1. $A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 2B$

(a) $\begin{cases} \frac{dA}{dt} = -k_1 A + k_2 B^2, & A(0)=A_0 \\ \frac{dB}{dt} = +2k_1 A - 2k_2 B^2, & B(0)=B_0 \end{cases}$

Note $\frac{d}{dt}(2A+B)=0$

then $2\bar{A}+\bar{B} = 2A_0+B_0 = M \cdots ①$

and by setting $\frac{dA}{dt}=\frac{dB}{dt}=0$

we get $\bar{A} = \frac{k_2}{k_1}\bar{B}^2 \cdots ②$

with ① and ②, we conclude

$$\bar{B} = \sqrt{\frac{k_1^2}{16k_2^2}+\frac{k_1}{2k_2}M} - \frac{k_1}{4k_2}$$

$$\bar{A} = \frac{k_2}{k_1}\left(\sqrt{\frac{k_1^2}{16k_2^2}+\frac{k_1}{2k_2}M} - \frac{k_1}{4k_2}\right)^2$$

(b) Coefficient matrix at $(\bar{A},\bar{B})$

$$J = \begin{bmatrix} -k_1 & 2k_2\bar{B} \\ 2k_1 & -4k_2\bar{B} \end{bmatrix}$$

$$P = -(-k_1 - 4k_2\bar{B}) = k_1 + 4k_2\bar{B} > 0$$

$$q = 4k_1k_2\bar{B} - 4k_1k_2\bar{B} = 0$$

临界情况，无法用 q,p 法判断稳定性.

(c) $S(t) = A(\ln(k_1A)-1) + B(\ln(\sqrt{k_2}B)-1)$

$$\frac{dS}{dt} = \ln(k_1A)\frac{dA}{dt} + \ln(\sqrt{k_2}B)\frac{dB}{dt}$$

$$= \ln(k_1A)(-k_1A+k_2B^2) - 2\ln(\sqrt{k_2}B)(-k_1A+k_2B^2)$$

$$= -(\ln(k_1A)-\ln(k_2B^2))(k_1A-k_2B^2)$$

Note $\ln(x)$ is an increasing function, so $\ln(a)-\ln(b)$ and $a-b$ have the same sign.

Hence, $\frac{dS}{dt} \le 0$.

2. $u_t + (1-u)u_x = 0$

(a) Lagrangian ODEs

$$\begin{cases} \frac{dX}{dt} = 1-U, & X(0)=X_0 \cdots ① \\ \frac{dU}{dt} = 0, & U(0)=f(X_0) \cdots ② \end{cases}$$

By ②, we get $U(t)=U(0)=f(X_0)$

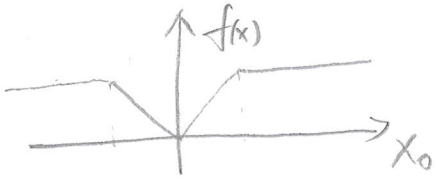
Then ① becomes

$$\frac{dX}{dt} = 1-f(X_0), \quad X(0)=X_0$$

Thus, $X(t) = X_0 + (1-f(X_0))t$

which is a straight line.

(b) $f(x)=\begin{cases}1 & |x|>1\\ |x| & |x|\le 1\end{cases}$



① when $X_0<-1$, $1-f(X_0)=0$

$X(t)=X_0$, $X(t)<-1$

② when $-1\le X_0\le 0$, $1-f(X_0)=1+X_0$

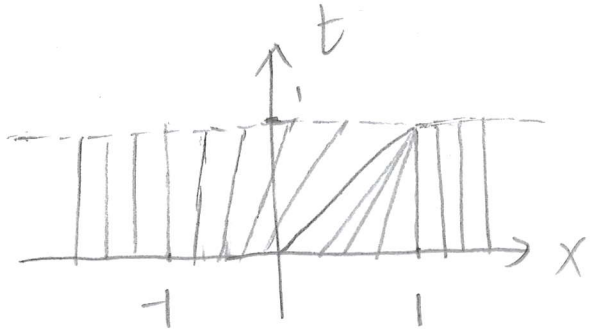
$X(t)=X_0+(1+X_0)t$, $-1\le X(t)\le t$

③ when $0\le X_0\le 1$, $1-f(X_0)=1-X_0$

$X(t)=X_0+(1-X_0)t$, $t\le X(t)\le 1$

④ when $X_0>1$ $1-f(X_0)=0$

$X(t)=X_0$, $X(t)>1$



① $u(x,t)=f(X_0)=1$

② $X_0=\frac{X-t}{1+t}$,

$u(x,t)=f(X_0)=-\frac{X-t}{1+t}$

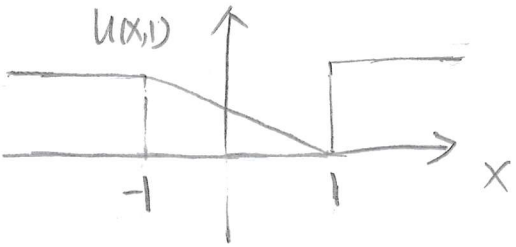
③ $X_0=\frac{X-t}{1-t}$

$u(x,t)=f(X_0)=\frac{X-t}{1-t}$

④ $u(x,t)=f(X_0)=1$

In sum, when $0\le t\le 1$

$$u(x,t)=\begin{cases}1 & x<-1\\ -\frac{x+t}{1+t} & -1\le x\le t\\ \frac{x+t}{1-t} & t\le x\le 1\\ 1 & x>1\end{cases}$$



(c) $u_t+(1-u)u_x=u_t+(u-\frac{1}{2}u^2)_x=0$

By R-H condition shock speed

$\dot{X}_S=1-\frac{1}{2}(u_l+u_r)$

$X_S(1)=1$

Here $u_l=-\frac{X_S-t}{1+t}$, $u_r=1$

then $\dot{X}_S=1-\frac{1}{2}(-\frac{X_S-t}{1+t}+1)$

$=\frac{1}{2}\frac{X_S-t}{1+t}+\frac{1}{2}$

$=\frac{1}{2}\frac{1}{1+t}X_S+\frac{1}{2}\cdot\frac{1}{1+t}$

$\Rightarrow X_S(t)=\pm\sqrt{1+t}-1$, $t\ge 1$

We conclude, when $t\ge 1$

$$u(x,t)=\begin{cases}1 & x>X_S(t) \text{ or } x<-1\\ -\frac{x-t}{1+t} & -1\le x\le X_S(t).\end{cases}$$

3. $E=\int_0^1 \left(\frac{1}{2}(u_x)^2+W(u)\right)dx$

(a) 令 $h(x)$, $0\le x\le 1$ 满足 $h(0)=0$, $h(1)=0$

则 $\int_0^1 \frac{\delta E}{\delta u} h(x)\,dx$

$=\left[\frac{d}{d\varepsilon} E(u+\varepsilon h)\right]_{\varepsilon=0}$

$=\left[\frac{d}{d\varepsilon}\int_0^1 \left[\frac{1}{2}(u_x+\varepsilon h_x)^2+W(u+\varepsilon h)\right]dx\right]_{\varepsilon=0}$

$=\int_0^1 [u_x h_x + h\,W'(u)]\,dx$

$= u_x h\Big|_0^1 + \int_0^1 (-u_{xx}h + W'(u)h)\,dx$

$=\int_0^1 (-u_{xx}+W'(u))\,h\,dx$

则 Euler-Lagrange 方程为

$$-u_{xx}-u+u^3=0$$

考虑边值条件 $u(0)=1$, $u(1)=1$

易知 $u\equiv 1$, $0\le x\le 1$ 为一个解.

(b) 由 $W'(u)=-u+u^3$

则 $u(x,t)$ 满足

$$u_t = u_{xx}+u-u^3$$

可写为 $u_t+(-u_x)_x = u-u^3$

通量函数 为 $-u_x$

源函数 为 $u-u^3$

$\frac{dE}{dt}=\int_0^1 (u_x u_{xt}+W'(u)u_t)\,dx$

$= u_x u_t\Big|_0^1$

$+\int_0^1 (-u_{xx}u_t+W'(u)u_t)\,dx$

$=\int_0^1 -(u_{xx}-W'(u))u_t\,dx$

$=\int_0^1 -(u_{xx}-W'(u))^2\,dx$

即 $\frac{dE}{dt}\le 0$.

4. $n_t = bn-pn$

令 $N=ne^{\int_0^t p(s)\,ds}$, 则

$$N_t = bN$$

又由 $N(0,x)=n(0,x)=n^0(x)$

则 $N(t,x)=e^{b(x)t}n^0(x)$ ---①

又由 $\frac{d}{dt}e^{\int_0^t p(s)ds}$

$= p(t)e^{\int_0^t p(s)ds}$

$=\int_a^b n(t,x)\,dx\, e^{\int_0^t p(s)ds}$

$=\int_a^b N(t,x)\,dx$ (by ①)

$=\int_a^b n^0(x)e^{b(x)t}\,dx$ ---②

且 $\frac{\partial}{\partial t}\left(\int_a^b \frac{n^0(x)}{b(x)}e^{b(x)t}\,dx\right)$

$=\int_a^b n^0(x)e^{b(x)t}\,dx$ ---③

由②③得

$$e^{\int_0^t \rho(s)ds} = \int_a^b \frac{n^0(x)}{b(x)} e^{b(x)t}dx + K \quad \cdots ④$$

代入 $t=0$ 得 $K = 1 - \int_a^b \frac{n^0(x)}{b(x)}dx$

由④解得

$$\rho(t) = \frac{\int_a^b n^0(x) e^{b(x)t}dx}{\int_a^b \frac{n^0(x)}{b(x)} e^{b(x)t}dx + K}$$

则 $$\rho(t) \le \frac{b(c)\int_a^b \frac{n^0(x)}{b(x)} e^{b(x)t}dx}{\int_a^b \frac{n^0(x)}{b(x)} e^{b(x)t}dx + K} \to b(c)$$

又 $\forall \varepsilon > 0$，令 $I_\varepsilon = \{x \mid b(x) > b(c) - \varepsilon\} \cap [a,b]$

则 $$\rho(t) \ge \frac{\int_{I_\varepsilon} n^0(x) e^{b(x)t}dx}{\int_a^b \frac{n^0(x)}{b(x)} e^{b(x)t}dx + K}$$

$$\ge \frac{(b(c)-\varepsilon)\int_{I_\varepsilon} \frac{n^0(x)}{b(x)} e^{b(x)t}dx}{\int_a^b \frac{n^0(x)}{b(x)} e^{b(x)t}dx + K}$$

$$= (b(c)-\varepsilon)/A_\varepsilon$$

这里 $$A_\varepsilon = \frac{\int_{I_\varepsilon} \frac{n^0(x)}{b(x)} e^{b(x)t}dx + \int_{[a,b]\setminus I_\varepsilon} \frac{n^0(x)}{b(x)} e^{b(x)t}dx + K}{\int_{I_\varepsilon} \frac{n^0(x)}{b(x)} e^{b(x)t}dx}$$

$$\to 1 \quad (t \to +\infty)$$

由 ε 任意，则 $t \to +\infty$ 时

$$\rho(t) \to b(c)$$

又由①知

$$n(x,t) = n^0(x) e^{b(x)t} e^{-\int_0^t \rho(s)ds}$$

$$= n^0(x) e^{\int_0^t (b(x)-\rho(s))ds}$$

若 $x \neq c$，由 $\rho(t) \to b(c)$

则 $\exists T$，s.t. 当 $t > T$，$\rho(t) > b(x)$

则 $n(x,t) \to 0 \quad (t \to +\infty)$.