第三次作业参考答案

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1. (1): 考虑在控制体 $[x_l, x_r] \times [t_b, t_a]$ 上对反应扩散方程进行积分得:

$$\int_{x_l}^{x_r} u(x,t_a) - u(x,t_b)\mathrm{d}x = \int_{t_b}^{t_a} \left[p(x_r)u_x(x_r,t) - p(x_l)u_x(x_l,t) + \int_{x_l}^{x_r} f(x)\mathrm{d}x \right] \mathrm{d}t.$$

取 $x_l = x_{j-\frac{1}{2}}, x_r = x_{j+\frac{1}{2}}, j = 0, 1, \cdots, t_b = t_m, t_a = t_{m+1}$, 对上式左端进行如下离散:

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t_{m+1}) - u(x,t_m)\mathrm{d}x \approx (U_j^{m+1} - U_j^m)\Delta x_j$$

其中 U_j^m 为 $u(x,t_m)$ 在 $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ 的积分平均 $\bar{u}_j^m$ 的近似

对右端我们有

$$\begin{aligned}
&\int_{t_m}^{t_{m+1}} \left[p(x_{j+\frac{1}{2}})u_x(x_{j+\frac{1}{2}},t) - p(x_{j-\frac{1}{2}})u_x(x_{j-\frac{1}{2}},t) + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(x)\mathrm{d}x \right] \mathrm{d}t \\
\approx &\left[p(x_{j+\frac{1}{2}}) \frac{U_{j+1}^m - U_j^m}{\Delta x_j} - p(x_{j-\frac{1}{2}}) \frac{U_j^m - U_{j-1}^m}{\Delta x_{j-1}} + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(x)\mathrm{d}x \right] \tau
\end{aligned}$$

则可得到如下显式格式:

$$\frac{U_j^{m+1} - U_j^m}{\tau} = \frac{p(x_{j+\frac{1}{2}})(U_{j+1}^m - U_j^m) - p(x_{j-\frac{1}{2}})(U_j^m - U_{j-1}^m)}{h^2} + \frac{\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(x)\mathrm{d}x}{h}.$$

若右端改为如下离散方式:

$$\begin{aligned}
&\int_{t_m}^{t_{m+1}} \left[p(x_{j+\frac{1}{2}})u_x(x_{j+\frac{1}{2}},t) - p(x_{j-\frac{1}{2}})u_x(x_{j-\frac{1}{2}},t) + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(x)\mathrm{d}x \right] \mathrm{d}t \\
\approx &\left[p(x_{j+\frac{1}{2}}) \frac{U_{j+1}^{m+1} - U_j^{m+1}}{\Delta x_j} - p(x_{j-\frac{1}{2}}) \frac{U_j^{m+1} - U_{j-1}^{m+1}}{\Delta x_{j-1}} + \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(x)\mathrm{d}x \right]
\end{aligned}$$

则可得到隐式格式:

$$\frac{U_j^{m+1} - U_j^m}{\tau} = \frac{p(x_{j+\frac{1}{2}})(U_{j+1}^{m+1} - U_j^{m+1}) - p(x_{j-\frac{1}{2}})(U_j^{m+1} - U_{j-1}^{m+1})}{h^2} + \frac{\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(x)\mathrm{d}x}{h}.$$

(2) 初值条件:

$$U_j^0 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} g(x)\mathrm{d}x$$

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边值条件: $x=1$ 处

$$U_N^m = 0 \text{ (矩形积分公式逼近)}$$

在 $[x_0, x_{\frac{1}{2}}] \times [t_m, t_{m+1}]$ 上对控制体进行积分

$$\int_{x_0}^{x_{\frac{1}{2}}} [u(x,t_{m+1}) - u(x,t_m)]\,\mathrm{d}x = \int_{t_m}^{t_{m+1}} \left[p(x_{\frac{1}{2}}) u_x(x_{\frac{1}{2}},t) + \alpha_0(t)u(0,t) + \alpha_1(t) + \int_{x_0}^{x_{\frac{1}{2}}} f(x)\mathrm{d}x \right] \mathrm{d}t$$

分别取显式、隐式离散可得

$$\frac{(\frac{3}{8}U_0^{m+1} + \frac{1}{8}U_1^{m+1}) - (\frac{3}{8}U_0^m + \frac{1}{8}U_1^m)}{\tau} = \frac{p(x_{\frac{1}{2}})(U_1^m - U_0^m)}{h^2} + \frac{\alpha_0(t_m)U_0^m + \int_{x_0}^{x_{\frac{1}{2}}} f(x)\mathrm{d}x}{h} + \frac{\int_{t_m}^{t_{m+1}} \alpha_1(t)\mathrm{d}t}{\tau h},$$

以及

$$\frac{(\frac{3}{8}U_0^{m+1} + \frac{1}{8}U_1^{m+1}) - (\frac{3}{8}U_0^m + \frac{1}{8}U_1^m)}{\tau} = \frac{p(x_{\frac{1}{2}})(U_1^{m+1} - U_0^{m+1})}{h^2} + \frac{\alpha_0(t_{m+1})U_0^{m+1} + \int_{x_0}^{x_{\frac{1}{2}}} f(x)\mathrm{d}x}{h} + \frac{\int_{t_m}^{t_{m+1}} \alpha_1(t)\mathrm{d}t}{\tau h}$$

2. 2.10: 对于任意的 $\mu > 0$, 格式右端各项系数均为正数，且其和等于左端的系数。则由定理 1.2 及推论 1.1 可得

$$\max_{1\le j\le N-1} |U_j^{m+1}| \le \max\left\{ \max_{0\le j\le N} |U_j^0|, \max_{1\le l\le m+1} \max\{|U_0^l|, |U_N^l|\} \right\}.$$

由误差方程:

$$(1+2\mu)e_j^{m+1} = e_j^m + \mu\left(e_{j-1}^{m+1} + e_{j+1}^{m+1}\right) - \tau T_j^{m+1}.$$

则对任意 $1 \le j \le N-1$ 都有

$$(1+2\mu)|e_j^{m+1}| \le \max_{0\le j\le N} |e_j^m| + 2\mu\left(\max_{0\le j\le N} |e_j^{m+1}|\right) + \tau T^{m+1}$$

即 $\max|e_j^{m+1}| \le \max\{|e_j^m|, \max\{|e_0^{m+1}|, |e_N^{m+1}|\}\} + \tau T^{m+1}$. 递推可得。

3. 2.16: 由书上 (2.3.17) 式可知，当 (2.3.15) 式成立时

$$\begin{aligned} \|e^{m+1}\|_\infty &\le \|e^m\|_\infty + \tau T^m + \bar{\mu} h^2 \|a' u_{xx}\|_\infty \|e\|_\infty \\ &\le (1 + K\tau)\|e\|_\infty + \tau T^m. \end{aligned}$$

其中 $K = \|a'u_{xx}\|_\infty$。若给定时间 $t_{\max}$ 则在 $t_{\max}$ 之内的最大模误差

$$\begin{aligned} \|e^{t_{\max}/\tau}\|_\infty \le& (1+K\tau)^{t_{\max}/\tau}\|e^0\|_\infty + \frac{(1+K\tau)^{t_{\max}/\tau} - 1}{K\tau}\tau\|T\|_\infty \\ \le& e^{Kt_{\max}}\|e^0\|_\infty + \frac{e^{t_{\max}K}}{K}\|T\|_\infty \\ \le& e^{Kt_{\max}}\left(\|e\|_\infty + \frac{1}{K}\|T\|_\infty\right). \end{aligned}$$

取 $C_1 = 1$, $C_2 = \frac{1}{K}$ 即可.

4. 2.17: 由 (2.3.29) 式可得 $V + N^2 A_h \frac{1}{2}\tau V = -\frac{1}{h^2}A_h U^m$, 即 $V = -\frac{1}{h^2}(I + \frac{1}{2}\mu A_h)^{-1} A_h U^m$。而由 (2.3.31) 式，可知 $U^{m+1/2} = (I + \frac{1}{2}\mu A_h)^{-1} U^m$. 注意到 $(I + \frac{1}{2}\mu A_h)^{-1} A_h = A_h (I + \frac{1}{2}\mu A_h)^{-1}$ 可得结论.

5. 2.22: 由 (2.4.27c) 可得

$$(1-\frac{1}{2}\mu_z\delta_z^2)U_{j,k,l}^{m+1}=U_{j,k,l}^{m+1**}-\frac{1}{2}\mu_z\delta_z^2U_{j,k,l}^m$$

两边作用 $(1-\frac{1}{2}\mu_x\delta_x^2)(1-\frac{1}{2}\mu_y\delta_y^2)$ 可得

$$\begin{aligned}
&(1-\frac{1}{2}\mu_x\delta_x^2)(1-\frac{1}{2}\mu_y\delta_y^2)(1-\frac{1}{2}\mu_z\delta_z^2)U_{j,k,l}^{m+1}\\
=&(1-\frac{1}{2}\mu_x\delta_x^2)(1-\frac{1}{2}\mu_y\delta_y^2)(U_{j,k,l}^{m+1**}-\frac{1}{2}\mu_z\delta_z^2U_{j,k,l}^m)\\
=&(1-\frac{1}{2}\mu_x\delta_x^2)\left(U_{j,k,l}^{m+1*}-\frac{1}{2}\mu_y\delta_y^2U_{j,k,l}^m\right)-\frac{1}{2}(1-\frac{1}{2}\mu_x\delta_x^2)(1-\frac{1}{2}\mu_y\delta_y^2)\mu_z\delta_z^2U_{j,k,l}^m\\
=&U_{j,k,l}^m+\frac{1}{2}\mu_x\delta_x^2U_{j,k,l}^m+\mu_y\delta_y^2U_{j,k,l}^m+\mu_z\delta_z^2U_{j,k,l}^m\\
&-\frac{1}{2}(1-\frac{1}{2}\mu_x\delta_x^2)\mu_y\delta_y^2U_{j,k,l}^m-\frac{1}{2}(1-\frac{1}{2}\mu_x\delta_x^2)(1-\frac{1}{2}\mu_y\delta_y^2)\mu_z\delta_z^2U_{j,k,l}^m\\
&=U_{j,k,l}^m+\frac{1}{2}\mu_x\delta_x^2U_{j,k,l}^m+\frac{1}{2}\mu_y\delta_y^2U_{j,k,l}^m+\frac{1}{2}\mu_z\delta_z^2U_{j,k,l}^m\\
&+\frac{1}{4}\mu_x\delta_x^2\mu_y\delta_y^2U_{j,k,l}^m+\frac{1}{4}\mu_x\delta_x^2\mu_z\delta_z^2U_{j,k,l}^m+\frac{1}{4}\mu_y\delta_y^2\mu_z\delta_z^2U_{j,k,l}^m-\frac{1}{8}\mu_x\delta_x^2\mu_y\delta_y^2\mu_z\delta_z^2U_{j,k,l}^m\\
&=(1+\frac{1}{2}\mu_x\delta_x^2)(1+\frac{1}{2}\mu_y\delta_y^2)(1+\frac{1}{2}\mu_z\delta_z^2)U_{j,k,l}^m-\frac{1}{4}\mu_x\delta_x^2\mu_y\delta_y^2\mu_z\delta_z^2U_{j,k,l}^m
\end{aligned}$$

6. 不同格式的局部截断误差和增长因子.

(1) 显式格式:

$$\begin{aligned}
T&=\left(\frac{\Delta_{+t}}{\tau}-\frac{\delta_x^2}{h^2}\right)u-\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)u\\
&=\left(\frac{\Delta_{+t}}{\tau}-\frac{\partial}{\partial t}-\left(\frac{\delta_x^2}{h^2}\right)u-\frac{\partial^2}{\partial x^2}\right)u\\
&=\frac{1}{2}u_{tt}\tau-\frac{1}{12}u_{xxxx}h^2+O(\tau^2+h^4).
\end{aligned}$$

将 $U_j^m=\lambda_k^me^{ik\pi\frac{j}{N}}$ 带入格式

$$U_j^{m+1}=(1-2\mu)U_j^m+\mu(U_{j-1}^m+U_{j+1}^m).$$

可得

$$\lambda_k=1+\mu\left(e^{ikh\pi}+e^{-ikh\pi}-2\right)=1-4\mu\sin^2(kh\pi/2)$$

(2) 隐式格式

$$\begin{aligned}
T&=\left(\frac{\Delta_{-t}}{\tau}-\frac{\delta_x^2}{h^2}\right)u-\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)u\\
&=\left(\frac{\Delta_{-t}}{\tau}-\frac{\partial}{\partial t}-\left(\frac{\delta_x^2}{h^2}\right)u-\frac{\partial^2}{\partial x^2}\right)u\\
&=-\frac{1}{2}u_{tt}\tau-\frac{1}{12}u_{xxxx}h^2+O(\tau^2+h^4).
\end{aligned}$$

将 $U_j^m=\lambda_k^me^{ik\pi\frac{j}{N}}$ 带入格式

$$(1+2\mu)U_j^{m+1}=U_j^m+\mu(U_{j-1}^{m+1}+U_{j+1}^{m+1}).$$

可得

$$(1+2\mu)\lambda_k=1+\mu\lambda_k\left(e^{ikh\pi}+e^{-ikh\pi}\right)=1+2\mu\lambda_k\cos(kh\pi)$$

因此

$$\lambda_k = \frac{1}{1+2\mu(1-\cos(kh\pi))} = \frac{1}{1+4\mu\sin^2(kh\pi/2)}$$

(3) Crank-Nicolson 格式: 在 $(x_j, t_{m+1/2})$ 处 Taylor 展开可得书上 (2.2.57) 式. 将 $U_j^m = \lambda_k^m e^{ik\pi\frac{j}{N}}$ 带入格式

$$\frac{U_j^{m+1}-U_j^m}{\tau} = \frac{1}{2}\left(\frac{U_{j+1}^{m+1}-2U_j^{m+1}+U_{j-1}^{m+1}}{h^2} + \frac{U_{j+1}^m-2U_j^m+U_{j-1}^m}{h^2}\right)$$

可得

$$\lambda_k - 1 = \mu(\cos(kh\pi)-1)(1+\lambda_k)$$

因此

$$\lambda_k = \frac{1-2\mu\sin^2(kh\pi/2)}{1+2\mu\sin^2(kh\pi/2)}.$$