高等数学A第十二次作业习题答案

课程助教组

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6.1

证明. 由题意$F'(x)=f(x)$, 故$a\neq 0$时, $\frac{1}{a}F'(ax+b)=f(ax+b)$, 从而题述不定积分成立. □

6.2

(1). 直接计算,

$$\int\left(\sqrt{2x}+\sqrt[3]{3x}\right)\mathrm{d}x=\frac{1}{3}(2x)^{\frac{3}{2}}+\frac{1}{4}(3x)^{\frac{4}{3}}+C.$$

(3). 利用$(a^x)'=a^x\ln a$,

$$\text{原式}=\frac{1}{\ln 3}3^{x+1}-\frac{1}{\ln 3}3^{-x}+\frac{1}{3}e^x+C.$$

(6). 用换元法,

$$\text{原式}=4\int\frac{\cos 2x}{(2\sin x\cos x)^2}\mathrm{d}x=2\int\frac{1}{(\sin 2x)^2}\mathrm{d}(\sin 2x)=-\frac{2}{\sin 2x}+C.$$

(10). 当$x\in[k,k+1)$时, $[x]=k$, 故原式$=kx+C$.

6.3

(4). 令$t=\tan(\frac{x}{2})$, 则$\mathrm{d}t=\frac{t^2+1}{2}\mathrm{d}x$,

$$\text{原式}=\int\frac{2}{(1+t)^2}\mathrm{d}t=-\frac{2}{1+t}+C=-\frac{2}{1+\tan(\frac{x}{2})}+C.$$

(7). 令$y=\frac{1}{2}x^2$, 则$\mathrm{d}y=x\mathrm{d}x$,

$$\text{原式}=\frac{1}{4}\int\frac{1}{1+y^2}\mathrm{d}y=\frac{1}{4}\arctan\frac{x^2}{2}+C.$$

(11). 利用$\cos 4x-\cos 2x=-2\sin x\sin 3x$,

$$\text{原式}=\frac{1}{4}\sin 2x-\frac{1}{8}\sin 4x+C.$$

(14). 令$y=\sin^2 x$, 则$\mathrm{d}y=2\sin x\cos x\mathrm{d}x$, 且用$\cos^2 x=1-\sin^2 x$,

$$\text{原式}=\frac{1}{2}\int\frac{\mathrm{d}y}{\sqrt{(a^2-b^2)y+b^2}}=\frac{1}{a^2-b^2}\sqrt{(a^2-b^2)\sin^2 x+b^2}+C.$$

(17). 设$x=a\tan\theta$, 则

$$\text{原式}=\frac{1}{a^2}\int\cos\theta\mathrm{d}\theta=\frac{1}{a^2}\sin\theta+C=\frac{x}{a^2\sqrt{a^2+x^2}}+C.$$

(20). 设$x=2\sin\theta$, $t=\tan\frac{\theta}{2}$, 则

$$\text{原式}=\int\frac{\mathrm{d}\theta}{2\sin\theta}=\int\frac{\mathrm{d}t}{2t}=\frac{1}{2}\ln|\tan\frac{\theta}{2}|+C=\frac{1}{4}\ln|\tan\frac{\theta}{2}|^2+C=\frac{1}{4}\ln\frac{|\sqrt{4-x^2}-2|}{\sqrt{4-x^2}+2}+C.$$

6.4

(2). 利用$\mathrm{d}(e^{-3x}) = -3e^{-3x}$,

$$原式 = -\frac{1}{3}\int x\mathrm{d}(e^{-3x}) = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C.$$

(4). 直接分部积分,

$$原式 = x\arcsin x - \int x\mathrm{d}(\arcsin x) = x\arcsin x - \frac{1}{2}\int \frac{\mathrm{d}(x^2)}{\sqrt{1-x^2}} = x\arcsin x + \sqrt{1-x^2} + C.$$

(5). 利用$\mathrm{d}(\cot x) = -\frac{1}{\sin^2 x}$,

$$原式 = -\int x\mathrm{d}(\cot x) = -x\cot x + \int \frac{1}{\sin x}\mathrm{d}\sin x = -x\cot x + \ln|\sin x| + C.$$

(9). 做两次分部积分,

$$原式 = x\sin(\ln x) - \int \cos(\ln x)\mathrm{d}x = x\left(\sin(\ln x) - \cos(\ln x)\right) - \int \sin(\ln x)\mathrm{d}x.$$

移项后得到答案为$\frac{1}{2}x\left(\sin(\ln x) - \cos(\ln x)\right) + C$.

6.5

(2). 对指数项分部积分,

$$I_n = -\int x^n \mathrm{d}(e^{-x}) = -x^n e^{-x} + nI_{n-1}.$$

(3). 注意到

$$\begin{aligned}(x^{n-1}\sqrt{1-x^2})' &= (n-1)x^{n-2}\sqrt{1-x^2} - \frac{x^n}{\sqrt{1-x^2}} \\ &= \frac{1}{\sqrt{1-x^2}}\left((n-1)x^{n-2}(1-x^2) - x^n\right) \\ &= \frac{1}{\sqrt{1-x^2}}\left((n-1)x^{n-2} - nx^n\right),\end{aligned}$$

两边积分, 整理后即得递推式:

$$I_n = -\frac{1}{n}x^{n-1}\sqrt{1-x^2} + \frac{n-1}{n}I_{n-2}.$$

6.6

(1). 容易分解得到

$$原式 = \int \frac{1}{2(x-1)} - \frac{1}{2(x+1)} + \frac{1}{(x+1)^2}\mathrm{d}x = \frac{1}{2}\ln\frac{|x-1|}{|x+1|} - \frac{1}{x+1} + C.$$

(7). 用待定系数法将有理分式化为最简真分式, 设

$$\frac{1}{(x-2)^2(x+3)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)^3}.$$

可以按如下步骤确定系数: 两边乘以$(x-2)^2$并令$x=2$, 得$B = 5^{-3}$, 再求一次导并令$x=2$得$A = -3*5^{-4}$; 两边乘以$(x+3)^3$并让$x=-3$, 得$E = 5^{-2}$, 求导后让$x=-3$得$D = 2*5^{-3}$; 两边乘以$x-2$并令$x\to+\infty$, 得$C = -A = 3*5^{-4}$. 故得到

$$原式 = \frac{3}{625}\ln\frac{|x+3|}{|x-2|} - \frac{1}{125(x-2)} - \frac{2}{125(x+3)} - \frac{1}{50(x+3)^2} + C.$$

6.7

(2). 注意解答6.3(20)时已经给出$\frac{1}{\sin x}$的不定积分, 故

$$\text{原式} = \int \frac{\mathrm{d}\tan x}{\sin x} = \frac{1}{\cos x} + \int \frac{\mathrm{d}x}{\sin x} = \frac{1}{\cos x} + \ln|\tan\frac{x}{2}| + C.$$

(6). 令$y = \cos x$, 则

$$\text{原式} = -\int \frac{(1-\cos^2 x)^2}{\cos^4 x}\mathrm{d}\cos x = -\int \left(y^{-4} - 2y^{-2} + 1\right)\mathrm{d}y = -\cos x - 2\sec x + \frac{1}{3\cos^3 x} + C.$$

(10). 令$t = \tan\frac{x}{2}$, 代入得

$$\begin{aligned}
\text{原式} &= \int \frac{4t}{(3t^2+2t+1)(t^2+1)}\mathrm{d}t \\
&= \int \frac{3t-1}{3t^2+2t+1}\mathrm{d}t + \int \frac{1-t}{t^2+1}\mathrm{d}t \\
&= \frac{1}{2}\ln\frac{|3t^2+2t+1|}{|3t^2+3|} - \sqrt{2}\arctan\frac{3t+1}{\sqrt{2}} + \arctan t + C \\
&= \frac{1}{2}\ln\frac{|2+\sin x-\cos x|}{3} - \sqrt{2}\arctan\frac{3\tan\frac{x}{2}+1}{\sqrt{2}} + \frac{x}{2} + C.
\end{aligned}$$

其中倒数第二个等式请参考课本6.3.1章.