第一次作业参考答案

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1. (1) 对特征值问题 $\phi''(x)=\lambda\phi(x)$, 考虑其特征方程

$$r^2-\lambda=0.$$

- 若 $\lambda<0$, 则 $r_1=i\sqrt{-\lambda}, r_2=-i\sqrt{-\lambda}$. 通解形式为 $\phi(x)=C_1\cos(\sqrt{-\lambda}x)+C_2\sin(\sqrt{-\lambda}x)$, 带入边值条件: $\phi(0)=\phi(1)=0$ 可得

$$C_1=0,\quad C_2=\begin{cases}0 & \sqrt{-\lambda}\neq k\pi,\ k\in\mathbb{Z},\\ C_2 & \sqrt{-\lambda}=k\pi,\ k\in\mathbb{Z}.\end{cases}$$

- 若 $\lambda=0$, 即 $\phi''(x)=0$, 则 $\phi(x)=kx+b$, 代入边值条件可得 $k=b=0$, $\phi(x)\equiv 0$.
- 若 $\lambda>0$, 则 $r_1=\sqrt{\lambda}, r_2=-\sqrt{\lambda}$. 通解形式为 $\phi(x)=C_1e^{\sqrt{\lambda}x}+C_2e^{-\sqrt{\lambda}x}$, 代入边值条件得 $C_1=C_2=0$.

综上，只有当 $\lambda=-(k\pi)^2$ 时, 有非平凡的特征函数 $\phi(x)=C\sin(k\pi x)$.

(2) 记 $F_j=f(j/N), j=0,\cdots,N$, 则边值问题的差分格式如下:

$$\begin{cases}\dfrac{U_{j+1}-2U_j+U_{j-1}}{h^2}=F_j,\\ U_0=0,\\ U_N=0.\end{cases}$$

整理为 $AU=f$ 的形式，其中

$$A_{(N-1)\times(N-1)}=\frac{1}{h^2}\begin{pmatrix}-2&1&0&\cdots&0&0\\1&-2&1&\cdots&0&0\\\vdots&\vdots&\vdots\cdots&\vdots&\vdots&\\0&0&0&\cdots&-2&1\\0&0&0&\cdots&1&-2\end{pmatrix},\quad U=\begin{pmatrix}U_1\\U_2\\\vdots\\U_{N-1}\end{pmatrix},\quad f=\begin{pmatrix}F_1\\F_2\\\vdots\\F_{N-1}\end{pmatrix}$$

设 λ 为 A 对应的特征值，$x=(x_1,x_2,\cdots,x_{N-1})^T$ 为 λ 所对应的特征向量，则有 $Ax=\lambda x$. 若令 $x_0=x_N=0$ 则代入可得

$$x_{i+1}-2x_i+x_{i-1}=-2(1-\alpha)x_i,\quad i=1,2,\cdots,N-1.$$

虚数 α 满足 $-2\frac{1-\alpha}{h^2}=\lambda$ 假设 $x_l=\xi^l$, 可得特征方程

$$\xi^2-2\alpha\xi+1=0$$

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两个特征根为

$$\xi_\pm = \alpha \pm \sqrt{\alpha - 1}.$$

并且满足 $\xi_+\xi_- = 1$. 因此 x_l 有如下形式

$$x_l = A_+\xi_+^l + A_-\xi_-^l$$

由边值条件 $x_0 = 0$，可得 $A_+ + A_- = 0$, 因此 $x_l = A_+(\xi_+^l - \xi_-^l)$.

由边值条件 $x_N = A_+(\xi_+^N - \xi_-^N) = 0$, 可得 $(\frac{\xi_+}{\xi_-})^N = 1$, 又因为 $\xi_+\xi_- = 1$, 所以有 $\xi_+^{2N} = 1$. 因此 ξ_+ (ξ_-) 是 $2N$ 次单位根, 即 $\xi_+ = e^{i\pi k/N}$, $0 < k < N, k \in \mathbb{Z}$.

而

$$\alpha = \frac{1 + \xi_+^2}{2\xi} = \cos\frac{\pi k}{N},\ 0 < k < N, k \in \mathbb{Z}.$$

因此可得

$$\lambda_k = -2\frac{1 - \cos\frac{\pi k}{N}}{h^2}, k \in \mathbb{Z}, 0 < k < N.$$

取 $A_+ = 1/(2i)$, 则 λ_k 所对应的特征向量 $x^{(k)}$ 的第 l 个分量 $x_l^{(k)} = A_+(\xi_+^l - \xi_-^l) = \sin(\pi kl/N)$, $0 < k, l < N$.

观察，当网格尺度 $h \to 0$ 时, $\lambda_k \to -k^2\pi^2$. 特征向量为连续问题特征函数在均匀网格上格点处取值.

2. 若

$$L_hU_{i,j} = \frac{U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}}{h^2} \geq 0 \quad \forall (i,j) \in J_\Omega.$$

若 $U_{i,j}$ 在 $(i_0, j_0) \in J_\Omega$ 上取得非负最大值，则可得 $U_{i_0,j_0} = U_{i_0\pm1,j_0} = U_{i_0,j_0\pm1}$, 继而可得 $U_{i,j}$ 在 J_Ω 上为常数.

3. (1) 在 (i,j) 处进行泰勒展开，可得截断误差

$$T_{i,j} = D_9^2u_{i,j} - (\Delta u)_{i,j} = \frac{h^2}{12}((\Delta)^2u)_{i,j} + O(h^4)$$

由于 $\Delta u = f$, 则当 $\Delta f = \Delta^2 u = 0$ 时，$O(h^2)$ 的误差主项会消失。

(2) 由 (1) 可知

$$\begin{aligned} D_9^2u_{i,j} &= (\Delta u)i, j + \frac{h^2}{12}((\Delta^2)u)_{i,j} + O(h^4) \\ &= f_{i,j} + \frac{h^2}{12}(\Delta f)_{i,j} + O(h^4) \\ &= f_{i,j} + \frac{h^2}{12}D_5^2f_{i,j} + O(h^4). \end{aligned}$$

则可以取 $F_{i,j} = f_{i,j} + \frac{h^2}{12}D_5^2f_{i,j}$, 也可以选 $F_{i,j} = f_{i,j} + \frac{h^2}{12}(\Delta f)_{i,j}$.

4. 由书上 (1.3.10) 式可得

$$\begin{aligned}\int_{\partial\omega} a(x,y)\nabla u(x,y)\cdot\boldsymbol{\nu}(x,y)\mathrm{d}s \approx & a_{i+1/2,j}\frac{U_{i+1,j}-U_{i,j}}{\Delta x}\Delta y - a_{i-1/2,j}\frac{U_{i,j}-U_{i-1,j}}{\Delta x}\Delta y\\ & + a_{i,j+1/2}\frac{U_{i,j+1}-U_{i,j}}{\Delta y}\Delta x - a_{i,j-1/2}\frac{U_{i,j}-U_{i,j-1}}{\Delta y}\Delta x,\\ \int_{\partial\omega} u(x,y)\boldsymbol{v}(x,y)\cdot\boldsymbol{\nu}(x,y)\mathrm{d}s \approx & \frac{U_{i+1,j}+U_{i,j}}{2}v^1_{i+1/2,j}\Delta y - \frac{U_{i,j}+U_{i-1,j}}{2}v^1_{i-1/2,j}\Delta y\\ & + \frac{U_{i,j+1}+U_{i,j}}{2}v^2_{i,j+1/2}\Delta x - \frac{U_{i,j}+U_{i,j-1}}{2}v^2_{i,j-1/2}\Delta x,\\ \int_{\omega} f(x,y)\mathrm{d}x\mathrm{d}y = & f_{i,j}\Delta x\Delta y.\end{aligned}$$

整理可得 (1.3.12) 式.

5. 截断误差均为 2 阶，泰勒展开即可，以下为同学解答过程。

5. 记差分格式 (1.3.9) 和 (1.3.12) 对应的差分算子分别为 L^1, L^2, 定常对流扩散方程对应的微分算子为 L.

设方程的解 u 足够光滑, 则计算可得

$$\begin{aligned}&a_{i+\frac{1}{2},j}(u_{i+1,j}-u_{i,j}) - a_{i-\frac{1}{2},j}(u_{i,j}-u_{i-1,j})\\ =&a_{i+\frac{1}{2},j}\left((\partial_x u)_{i+\frac{1}{2},j}\Delta x + \frac{1}{24}(\partial_x^3 u)_{i+\frac{1}{2},j}(\Delta x)^3 + O((\Delta x)^5)\right)\\ &- a_{i-\frac{1}{2},j}\left((\partial_x u)_{i-\frac{1}{2},j}\Delta x + \frac{1}{24}(\partial_x^3 u)_{i-\frac{1}{2},j}(\Delta x)^3 + O((\Delta x)^5)\right)\\ =&\left((\partial_x(a\partial_x u))_{i,j}\Delta x + \frac{1}{24}(\partial_x^3(a\partial_x u))_{i,j}(\Delta x)^3 + O((\Delta x)^5)\right)\Delta x + \frac{1}{24}(\partial_x(a\partial_x^3 u))_{i,j}(\Delta x)^4 + O((\Delta x)^6)\\ =&(\partial_x(a\partial_x u))_{i,j}(\Delta x)^2 + \frac{1}{24}\left[(\partial_x^3(a\partial_x u))_{i,j} + (\partial_x(a\partial_x^3 u))_{i,j}\right](\Delta x)^4 + O((\Delta x)^6).\end{aligned}$$

同理可得

$$\begin{aligned}&a_{i,j+\frac{1}{2}}(u_{i,j+1}-u_{i,j}) - a_{i,j-\frac{1}{2}}(u_{i,j}-u_{i,j-1})\\ =&(\partial_y(a\partial_y u))_{i,j}(\Delta y)^2 + \frac{1}{24}\left[(\partial_y^3(a\partial_y u))_{i,j} + (\partial_y(a\partial_y^3 u))_{i,j}\right](\Delta y)^4 + O((\Delta y)^6).\end{aligned}$$

在差分格式 (1.3.9) 中,

$$\begin{aligned}(uv^1)_{i+1,j} - (uv^1)_{i-1,j} &= (\partial_x(uv^1))_{i,j}\Delta x + \frac{1}{3}(\partial_x^3(uv^1))_{i,j}(\Delta x)^3 + O((\Delta x)^5),\\ (uv^2)_{i,j+1} - (uv^2)_{i,j-1} &= (\partial_y(uv^2))_{i,j}\Delta y + \frac{1}{3}(\partial_y^3(uv^2))_{i,j}(\Delta y)^3 + O((\Delta y)^5).\end{aligned}$$

因此差分格式 (1.3.9) 的局部截断误差为

$$\begin{aligned}T_{i,j}^1 =&((L^1-L)u)_{i,j}\\ =&\frac{1}{24}\left[(\partial_x^3(a\partial_x u))_{i,j}+(\partial_x(a\partial_x^3 u))_{i,j}+8(\partial_x^3(uv^1))_{i,j}\right](\Delta x)^2\\ &+\frac{1}{24}\left[(\partial_y^3(a\partial_y u))_{i,j}+(\partial_y(a\partial_y^3 u))_{i,j}+8(\partial_y^3(uv^2))_{i,j}\right](\Delta y)^2+O((\Delta x)^4+(\Delta y)^4).\end{aligned}$$

在差分格式 (1.3.12) 中,

$$\begin{aligned}&(u_{i+1,j}+u_{i,j})v^1_{i+\frac{1}{2},j}-(u_{i,j}+u_{i-1,j})v^1_{i-\frac{1}{2},j}\\ =&\left(2u_{i+\frac{1}{2},j}+\frac{1}{4}(\partial_x^2u)_{i+\frac{1}{2},j}(\Delta x)^2+O((\Delta x)^4)\right)v^1_{i+\frac{1}{2},j}-\left(2u_{i-\frac{1}{2},j}+\frac{1}{4}(\partial_x^2u)_{i-\frac{1}{2},j}(\Delta x)^2+O((\Delta x)^4)\right)v^1_{i-\frac{1}{2},j}\\ =&2(\partial_x(uv^1))_{i,j}\Delta x+\left(\frac{1}{12}(\partial_x^3(uv^1))_{i,j}+\frac{1}{4}(\partial_x(v^1\partial_x^2u))_{i,j}\right)(\Delta x)^3+O((\Delta x)^5).\end{aligned}$$

最后一项的计算同理. 因此差分格式 (1.3.12) 的局部截断误差为

$$\begin{aligned}T_{i,j}^2 =&((L^2-L)u)_{i,j}\\ =&\frac{1}{24}\left[(\partial_x^3(a\partial_x u))_{i,j}+(\partial_x(a\partial_x^3 u))_{i,j}+(\partial_x^3(uv^1))_{i,j}+3(\partial_x(v^1\partial_x^2u))_{i,j}\right](\Delta x)^2\\ &+\frac{1}{24}\left[(\partial_y^3(a\partial_y u))_{i,j}+(\partial_y(a\partial_y^3 u))_{i,j}+(\partial_y^3(uv^2))_{i,j}+3(\partial_y(v^2\partial_y^2u))_{i,j}\right](\Delta y)^2+O((\Delta x)^4+(\Delta y)^4).\end{aligned}$$