$