ISSN 0012-9593

quatrième série - tome 59 *fascicule 3* *mai-juin 2026*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Davesh MAULIK, Junliang SHEN & Qizheng YIN

Algebraic cycles and Hitchin systems

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Responsable du comité de rédaction / *Editor-in-chief*

Jean-François Quint

Publication fondée en 1864 par Louis Pasteur

Continuée de 1872 à 1882 par H. Sainte-Claire Deville
de 1883 à 1888 par H. Debray
de 1889 à 1900 par C. Hermite
de 1901 à 1917 par G. Darboux
de 1918 à 1941 par É. Picard
de 1942 à 1967 par P. Montel

Comité de rédaction au 27 mars 2026

T. Alazard
S. Cantat
K. Cesnavicius
Y. de Cornulier
C. Fermanian Kammerer
J. Fresán
S. Gouëzel
Y. Harpaz
A. Keating
G. Miermont
S. Riche
A. Song

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88.
Email : annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 514 euros.
Abonnement avec supplément papier :
Europe : 722 €. Hors Europe : 813 € ($ 985). Vente au numéro : 77 €.

© 2026 Société Mathématique de France, Paris

En application de la loi du 1er juillet 1992, il est interdit de reproduire, même partiellement, la présente publication sans l'autorisation de l'éditeur ou du Centre français d'exploitation du droit de copie (20, rue des Grands-Augustins, 75006 Paris).
All rights reserved. No part of this publication may be translated, reproduced, stored in a retrieval system or transmitted in any form or by any other means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the publisher.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directrice de la publication : Isabelle Gallagher
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 59, 2026, p. 637 à 682

ALGEBRAIC CYCLES AND HITCHIN SYSTEMS

BY DAVESH MAULIK, JUNLIANG SHEN AND QIZHENG YIN

ABSTRACT. – The purpose of this paper is to study motivic aspects of the Hitchin system for GL_n. Our results include the following. (a) We prove the motivic decomposition conjecture of Corti-Hanamura for the Hitchin system; in particular, the decomposition theorem associated with the Hitchin system is induced by algebraic cycles. This yields an unconditional construction of the motivic perverse filtration for the Hitchin system, which lifts the cohomological/sheaf-theoretic perverse filtration. (b) We prove that the inverse of the relative hard Lefschetz symmetry is induced by a relative algebraic correspondence, confirming the relative Lefschetz standard conjecture for the Hitchin system. (c) We show a strong perversity bound for the normalized Chern classes of a universal bundle with respect to the motivic perverse filtration; this specializes to the sheaf-theoretic result obtained earlier by Maulik-Shen. (d) We prove a χ-independence result for the relative Chow motives associated with Hitchin systems.

Our methods combine Fourier transforms for compactified Jacobian fibrations associated with integral locally planar curves, nearby and vanishing cycle techniques, and a Springer-theoretic interpretation of parabolic Hitchin moduli spaces.

RÉSUMÉ. – L'objectif de cet article est d'étudier des aspects motiviques du système de Hitchin pour GL_n. Nos résultats incluent les suivants. (a) Nous prouvons la conjecture de décomposition motivique de Corti-Hanamura pour le système de Hitchin ; en particulier, le théorème de décomposition associé au système de Hitchin est induit par des cycles algébriques. Cela fournit une construction inconditionnelle de la filtration perverse motivique pour le système de Hitchin, qui relève la filtration perverse cohomologique/faisceautique. (b) Nous prouvons que l'inverse de la symétrie de Lefschetz difficile relative est induit par une correspondance algébrique relative, ce qui confirme la conjecture standard de Lefschetz relative pour le système de Hitchin. (c) Nous montrons une borne de perversité forte pour les classes de Chern normalisées d'un fibré universel par rapport à la filtration perverse motivique ; cela se restreint au résultat faisceautique obtenu précédemment par Maulik-Shen. (d) Nous prouvons un résultat de χ-indépendance pour les motifs de Chow relatifs associés aux systèmes de Hitchin.

Nos méthodes combinent les transformées de Fourier pour les fibrations jacobiennes compactifiées associées aux courbes intégrales localement planes, des techniques de cycles proches et évanescents, et une interprétation en termes de la théorie de Springer des espaces de modules de Hitchin paraboliques.

0012-9593/03/© 2026 Société Mathématique de France. Tous droits réservés
doi:10.24033/asens.1645

1. Introduction

The topology of the Hitchin system has been intensively studied over decades. In [54], Ngô proved the fundamental lemma of the Langlands program by reducing it to the study of the decomposition theorem [8] for the Hitchin system; the topological mirror symmetry conjecture of Hausel-Thaddeus [32] was proven by comparing the decomposition theorem for Hitchin systems associated with the Langlands dual groups SL_n and PGL_n [29, 44]; the decomposition theorem for the Hitchin system further yields a perverse filtration on the singular cohomology of the Hitchin moduli space, whose interaction with the non-abelian Hodge theory is the main theme of the P=W conjecture [12], recently proven [46, 30, 47]; the decomposition theorem for the Hitchin system also appears naturally in enumerative geometry, particularly in the study of BPS invariants [18, 45, 36, 20].

Most of these developments relied heavily on topological and sheaf-theoretic approaches. Nevertheless, hints from several recent results [28, 41, 35, 47] suggest that there should be a motivic theory hidden behind the cohomological structures mentioned above. The purpose of this work is to study motivic aspects of the decomposition theorem and the perverse filtration for the Hitchin system. The upshot is that the geometry of the Hitchin system provides us powerful tools to construct algebraic cycles on the moduli of Higgs bundles, lifting many cohomological statements to the motivic level.

1.1. Hitchin systems

Throughout, we work over the complex numbers $\mathbb{C}$; all Chow groups are taken with rational coefficients.

Let Σ be a nonsingular projective curve of genus $g \geq 2$. Let $M_{n,d}$ be the moduli space of stable Higgs bundles with coprime rank n and degree d on Σ; it carries the structure of an integrable system, known as the *Hitchin system*:

$$f : M_{n,d} \to B, \tag{1}$$

which is a proper and surjective morphism to an affine space B. We use d_f to denote the relative dimension of (1). By the decomposition theorem of Beilinson, Bernstein, Deligne, and Gabber [8], the derived pushforward $Rf_*\mathbb{Q}_{M_{n,d}} \in D^{\mathrm{b}}_{\mathrm{c}}(B)$ admits a decomposition into (shifted) semisimple perverse sheaves:

$$Rf_*\mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - d_f] \simeq \bigoplus_{i=0}^{2d_f} {}^{\mathfrak{p}}\mathcal{H}^i\left(Rf_*\mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - d_f]\right)[-i] \in D^{\mathrm{b}}_{\mathrm{c}}(B), \tag{2}$$

where the semisimple summands on the right-hand side satisfy the Lefschetz symmetry

$$\sigma^i : {}^{\mathfrak{p}}\mathcal{H}^{d_f - i}\left(Rf_*\mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - d_f]\right) \xrightarrow{\simeq} {}^{\mathfrak{p}}\mathcal{H}^{d_f + i}\left(Rf_*\mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - d_f]\right) \tag{3}$$

induced by the cup-product with the i-th power of a relative ample class σ. The decomposition (2) is an important invariant for the Hitchin system (1) and has played a crucial role in all the stories mentioned above. For example, recent resolutions of the P=W conjecture show that the (cohomological) perverse filtration

$$P_0H^*(M_{n,d}, \mathbb{Q}) \subset P_1H^*(M_{n,d}, \mathbb{Q}) \subset \cdots \subset H^*(M_{n,d}, \mathbb{Q}) \tag{4}$$

induced by (2) matches the double-indexed weight filtration associated with the corresponding character variety via non-abelian Hodge theory:

$$P_k H^*(M_{n,d}, \mathbb{Q}) = W_{2k} H^*(M_{\mathrm{char}}, \mathbb{Q}).$$

Furthermore, the Lefschetz symmetry (3) matches the *curious hard Lefschetz* for the character variety [31, 49].

Our first result is that both the decomposition (2) and the filtration (4) are motivic in a strong sense. In particular, this confirms the *motivic decomposition conjecture* of Corti-Hanamura [19] for the Hitchin system.

THEOREM 1.1 (Motivic decomposition). – *There exists a decomposition of the relative diagonal into orthogonal projectors:*

$$[\Delta_{M_{n,d}/B}] = \sum_{i=0}^{2d_f} \mathfrak{r}_i, \qquad \mathfrak{r}_i \circ \mathfrak{r}_i = \mathfrak{r}_i, \quad \mathfrak{r}_i \circ \mathfrak{r}_j = 0, \quad i \neq j,$$

which induces a sheaf-theoretic decomposition (2). *Here* $[\Delta_{M_{n,d}/B}]$ *and* $\mathfrak{r}_i$ *lie in the Chow group of the relative product* $M_{n,d} \times_B M_{n,d}$, *and the composition is taken as relative correspondences.*

Our method also yields immediately the relative Lefschetz standard conjecture for the Hitchin system, which shows the algebraicity of the inverse of (3).

THEOREM 1.2 (Relative Lefschetz). – *For any relative ample class* σ *and* $i > 0$, *the inverse of the Lefschetz symmetry* (3) *is induced by an algebraic cycle* $\mathfrak{Z}_{\sigma,i}$ *on* $M_{n,d} \times_B M_{n,d}$:

$$\mathfrak{Z}_{\sigma,i} : {}^{\mathfrak{p}}\mathcal{H}^{d_f+i}\left(Rf_*\mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - d_f]\right) \xrightarrow{\simeq} {}^{\mathfrak{p}}\mathcal{H}^{d_f-i}\left(Rf_*\mathbb{Q}_{M_{n,d}}[\dim M_{n,d} - d_f]\right).$$

Under the language of [19], we obtain from Theorem 1.1 a decomposition in the category of relative Chow motives over B:

$$h\left(M_{n,d}\right) = \bigoplus_i h_i(M_{n,d}) \in \mathrm{CHM}(B) \tag{5}$$

with

$$h(M_{n,d}) = (M_{n,d}, [\Delta_{M_{n,d}/B}], 0), \quad h_i(M_{n,d}) = (M_{n,d}, \mathfrak{r}_i, 0)$$

whose homological realization under the Corti-Hanamura functor

$$\mathrm{CHM}(B) \to D^{\mathrm{b}}_{\mathrm{c}}(B)$$

recovers (2). As a consequence, we obtain a motivic perverse filtration

$$P_\bullet h(M_{n,d}) \subset h(M_{n,d}); \tag{6}$$

see Section 3.1. This motivic perverse filtration specializes to both the cohomological perverse filtration (4) and a perverse filtration on the Chow groups of $M_{n,d}$; the latter has not yet been much studied.

REMARK 1.3. – We conclude Section 1.1 with some remarks.

(a) The decomposition (5) we constructed in Theorem 1.1 specializes to a particular sheaf-theoretic isomorphism (2). Conversely, it is interesting to explore whether the natural sheaf-theoretic decompositions [21, 10] admit motivic liftings.

(b) Motivated by the P=W conjecture, it is conjectured that the Hitchin system admits a multiplicative decomposition 5, i.e., a motivic decomposition compatible with cup-product; we refer to [4, 55] for detailed discussions. It is not clear whether the decomposition given by Theorem 1.1 is multiplicative.

(c) A natural conjectual enhancement of Theorem 1.4 is the relative Chow-Lefschetz standard conjecture for the Hitchin system, i.e., the Lefschetz symmetry for the motivic decomposition (5). Our argument does not prove that, and it is an interesting direction to explore.

1.2. Motivic strong perversity of Chern classes

Let $\mathcal{U}$ be a universal Higgs bundle over $\Sigma \times M_{n,d}$. After normalization, it induces a canonical Chern character

$$\widetilde{\mathrm{ch}}(\mathcal{U}) := \sum_k \widetilde{\mathrm{ch}}_k(\mathcal{U}) \in \mathrm{CH}^*(\Sigma \times M_{n,d}), \quad \widetilde{\mathrm{ch}}_k(\mathcal{U}) \in \mathrm{CH}^k(\Sigma \times M_{n,d}),$$

which is independent of the choice of the universal bundle $\mathcal{U}$. It was proven in [46] that the cohomology class $\widetilde{\mathrm{ch}}_k(\mathcal{U})$ has strong perversity k, i.e., its action on the direct image complex

$$\widetilde{\mathrm{ch}}_k(\mathcal{U}) : Rf_*\mathbb{Q}_{\Sigma\times M_{n,d}} \to Rf_*\mathbb{Q}_{\Sigma\times M_{n,d}}[2k]$$

via cup-product satisfies that

$$\widetilde{\mathrm{ch}}_k(\mathcal{U}) : {}^{\mathfrak{p}}\tau_{\le i} Rf_*\mathbb{Q}_{\Sigma\times M_{n,d}} \to \left({}^{\mathfrak{p}}\tau_{\le i+k} Rf_*\mathbb{Q}_{\Sigma\times M_{n,d}}\right)[2k]. \tag{7}$$

Here, we also use f to denote the product morphism $\Sigma \times M_{n,d} \to \Sigma \times B$ and consider the corresponding perverse filtration. Specializing to global cohomology, the equation (7) yields

$$\widetilde{\mathrm{ch}}_k(\mathcal{U}) : P_i H^*(\Sigma \times M_{n,d}, \mathbb{Q}) \to P_{i+k} H^{*+2k}(\Sigma \times M_{n,d}, \mathbb{Q}),$$

which further implies the P=W conjecture; we refer to [46, Section 2] for details on the reduction.

In this paper, we show that the sheaf-theoretic strong perversity (7) for Chern classes can be lifted motivically using the motivic perverse filtration (6).

THEOREM 1.4 (Motivic strong perversity of Chern classes). – *We have for any $i, k \in \mathbb{Z}$:*

$$\widetilde{\mathrm{ch}}_k(\mathcal{U}) : P_i h(\Sigma \times M_{n,d}) \to P_{i+k} h(\Sigma \times M_{n,d})(k) \in \mathrm{CHM}(\Sigma \times B).$$

Here the action is given by cup-product and (k) stands for the k-th Tate twist.

1.3. Motivic χ-independence

Now we consider two degrees d, d' which are both coprime to n. The Hitchin systems associated with $M_{n,d}, M_{n,d'}$ share the same base,

$$f_d : M_{n,d} \to B, \quad f_{d'} : M_{n,d'} \to B.$$

Here we use $f_\bullet$ to indicate the dependence on the degree.

The following result was predicted by the Hausel-Thaddeus conjecture on the topological mirror symmetry [32], and was first proven by Groechenig-Wyss-Ziegler [29] for perverse

sheaves via point counting over finite fields and Maulik-Shen [45] for Hodge modules using support theorem and the vanishing cycle functor:

$$Rf_{d*}\mathbb{Q}_{M_{n,d}} \simeq Rf_{d'*}\mathbb{Q}_{M_{n,d'}}. \tag{8}$$

This equation is an instance of the *χ-independence* phenomenon: the moduli space of Higgs bundles can be viewed as the moduli space of 1-dimensional stable sheaves properly supported on the surface $T^*\Sigma$ in the curve class $n[\Sigma]$, and (8) implies that the decomposition theorem is independent of the choice of the Euler characteristic $\chi\,(=\deg+(1-g)n)$ of the sheaves. This phenomenon has tight connections to enumerative geometry [56], and has been generalized to singular moduli spaces using cohomological Donaldson-Thomas theory [45, 36]. Recently, Hoskins and Pepin Lehalleur [35] proved the χ-independence of the Chow motives of the moduli of Higgs bundles:

$$h(M_{n,d}) = h(M_{n,d'}) \in \mathrm{CHM}(\mathrm{pt}). \tag{9}$$

We prove a motivic χ-independence result for the relative Chow motives over B, which enhances both the sheaf-theoretic result (8) and the motivic result (9).

THEOREM 1.5 (Motivic χ-independence). – *Assume $(n,d)=(n,d')=1$, we have an isomorphism of relative Chow motives:*

$$h(M_{n,d}) \simeq h(M_{n,d'}) \in \mathrm{CHM}(B) \tag{10}$$

preserving the motivic perverse filtrations

$$P_k h(M_{n,d}) \simeq P_k h(M_{n,d'}) \in \mathrm{CHM}(B).$$

REMARK 1.6. – (a) To the best of our knowledge, the isomorphism of the relative Chow motives (10) (even without the motivic perverse filtrations) was not known before. After pushing to a point, this recovers the isomorphism (9) of Hoskins and Pepin Lehalleur.

(b) By Proposition 3.4, we obtain that $f_d : M_{n,d} \to B$ and $f_{d'} : M_{n,d'} \to B$ have isomorphic motivic decompositions; i.e., there exist motivic decompositions

$$h(M_{n,d}) = \bigoplus_i h_i(M_{n,d}), \quad h(M_{M_{n,d'}}) = \bigoplus_i h_i(M_{n,d'})$$

satisfying that

$$h_i(M_{n,d}) \simeq h_i(M_{n,d'}) \in \mathrm{CHM}(B).$$

1.4. Background and relations to other work

In this section, we discuss briefly some background and relations to some previous work.

1.4.1. *Motivic decomposition conjecture*. – Let $f : X \to B$ be a proper morphism of nonsingular varieties. The motivic decomposition conjecture of Corti-Hanamura [19] predicts that in general the sheaf-theoretic decomposition

$$Rf_*\mathbb{Q}_X[\dim X - d_f] \simeq \bigoplus_{i=0}^{2d_f} {}^{\mathfrak{p}}\mathcal{H}^i\left(Rf_*\mathbb{Q}_X[\dim X - d_f]\right)[-i], \quad d_f := \dim X \times_B X - \dim X$$

admits a *motivic* lifting. More precisely, as in the statement of Theorem 1.1, they conjectured that there is a decomposition of the relative diagonal cycle into orthogonal projectors

$$[\Delta_{X/B}] = \sum_{i=0}^{2d_f} \mathfrak{r}_i, \qquad \mathfrak{r}_i \circ \mathfrak{r}_i = \mathfrak{r}_i, \quad \mathfrak{r}_i \circ \mathfrak{r}_j = 0, \quad i \neq j,$$

which induces a sheaf-theoretic decomposition as above. In general, the conjectural existence of the motivic decomposition is wide open. We note two major challenges in constructing such a decomposition. First, in the special case when X is a nonsingular projective variety and B is a point, the motivic decomposition conjecture is equivalent to the existence of a Chow-Künneth decomposition for X. This was conjectured by Murre [53], and has only been verified in very limited cases. Second, even if we are able to find orthogonal projectors over a Zariski dense open subset $U \subset B$ (e.g., the locus where f is smooth), it is unclear if these projectors over U can be extended to orthogonal projectors over B. In particular, the cycles given by the Zariski closures of the orthogonal projectors over U are not orthogonal projectors in general, and understanding the behavior of the "boundary" is crucial.

Recently there have been some attempts to apply the theory of motivic intermediate extension of Wildeshaus to construct motivic decompositions for certain proper morphisms with *Tate fibers*; we refer to [15] and references therein for more details.

1.4.2. *Derived equivalences*. – For a certain class of examples, one approach to the motivic decomposition conjecture is to use symmetries of the derived category of coherent sheaves. It was first found by Beauville [6] and Deninger-Murre [22] over 35 years ago that descending Mukai's Fourier transform [52] from the derived category of coherent sheaves to algebraic cycles yields a motivic decomposition for an abelian scheme $f : A \to B$. Later, the Fourier transform was further used by Künnerman [37] to construct a Lefschetz decomposition. In this case the morphism f is smooth and each summand appeared in the decomposition theorem is a (shifted) semisimple local system.

The case of abelian fibrations with singular fibers is more complicated, and the perverse t-structure comes into play. On the derived category side, Arinkin [3] extended the Fourier-Mukai transform to compactified Jacobian fibrations associated with a family of projective integral locally planar curves. Arinkin's work was motivated by an attempt to understand the classical limit of the geometric Langlands correspondence for Hitchin systems [3, Section 4.5]. Recently, motivated by the P=W conjecture for Hitchin moduli spaces [12, 9, 46, 30], a theory of Fourier transform for relative Chow motives was established in [47] using the Arinkin sheaf. As a consequence of the Fourier theory and Ngô's support theorem [54], it was obtained in [47] that the motivic decomposition conjecture holds for compactified Jacobians fibrations associated with projective integral locally planar curves.

We note that, for the singular fibers, this approach matches the complexity of the Arinkin sheaf with the complexity of intermediate extensions of the local systems.

1.4.3. *Hitchin systems*. – For the Hitchin system (1), the techniques described above imply the motivic decomposition conjecture for the Hitchin system over a Zariski open subset $B^{\mathrm{ell}} \subset B$, called *the elliptic locus*, formed by integral spectral curves. This open subset is exactly the locus over which the Hitchin fibers are integral. However, to the best of our knowledge, it is not yet known how to extend Arinkin's derived equivalence [3] from the elliptic locus to the total Hitchin base B. Even worse, the full support theorem *fails* for (1); the results of [13] suggest that the supports of the decomposition theorem associated with (1) are very complicated and there is no complete description by far. In view of the approach of [1], the lack of the control of the supports provides also the difficulty in proving the relative Lefschetz standard conjecture. The main point of this paper is to give a proof of these conjectures for the Hitchin system (1) in the absence of the aforementioned geometric ingredients.

1.4.4. *Idea of the proof of Theorem 1.1*. – The strategy for overcoming these difficulties is to interpret the Hitchin moduli space as a global analogue of a Lie algebra, thereby enabling the use of tools from Springer theory. Rather than extending the motivic decomposition from B^{ell}, we instead consider alternative Hitchin moduli spaces associated with certain parabolic structures, where all the Hitchin fibers are integral. The correspondences arising from Springer theory yield algebraic cycles that relate $M_{n,d}$ to the parabolic moduli space. Consequently, the motivic decomposition for $f : M_{n,d} \to B$ can be derived from that of the parabolic Hitchin system. Finally, to verify that the algebraic cycles we construct yield the desired homological realization, we combine techniques of nearby and vanishing cycles with Springer theory. As a byproduct, we show that Corti-Hanamura's motivic decomposition conjecture is preserved under specialization; see Corollary 5.2.

Springer theory has been applied to the study of moduli spaces of Higgs bundles and character varieties in connection with various questions and conjectures; see [23, 49, 46, 30].

1.5. Acknowledgements

J.S. gratefully acknowledges the hospitality of both the math department of MIT and the Isaac Newton Institute at Cambridge during his stay in the spring of 2024 where part of this work was completed.

D.M. was supported by a Simons Investigator Grant. J.S. was supported by the NSF grant DMS-2301474, a Sloan Research Fellowship, and a Simons Fellowship during his visit at the Isaac Newton Institute.

2. Moduli of Higgs bundles

In this section, we review the geometry of the moduli of Higgs bundles. Throughout, we fix a nonsingular projective curve Σ of genus $g \geq 2$, and fix two integers n, d with $(n, d) = 1$.

2.1. Moduli spaces

We denote by $M_{n,d}$ the moduli space of stable Higgs bundles

$$(\mathcal{E}, \theta), \quad \theta : \mathcal{E} \to \mathcal{E} \otimes \omega_\Sigma, \quad \mathrm{rk}(\mathcal{E}) = n, \quad \deg(\mathcal{E}) = d,$$

where the stability condition is with respect to the slope $\mu(\mathcal{E}, \theta) = \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E})$. The moduli space admits a proper morphism to an affine space

$$f : M_{n,d} \to B := \bigoplus_{i=1}^{n} H^0\left(\Sigma, \omega_\Sigma^{\otimes i}\right). \tag{11}$$

This map is given by calculating the characteristic polynomial of the Higgs field θ:

$$f(\mathcal{E}, \theta) = (\mathrm{tr}(\theta), \mathrm{tr}(\wedge^2\theta), \ldots, \det(\theta)) \in B,$$

and is known as the *Hitchin system*. The moduli space $M_{n,d}$ is a holomorphic symplectic variety and the Hitchin system f is a Lagrangian fibration [33, 34]. Later, when we need to consider the moduli spaces $M_{n,d}$ with different d, we will write f_d to indicate the dependence of the Hitchin system on the degree.

By the Beauville-Narasimhan-Ramanan correspondence [7], the Hitchin base B can be viewed as the parameter space of spectral curves; these are properly supported curves in the open surface $\mathrm{Tot}_\Sigma(\omega_\Sigma)$ that are degree n covers of the 0-section Σ. We define the elliptic locus $B^{\mathrm{ell}} \subset B$ to be the open subset formed by integral spectral curves. The results of [47, Theorem 0.3 and Section 4] thus establish the motivic decomposition for the restricted Hitchin system

$$f^{\mathrm{ell}} : M_{n,d}^{\mathrm{ell}} \to B^{\mathrm{ell}}.$$

As we commented in Section 1.4.2, at the level of algebraic cycles it is not easy to extend orthogonal projectors from B^{ell} to B. A plausible approach is to extend first Arinkin's Fourier-Mukai transform from $M_{n,d}^{\mathrm{ell}}$ to the total space $M_{n,d}$, and then extend the Fourier theory of relative Chow motives of [47] as well. Even if we are able to achieve this, a potential difficulty is the complexity of the supports (see Remark 2.1 below) which makes it hard to calculate the homological realization of the projectors.

Instead, our approach in this paper is to first construct orthogonal projectors for certain parabolic Hitchin moduli space which we review in the following sections; then we show that these projectors induce the desired orthogonal projectors of the original Hitchin system. This path of reduction using parabolic Hitchin moduli spaces was applied in proving the P=W conjecture: it was first used in the proof of Hausel-Mellit-Minets-Schiffmann [30], and was later also used in the proof of Maulik-Shen-Yin [47, Section 5.4].

REMARK 2.1. – We can also consider the moduli space of stable $\mathfrak{D}$-twisted Higgs bundles

$$(\mathcal{E}, \theta), \quad \theta : \mathcal{E} \to \mathcal{E} \otimes \omega_\Sigma(\mathfrak{D})$$

as in [16] with $\mathfrak{D} \geq 0$ an effective divisor. The case of $\mathfrak{D} = 0$ recovers the Hitchin system (11) we considered above. We note that the decomposition theorem for the Hitchin system behaves very differently when $\mathfrak{D} = 0$ and $\mathfrak{D} > 0$. For $\mathfrak{D} > 0$, Chaudouard-Laumon [16] showed that every simple summand that appeared in the decomposition theorem has full support B. This full support property has been generalized to singular moduli spaces without coprime assumption on n and d [45] and has many applications [44, 28, 35, 46].

However, when $D = 0$, the supports for the Hitchin system are very different. In [13], the authors classified all the supports over an open subset of the Hitchin base formed by *reduced* spectral curves. However it is not well understood whether there are supports outside this open subset. A question raised by Mauri-Migliorini [48, Question 1.5] asks if the full support property holds for the singular Hitchin moduli space of rank n and degree $d = 0$. In Section 5.3.5, we will discuss briefly how to extend the results of this paper to the $\mathfrak{D}$-twisted case.

2.2. Parabolic Higgs bundles

In this section, we review the parabolic Hitchin moduli space (*c.f.* [30, Section 8.4]). Our purpose is to describe a smooth family of moduli spaces, connecting the moduli of strongly parabolic Higgs bundles with nilpotent residue (which has complicated spectral curves) to a certain moduli space of parabolic Higgs bundles with general residue (which only has integral spectral curves).

We fix a point $p \in \Sigma$. Denote by M^{par} the moduli space of stable parabolic Higgs bundles $(\mathcal{E}, \theta, F_p)$, where $(\mathcal{E}, \theta)$ is a meromorphic Higgs bundle

$$(\mathcal{E}, \theta), \quad \theta : \mathcal{E} \to \mathcal{E} \otimes \omega_\Sigma(p), \quad \mathrm{rk}(\mathcal{E}) = n, \quad \deg(\mathcal{E}) = d,$$

and F_p is a complete flag of $\mathcal{E}$ over the point p such that the residue $\mathrm{res}_p(\theta)$ of the Higgs field over the point p preserves the flag F_p. The stability condition here requires that any parabolic Higgs subbundle has a smaller slope. We have the corresponding Hitchin map which is proper over an affine space

$$f^{\mathrm{par}} : M^{\mathrm{par}} \to B^{\mathrm{par}} := \bigoplus_{i=1}^{n} H^0\left(\Sigma, \omega_\Sigma(p)^{\otimes i}\right).$$

We record the following lemma which is a direct consequence of the Riemann-Roch formula.

LEMMA 2.2. – *We have*

$$H^0(\Sigma, \omega_\Sigma) = H^0(\Sigma, \omega_\Sigma(p)).$$

Since $\mathrm{res}_p(\theta)$ preserves a complete flag, its eigenvalues naturally lie in an affine space

$$\mathbb{A}^n := \left\{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{A}^1\right\},$$

and Lemma 2.2 ensures that the sum of the eigenvalues is always 0; in other words, the parameter space of (ordered) eigenvalues of $\mathrm{res}_p(\theta)$ lies in a codimension 1 hyperplane

$$\mathcal{A} = \left\{(\lambda_1, \ldots, \lambda_n) \mid \sum_i \lambda_i = 0\right\} \subset \mathbb{A}^n.$$

As a result, the moduli space admits a natural morphism

$$\chi_p : M^{\mathrm{par}} \to \mathcal{A} \simeq \mathbb{A}^{n-1}, \tag{12}$$

which is in fact smooth; see e.g., [39, Corollary 2.11]. Each closed fiber is a moduli of stable parabolic Higgs bundles with prescribed eigenvalues at p; this is a nonsingular symplectic variety.

On the other hand, the Hitchin base B^{par} only remembers the *unordered* eigenvalues of $\mathrm{res}_p(\theta)$ through its quotient by

$$B_0^{\mathrm{par}} = \bigoplus_{i=1}^{n} H^0\left(\Sigma, \omega_\Sigma^{\otimes i}((i-1)p)\right) \subset B^{\mathrm{par}},$$

see [42, Sections 3 and 4]. There is a natural identification

$$\mathcal{A} /\!\!/ \mathfrak{S}_n \simeq \overline{B}^{\mathrm{par}} := B^{\mathrm{par}}/B_0^{\mathrm{par}}.$$

We consider the *ordered* Hitchin map

$$h : M^{\mathrm{par}} \to W := \mathcal{A} \times_{\overline{B}^{\mathrm{par}}} B^{\mathrm{par}},$$

which can be thought of as a family of fibrations over $\mathcal{A}$:

$$\begin{array}{ccc} M^{\mathrm{par}} & \xrightarrow{\quad h \quad} & W \\ & \searrow \quad \swarrow & \\ & \mathcal{A}. & \end{array}$$

For our purposes, we take a general line $T \simeq \mathbb{A}^1 \subset \mathcal{A}$ passing through the origin and restrict the family (12) over T:

$$\chi_p : M_T^{\mathrm{par}} \to T.$$

The geometry of this family is described by [30, Sections 8.3 and 8.4] and [47, Section 5.4.2], connecting two types of Hitchin moduli spaces. We review in the following some details which are relevant for our purposes.

(a) Let $\eta \in T$ be a general point which represents n *distinct* eigenvalues satisfying the condition that no proper subset of the eigenvalues has sum 0. Let $C_\eta \to W_\eta$ be the universal spectral curve with fixed eigenvalues η at p. By the generality of η and Lemma 2.2, every curve in the family $C_\eta \to W_\eta$ is integral. The fiber $M_\eta^{\mathrm{par}} := \chi_p^{-1}(\eta)$ therefore admits a compactified Jacobian fibration

$$h_\eta : M_\eta^{\mathrm{par}} \to W_\eta.$$

Let $T^\circ \subset T$ be the Zariski open subset such that the spectral curves in W_η are integral for any $\eta \in T^\circ$. We thus obtain a family of compactified Jacobian fibrations over T°:

$$\begin{array}{ccc} M_{T^\circ}^{\mathrm{par}} & \xrightarrow{\quad h^\circ \quad} & W_{T^\circ} \\ & \searrow \quad \swarrow & \\ & T^\circ. & \end{array}$$

(b) The central fiber $M_0^{\mathrm{par}} := \chi_p^{-1}(0)$ together with its associated Hitchin system

$$h_0 : M_0^{\mathrm{par}} \to W_0$$

is a classical object, known as the moduli of stable strongly parabolic Higgs bundles. The variety M_0^{par} parameterizes triples $(\mathcal{E}, \theta, F_p)$ with similar condition as for stable parabolic Higgs bundle, but the main change is that the residue of the Higgs field preserves the flag in a stronger sense:

$$\mathrm{res}_p(\theta) : F_{p,j} \to F_{p,j+1}.$$

In other words, M_0^{par} is the moduli of stable parabolic Higgs bundles with nilpotent residue.

In conclusion, by specializing over T, we are able to connect the geometry of a compactified Jacobian fibration to the Hitchin system associated with the moduli of strongly parabolic Higgs bundles:

$$h_\eta : M_\eta^{\mathrm{par}} \to W_\eta \quad \rightsquigarrow \quad h_0 : M_0^{\mathrm{par}} \to W_0. \tag{13}$$

We now connect the latter to the more classical Hitchin system $f : M_{n,d} \to B$ via a correspondence following [30, Section 8.6].

2.3. Correspondences

We consider the closed subvariety parameterizing parabolic Higgs bundles with 0-residue at p,

$$\widetilde{M}_0 = \{(\mathcal{E}, \theta, F_p) \mid \mathrm{res}_p(\theta) = 0\} \subset M_0^{\mathrm{par}} = \{(\mathcal{E}, \theta, F_p) \mid \mathrm{res}_p(\theta) \text{ is nilpotent}\}. \tag{14}$$

In particular, every parabolic Higgs bundle in $\widetilde{M}_0$ has no pole at p. Hence there is a well-defined forgetful map:

$$\widetilde{M}_0 \to M_{n,d}, \quad (\mathcal{E}, \theta, F_p) \mapsto (\mathcal{E}, \theta), \tag{15}$$

where every closed fiber is isomorphic to a flag variety Fl_n parameterizing all possible choices of flags.

We summarize the discussion above in the diagram

$$\begin{array}{ccc} \widetilde{M}_0 & \xrightarrow{\ \iota\ } & M_0^{\mathrm{par}} \\ \downarrow{\scriptstyle \pi} & & \\ M_{n,d}, & & \end{array} \tag{16}$$

where ι is the closed embedding (14) and π is the smooth projection (15). All the three moduli spaces in (16) map to a common base W_0 with all the induced natural diagrams commuting: the map from M_0^{par} is the parabolic Hitchin map h_0, the map from $\widetilde{M}_0$ is the induced map, and the map from $M_{n,d}$ is the Hitchin map $f : M_{n,d} \to B$ composed with the natural inclusion of the linear subspace $B \subset W_0$.

Our purpose is to construct certain algebraic cycles on $M_{n,d} \times_B M_{n,d}$ which serve as orthogonal projectors. The diagram (16) allows us to construct cycles for $M_{n,d}$ from M_0^{par}, i.e., we have

$$\Gamma : \mathrm{CH}_*(M_0^{\mathrm{par}} \times_{W_0} M_0^{\mathrm{par}}) \to \mathrm{CH}_*(M_{n,d} \times_B M_{n,d}) \tag{17}$$

given by the formula

$$\Gamma(\alpha) := {}^{\mathrm{t}}[\widetilde{M}_0] \circ \alpha \circ [\widetilde{M}_0].$$

Here we view $[\widetilde{M}_0]$ to be a correspondence from $M_{n,d}$ to M_0^{par} via (16) and ${}^{\mathrm{t}}[\widetilde{M}_0]$ to be its transpose. The change of indices of the Chow groups through Γ is given by the dimension formula:

$$\mathrm{codim}(\iota) = \mathrm{rel.dim}(\pi) = \binom{n}{2};$$

indeed, each closed fiber of π is a flag variety whose dimension is $\binom{n}{2}$. In particular, Γ sends $\mathrm{CH}_{\dim M_0^{\mathrm{par}}}(M_0^{\mathrm{par}} \times_{W_0} M_0^{\mathrm{par}})$ to $\mathrm{CH}_{\dim M_{n,d}}(M_{n,d} \times_B M_{n,d})$.

The following lemma gives a sufficient condition for $\Gamma(\alpha)$ to vanish.

LEMMA 2.3. – *Suppose we are given a flat, proper surjective morphism $p : Z \to W_0$ with Z irreducible and nonsingular and that there exists a factorization of α*

$$\alpha := \alpha_2 \circ \alpha_1 \in \mathrm{CH}_{\dim M_0^{\mathrm{par}}}(M_0^{\mathrm{par}} \times_{W_0} M_0^{\mathrm{par}}),$$

such that either

$$\alpha_1 \in \mathrm{CH}_{>\dim(M_0^{\mathrm{par}} \times_{W_0} Z) - \binom{n}{2}}(M_0^{\mathrm{par}} \times_{W_0} Z) \text{ or } \alpha_2 \in \mathrm{CH}_{>\dim(M_0^{\mathrm{par}} \times_{W_0} Z) - \binom{n}{2}}(Z \times_{W_0} M_0^{\mathrm{par}}).$$

Then we have

$$\Gamma(\alpha) = 0 \in \mathrm{CH}_{\dim M_{n,d}}(M_{n,d} \times_{W_0} M_{n,d}) = \mathrm{CH}_{\dim M_{n,d}}(M_{n,d} \times_B M_{n,d}).$$

Proof. – We prove the vanishing under the first hypothesis; the argument under the second hypothesis is identical. If we consider the composition $\alpha_1 \circ [\widetilde{M_0}] \in \mathrm{CH}_*(M_{n,d} \times_{W_0} Z)$, the assumption on α_1 implies that this lives in degree strictly larger than

$$\dim(M_0^{\mathrm{par}} \times_{W_0} Z) - 2\binom{n}{2}.$$

The assumption on p implies that this bound equals

$$\dim M_0^{\mathrm{par}} + \dim Z - \dim W_0 - 2\binom{n}{2} = \dim M_{n,d} + \dim Z - \dim W_0,$$

which is the dimension of $M_{n,d} \times_{W_0} Z$. As a result, we have the vanishing of $\alpha_1 \circ [\widetilde{M_0}]$ so $\Gamma(\alpha)$ vanishes as well. □

We conclude this section by giving an overview of the rest of the paper. Our strategy is to first construct projectors for compactified Jacobian fibrations and show that these algebraic cycles satisfy the desired properties as in Theorems 1.4 and 1.5 using the techniques of [47]. We will discuss this in Section 3, which solves the problems for $h_\eta : M_\eta^{\mathrm{par}} \to W_\eta$ of Section 2.2(a). Then applying the specialization and the correspondence Γ, we obtain algebraic cycles for $M_{n,d}$; we further apply a Springer theory argument to show that these cycles form orthogonal projectors as desired. This is treated in Section 4. Finally, in Section 5, we show that the projectors obtained via Γ have the correct homological realization under the Corti-Hanamura functor.

3. Projectors for compactified Jacobians

In this section, we first introduce the notion of motivic perverse filtrations. Then we treat the case of compactified Jacobians associated with locally planar curves, where our main tool is the Fourier theory developed in [47]. For our purpose, we need a slight extension of the results of [47] in Section 3.3. Then we apply these results to the parabolic Hitchin system of Section 2.2(a), which further allows us to prove analogues of Theorems 1.1, 1.4, and 1.5 for $h_0 : M_0^{\mathrm{par}} \to W_0$.

3.1. Motivic background

We begin by reviewing the motivic background necessary for our main results, and by introducing the notion of a *motivic perverse filtration*.

3.1.1. *Relative Chow motives.* – The theory of relative Chow motives of Corti-Hanamura [19] is built to be compatible with the decomposition theorem, adapts well to non-proper bases, and admits natural Chow/homological realizations [27]. We briefly recall the theory below, and refer to [47, Section 2.2] for more details.

We work over a nonsingular base variety B. Recall that the group of degree k relative correspondences between two nonsingular proper B-schemes X, Y with Y irreducible is

$$\mathrm{Corr}_B^k(X,Y) := \mathrm{CH}_{\dim Y-k}(X \times_B Y).$$

For reducible Y, we decompose $Y = \sqcup_\alpha Y_\alpha$ and define the group of correspondences accordingly. Compositions of relative correspondences are defined via refined intersection theory. The category of relative Chow motives $\mathrm{CHM}(B)$ consists of objects triples $(X, \mathfrak{p}, m)$ where X is a nonsingular proper B-scheme, $\mathfrak{p} \in \mathrm{Corr}_B^0(X,X)$ is a projector, and $m \in \mathbb{Z}$. For example, the motive of X is

$$h(X) := (X, [\Delta_{X/B}], 0),$$

where $\Delta_{X/B}$ is the relative diagonal. Morphisms between two motives $M = (X, \mathfrak{p}, m)$, $N = (Y, \mathfrak{q}, n)$ are

$$\mathrm{Hom}_{\mathrm{CHM}(B)}(M,N) := \mathfrak{q} \circ \mathrm{Corr}_B^{n-m}(X,Y) \circ \mathfrak{p}.$$

Note that in [47, Section 2.3.3], we also introduced relative binary correspondences

$$\mathrm{Corr}_B^k(X,Y;Z) := \mathrm{CH}_{\dim Z-k}(X \times_B Y \times_B Z),$$

where X, Y, Z are nonsingular proper B-schemes. Binary morphisms from $M = (X, \mathfrak{p}, m)$, $N = (Y, \mathfrak{q}, n)$ to $P = (Z, \mathfrak{r}, p)$ are given by

$$\mathfrak{r} \circ \mathrm{Corr}_B^{p-m-n}(X,Y;Z) \circ (\mathfrak{p} \times \mathfrak{q}).$$

For example, the motivic cup-product

$$\cup : h(X) \times h(X) \to h(X)$$

is induced by the class of the relative small diagonal

$$[\Delta_{X/B}^{\mathrm{sm}}] \in \mathrm{Corr}_B^0(X,X;X);$$

see [47, Section 2.3.3] for more details. Finally, the degree k Chow group of $M = (X, \mathfrak{p}, m)$ is

$$\mathrm{CH}^k(M) := \mathrm{Hom}_{\mathrm{CHM}(B)}(h(B/B), M(k)),$$

where $M(k) := (X, \mathfrak{p}, m+k)$ is the k-th Tate twist of M.

3.1.2. *Motivic perverse filtrations.* – There is a realization functor

$$\mathrm{CHM}(B) \to D_c^b(B), \tag{18}$$

which sends the motive $h(X)$ for $f : X \to B$ to the direct image complex $Rf_*\mathbb{Q}_X$. The latter admits a natural (sheaf-theoretic) perverse filtration induced by the perverse truncation functor. We introduce the notion of a *motivic perverse filtration* lifting the perverse truncation of $Rf_*\mathbb{Q}_X \in D_c^b(X)$ to the object $h(X) \in \mathrm{CHM}(B)$. Then we prove that, although a motivic perverse filtration is *a priori* weaker than a motivic decomposition in the sense of Corti-Hanamura [19], they are in fact equivalent.

Let $f : X \to B$ be a proper morphism with X irreducible and nonsingular, and let d_f be the defect of semismallness $\dim X \times_B X - \dim X$.

Definition 3.1. – We say that a sequence of motives

$$P_k h(X) = (X, \mathfrak{p}_k, 0), \quad 0 \le k \le 2d_f,$$

which are all summands of $h(X)$, form a *motivic perverse filtration* for f, if

(a) (Termination) $P_{2d_f} h(X) = h(X)$;

(b) (Realization) under the Corti-Hanamura functor (18) the natural inclusion

$$P_\bullet h(X) \hookrightarrow h(X)$$

specializes to the natural inclusion induced by the perverse truncation functor

$${}^{\mathfrak{p}}\tau_{\le \bullet + (\dim X - d_f)} Rf_*\mathbb{Q}_X \hookrightarrow Rf_*\mathbb{Q}_X;$$

(c) (Semi-orthogonality) for any k we have the relation

$$\mathfrak{p}_{k+1} \circ \mathfrak{p}_k = \mathfrak{p}_k.$$

We note that (c) together with an easy induction implies

$$\mathfrak{p}_l \circ \mathfrak{p}_k = \mathfrak{p}_k, \quad k < l. \tag{19}$$

Indeed, we have

$$\mathfrak{p}_l \circ \mathfrak{p}_k = \mathfrak{p}_l \circ \mathfrak{p}_{k+1} \circ \mathfrak{p}_k = \mathfrak{p}_l \circ \mathfrak{p}_{l-1} \circ \cdots \circ \mathfrak{p}_{k+1} \circ \mathfrak{p}_k = \mathfrak{p}_{l-1} \circ \cdots \circ \mathfrak{p}_{k+1} \circ \mathfrak{p}_k = \mathfrak{p}_k.$$

On the other hand, this does not mean that $P_k h(X)$ is a summand of $P_l h(X)$ if $k < l$. So if we define directly

$$\mathfrak{r}_k := \mathfrak{p}_k - \mathfrak{p}_{k-1},$$

they are only semi-orthogonal:

$$\mathfrak{r}_l \circ \mathfrak{r}_k = 0, \quad k < l,$$

and we lose control of the other half of the orthogonality

$$\mathfrak{r}_l \circ \mathfrak{r}_k \overset{?}{=} 0, \quad k \ge l.$$

In particular, the $\mathfrak{r}_k$ are not necessarily projectors. However, the following proposition in [47] shows that the existence of a motivic perverse filtration is equivalent to the existence of a motivic decomposition.

PROPOSITION 3.2 (Motivic perverse filtration = motivic decomposition). –
A motivic perverse filtration induces a motivic decomposition

$$h(X) = \bigoplus_{i=0}^{2d_f} h_i(X) \in \mathrm{CHM}(B).$$

Proof. – The proof was given in the [47, Section 2.5.5]; we include it here for completeness. Assume that the motives $P_k h(X) = (X, \mathfrak{p}_k, 0)$ induce a motivic perverse filtration. We consider the modification of projectors:

$$\widetilde{\mathfrak{p}}_k := \mathfrak{p}_k \circ \cdots \circ \mathfrak{p}_{2d_f}, \quad \overline{\mathfrak{p}}_k := \widetilde{\mathfrak{p}}_k - \widetilde{\mathfrak{p}}_{k-1}.$$

Using (19), it is straightforward to verify that both $\widetilde{\mathfrak{p}}_k$ and $\overline{\mathfrak{p}}_k$ are projectors satisfying the relations

$$\widetilde{\mathfrak{p}}_l \circ \widetilde{\mathfrak{p}}_k = \widetilde{\mathfrak{p}}_k \circ \widetilde{\mathfrak{p}}_l = \widetilde{\mathfrak{p}}_k, \quad k < l.$$

In particular, we have

$$\widetilde{P}_k h(X) = \widetilde{P}_{k-1} h(X) \oplus h_k(X) \in \mathrm{CHM}(B)$$

with

$$\widetilde{P}_k h(X) := (X, \widetilde{\mathfrak{p}}_k, 0), \quad h_k(X) := (X, \overline{\mathfrak{p}}_k, 0).$$

We see from (b) that both motives $\widetilde{P}_k h(X)$ and $P_k h(X)$ have the same realization under (18), which is the one given by the perverse truncation functor. In particular, the homological realization of the projector $\overline{\mathfrak{p}}_k$ is taking the corresponding perverse cohomology of the complex $Rf_*\mathbb{Q}_X$. This finishes the proof of the proposition. □

In the following, we further extend the equivalence between motivic perverse filtrations and motivic decompositions of Proposition 3.2 to isomorphism classes.

Assume that

$$f : X \to B, \quad f' : X' \to B$$

admit motivic perverse filtrations

$$P_k h(X) = (X, \mathfrak{p}_k, 0), \quad P_k h(X') = (X', \mathfrak{p}'_k, 0), \quad 0 \le k \le 2d_f = 2d_{f'}. \tag{20}$$

DEFINITION 3.3. – We say that an isomorphism of relative Chow motives

$$\mathfrak{C} : h(X) \xrightarrow{\simeq} h(X'), \qquad \mathfrak{C}^{-1} : h(X') \xrightarrow{\simeq} h(X)$$

with

$$\mathfrak{C} \in \mathrm{Corr}^0_B(X, X'), \qquad \mathfrak{C}^{-1} \in \mathrm{Corr}^0_B(X', X)$$

preserves the motivic perverse filtrations (20), if for any k we have

$$\mathfrak{q}'_{k+1} \circ \mathfrak{C} \circ \mathfrak{p}_k = 0, \quad \mathfrak{q}_{k+1} \circ \mathfrak{C}^{-1} \circ \mathfrak{p}'_k = 0. \tag{21}$$

Here

$$\mathfrak{q}_{k+1} := [\Delta_{X/B}] - \mathfrak{p}_k, \quad \mathfrak{q}'_{k+1} := [\Delta_{X'/B}] - \mathfrak{p}'_k.$$

The following proposition shows that, for two relative Chow motives, an isomorphism preserving motivic filtrations automatically yields isomorphic motivic decompositions as well.

PROPOSITION 3.4. – *With the notation as above, assume that there is an isomorphism between the relative Chow motives $h(X)$ and $h(X')$ preserving the motivic perverse filtrations* (20)*. Then the following holds:*

(a) *For any k we have an isomorphism*

$$P_k h(X) \simeq P_k h(X') \in \mathrm{CHM}(B)$$

compatible with the inclusions $P_k h(X) \hookrightarrow h(X)$ and $P_k h(X') \hookrightarrow h(X')$.

(b) *The motivic decompositions of $h(X), h(X')$ given by Proposition 3.2 are isomorphic, i.e., for any k we have an isomorphism*

$$h_k(X) \simeq h_k(X') \in \mathrm{CHM}(B).$$

Proof. – We first prove (a). Assume that the correspondences $\mathfrak{C}, \mathfrak{C}^{-1}$ induce an isomorphism between $h(X), h(X')$ preserving the perverse filtrations. We claim that the morphisms between the relative Chow motives

$$\mathfrak{p}'_k \circ \mathfrak{C} \circ \mathfrak{p}_k : P_k h(X) \to P_k h(X'), \quad \mathfrak{p}_k \circ \mathfrak{C}^{-1} \circ \mathfrak{p}'_k : P_k h(X') \to P_k h(X)$$

are inverse to each other. In fact, we have

$$\begin{aligned} \mathfrak{p}_k = \mathfrak{p}_k \circ \mathfrak{C}^{-1} \circ \mathfrak{C} \circ \mathfrak{p}_k &= \mathfrak{p}_k \circ \mathfrak{C}^{-1} \circ (\mathfrak{p}'_k + \mathfrak{q}'_{k+1}) \circ \mathfrak{C} \circ \mathfrak{p}_k \\ &= \mathfrak{p}_k \circ \mathfrak{C}^{-1} \circ \mathfrak{p}'_k \circ \mathfrak{C} \circ \mathfrak{p}_k, \end{aligned}$$

where we used $\mathfrak{q}'_{k+1} \circ \mathfrak{C} \circ \mathfrak{p}_k = 0$ in the last equation. This proves

$$\mathfrak{p}_k \circ \mathfrak{C}^{-1} \circ \mathfrak{p}'_k \circ \mathfrak{C} \circ \mathfrak{p}_k = \mathfrak{p}_k;$$

the other identity is parallel. To show that this isomorphism is compatible with $\mathfrak{C}$ and the inclusions into $h(X)$ and $h(X')$, it suffices to show that

$$\mathfrak{C} \circ \mathfrak{p}_k = \mathfrak{p}'_k \circ (\mathfrak{p}'_k \circ \mathfrak{C} \circ \mathfrak{p}_k),$$

which again follows via the condition that $\mathfrak{C}$ preserves the motivic filtration. We have completed the proof of (a).

Now we prove (b); we follow the notation in the proof of Proposition 3.2. As in (a), it suffices to show that

$$(\widetilde{\mathfrak{p}}_k - \widetilde{\mathfrak{p}}_{k-1}) \circ \mathfrak{C}^{-1} \circ (\widetilde{\mathfrak{p}}'_k - \widetilde{\mathfrak{p}}'_{k-1}) \circ \mathfrak{C} \circ (\widetilde{\mathfrak{p}}_k - \widetilde{\mathfrak{p}}_{k-1}) = \widetilde{\mathfrak{p}}_k - \widetilde{\mathfrak{p}}_{k-1}; \tag{22}$$

the other identity follows from a parallel argument. Expanding the left-hand side of (22), we obtain 8 terms, each of which is of the form

$$\pm \widetilde{\mathfrak{p}}_{i_1} \circ \mathfrak{C}^{-1} \circ \widetilde{\mathfrak{p}}'_{i_2} \circ \mathfrak{C} \circ \widetilde{\mathfrak{p}}_{i_3}, \quad i_1, i_2, i_3 \in \{k-1, k\}.$$

We claim that

$$\widetilde{\mathfrak{p}}_{i_1} \circ \mathfrak{C}^{-1} \circ \widetilde{\mathfrak{p}}'_{i_2} \circ \mathfrak{C} \circ \widetilde{\mathfrak{p}}_{i_3} = \widetilde{\mathfrak{p}}_{\min\{i_1, i_2, i_3\}}. \tag{23}$$

Since at least two of the three indices i_1, i_2, i_3 are equal, without loss of generality we treat here the case $i_2 = i_3$. By (19) and the first equation of (21), we have

$$\mathfrak{C} \circ \mathfrak{p}_k = \mathfrak{p}'_l \circ \mathfrak{C} \circ \mathfrak{p}_k, \quad k \le l,$$

which further yields

$$\begin{aligned}\widetilde{\mathfrak{p}}'_{i_2}\circ\mathfrak{C}\circ\widetilde{\mathfrak{p}}_{i_2} &= \mathfrak{p}'_{i_2}\circ\cdots\circ\mathfrak{p}'_{2d_f}\circ\mathfrak{C}\circ\mathfrak{p}_{i_2}\circ\cdots\circ\mathfrak{p}_{2d_f}\\ &= \mathfrak{C}\circ\mathfrak{p}_{i_2}\circ\cdots\circ\mathfrak{p}_{2d_f}\\ &= \mathfrak{C}\circ\widetilde{\mathfrak{p}}_{i_2}.\end{aligned}$$

Consequently, (23) follows:

$$\begin{aligned}\widetilde{\mathfrak{p}}_{i_1}\circ\mathfrak{C}^{-1}\circ\widetilde{\mathfrak{p}}'_{i_2}\circ\mathfrak{C}\circ\widetilde{\mathfrak{p}}_{i_2} &= \widetilde{\mathfrak{p}}_{i_1}\circ\mathfrak{C}^{-1}\circ\mathfrak{C}\circ\widetilde{\mathfrak{p}}_{i_2}\\ &= \widetilde{\mathfrak{p}}_{i_1}\circ\widetilde{\mathfrak{p}}_{i_2}\\ &= \widetilde{\mathfrak{p}}_{\min\{i_1,i_2\}}.\end{aligned}$$

Finally, (22) is deduced immediately from (23), which completes the proof of (b). □

In many cases, it is more natural to work with a motivic perverse filtration, rather than a motivic decomposition. This is because they carry the same information, but the extra complexity of obtaining the orthogonal projectors from the semi-orthogonal projectors, as in the proof of Proposition 3.2, is formal and not essential. This in particular is the case for compactified Jacobian fibrations [47, Section 2.5.5] as we will also review below, where the semi-orthogonal projectors are of better form; see Theorem 3.5.

3.2. Compactified Jacobians

Throughout this section, we work with the general setting that $C \to B$ is a flat family of integral projective curves of arithmetic genus g with planar singularities over an irreducible base B; we will specialize to the case of spectral curves in Section 3.4. We assume that the total space C is nonsingular with a multi-section

$$D \subset C \to B$$

of degree r which is finite and flat over B. For convenience, we also assume that for any d, the degree d compactified Jacobian $\overline{J}^d_C$ is a quasi-projective variety. This is, for example, satisfied when $C \to B$ arises as a linear system of a nonsingular surface, where the quasi-projectivity follows from the fact that the compactified Jacobian arises as the moduli space of certain semistable torsion sheaves on the surface; see e.g., [38]. We further assume that $\overline{J}^d_C$ is nonsingular. In order to work with the universal family, we also need to consider the moduli stack of the degree d compactified Jacobian, which is a $\mathbb{G}_m$-gerbe over $\overline{J}^d_C$. Following [47, Section 4.2], we rigidify this stack by imposing the condition that the universal family is *trivialized* along the multi-section; this gives a Deligne-Mumford stack which is a μ_r-gerbe over the scheme:

$$\overline{\mathcal{J}}^d_C \to \overline{J}^d_C;$$

see [47, Proposition 4.1]. The rigidification condition we imposed implies that there is a universal sheaf $\mathcal{F}^d$ over $C \times_B \overline{\mathcal{J}}^d_C$ satisfying

$$\det\left(p_{J*}\left(\mathcal{F}^d|_{\overline{\mathcal{J}}^d_C \times_C D}\right)\right) \simeq \mathcal{O}_{\overline{\mathcal{J}}^d_C} \in \mathrm{Pic}\left(\overline{\mathcal{J}}^d_C\right).$$

Here p_J is the projection to the compactified Jacobian factor.

Now we consider two integers d, e. Arinkin's work [3] shows that the universal sheaves $\mathcal{F}^d$, $\mathcal{F}^e$ induce a pair of Fourier-Mukai kernels

$$\mathcal{P}_{e,d} \in D^{\mathrm{b}}\mathrm{Coh}\left(\overline{\mathcal{J}}^e_C \times \overline{\mathcal{J}}^d_C\right), \quad \mathcal{P}^{-1}_{e,d} \in D^{\mathrm{b}}\mathrm{Coh}\left(\overline{\mathcal{J}}^d_C \times \overline{\mathcal{J}}^e_C\right). \tag{24}$$

The associated Fourier-Mukai transforms are equivalences of categories

$$\mathrm{FM}_{\mathcal{P}_{e,d}} : D^{\mathrm{b}}\mathrm{Coh}(\overline{\mathcal{J}}^e_C)_{(-d)} \xrightarrow{\simeq} D^{\mathrm{b}}\mathrm{Coh}(\overline{\mathcal{J}}^d_C)_{(e)}, \quad \mathrm{FM}_{\mathcal{P}^{-1}_{e,d}} = \mathrm{FM}^{-1}_{\mathcal{P}_{e,d}},$$

where $D^{\mathrm{b}}\mathrm{Coh}(\overline{\mathcal{J}}^m_C)_{(k)}$ is the isotypic category consisting of objects for which the action of μ_r on the fibers is given by the character $\lambda \mapsto \lambda^k$ of μ_r. The objects (24) induce (Chow-theoretic) Fourier transforms

$$\mathfrak{F}_{e,d} := \mathrm{td}\left(-T_{\overline{\mathcal{J}}^e_C \times_B \overline{\mathcal{J}}^d_C}\right) \cap \tau(\mathcal{P}_{e,d}) \in \mathrm{CH}_*(\overline{\mathcal{J}}^e_C \times_B \overline{\mathcal{J}}^d_C) = \mathrm{CH}_*(\overline{J}^e_C \times_B \overline{J}^d_C),$$

$$\mathfrak{F}^{-1}_{e,d} := \mathrm{td}(-(\pi_d \times_B \pi_e)^* T_B) \cap \tau(\mathcal{P}^{-1}_{e,d}) \in \mathrm{CH}_*(\overline{\mathcal{J}}^d_C \times_B \overline{\mathcal{J}}^e_C) = \mathrm{CH}_*(\overline{J}^d_C \times_B \overline{J}^e_C).$$

Here $\pi_\bullet$ is the natural projection to the base B, $T_{\overline{\mathcal{J}}^e_C \times_B \overline{\mathcal{J}}^d_C}$ is the virtual tangent bundle, $\tau(-)$ is the tau-class used in the Riemann-Roch theorem of Baum-Fulton-MacPherson [5], and we identify the (rational) Chow groups of the coarse moduli spaces and the gerbes; we refer to [47, Section 4] for more details.

The correspondences $\mathfrak{F}_{e,d}, \mathfrak{F}^{-1}_{e,d}$ are of mixed degrees, and we have decompositions

$$\mathfrak{F}_{e,d} = \sum_{i \geq 0} (\mathfrak{F}_{e,d})_i, \qquad (\mathfrak{F}_{e,d})_i \in \mathrm{Corr}^{i-g}_B(\overline{J}^e_C, \overline{J}^d_C),$$

$$\mathfrak{F}^{-1}_{e,d} = \sum_{i \geq 0} (\mathfrak{F}^{-1}_{e,d})_i, \qquad (\mathfrak{F}^{-1}_{e,d})_i \in \mathrm{Corr}^{i-g}_B(\overline{J}^d_C, \overline{J}^e_C).$$

For a fixed d, in order to define a motivic perverse filtration associated with $\pi_d : \overline{J}^d_C \to B$ using the Fourier transform, we need to choose a degree e, whose associated compactified Jacobian fibration will serve as the "dual" side; see Step 1 of [47, Section 4.4]. In this paper, we choose this degree uniformly to be 1; this choice will be important for applying the Abel-Jacobi map $C \to \overline{J}^1_C$ to build a connection between the Fourier transform and the normalized Chern classes of a universal bundle for a Hitchin-type moduli space. We define

$$\mathfrak{p}_k := \sum_{i \leq k} (\mathfrak{F}_{1,d})_i \circ (\mathfrak{F}^{-1}_{1,d})_{2g-i} \in \mathrm{Corr}^0_B(\overline{J}^d_C, \overline{J}^d_C). \tag{25}$$

The following theorem proven in [47] confirmed the motivic decomposition conjecture for the compactified Jacobian fibration associated with $C \to B$, with the projectors constructed above.

THEOREM 3.5 ([47, Corollary 4.6(i)]). – *For every $0 \leq k \leq 2g$, the self-correspondence $\mathfrak{p}_k$ is a projector for $\pi_d : \overline{J}^d_C \to B$, and the motives*

$$P_k h(\overline{J}^d_C) := (\overline{J}^d_C, \mathfrak{p}_k, 0) \in \mathrm{CHM}(B), \quad 0 \leq k \leq 2g$$

form a motivic perverse filtration in the sense of Definition 3.1. Moreover, every submotive $P_k h(\overline{J}^d_C)$ admits a canonical orthogonal complement $Q_{k+1} h(\overline{J}^d_C)$:

$$h(\overline{J}^d_C) = P_k h(\overline{J}^d_C) \oplus Q_{k+1} h(\overline{J}^d_C), \qquad Q_{k+1} h(\overline{J}^d_C) = (\overline{J}^d_C, \mathfrak{q}_{k+1}, 0)$$

with

$$\mathfrak{q}_{k+1} := [\Delta_{\overline{J}^d_C/B}] - \mathfrak{p}_k = \sum_{i \geq k+1} (\mathfrak{F}_{1,d})_i \circ (\mathfrak{F}_{1,d}^{-1})_{2g-i}.$$

The orthogonal complements $Q_\bullet h(\overline{J}^d_C)$ allow us to directly formulate the condition that a correspondence to $h(\overline{J}^d_C)$ factors through a certain piece of the motivic perverse filtration; we refer to the statements in Theorem 3.6 below as examples.

Now we consider two natural operations on the relative Chow motive $h(\overline{J}^d_C)$. The first is induced by the correspondence $(\mathfrak{F}_{1,d})_k$:

$$(\mathfrak{F}_{1,d})_k : h(\overline{J}^1_C)(g-k) \to h(\overline{J}^d_C). \tag{26}$$

The second is the multiplicative structure given by cup-product:

$$\cup : h(\overline{J}^d_C) \times h(\overline{J}^d_C) \to h(\overline{J}^d_C).$$

The following theorem shows the compatibility between the motivic perverse filtration of Theorem 3.5 and the two operations above.

THEOREM 3.6. – *The following hold for* $\pi_d : \overline{J}^d_C \to B$.

(a) *The morphism* $(\mathfrak{F}_{1,d})_k$ *in* (26) *factors through*

$$(\mathfrak{F}_{1,d})_k : h(\overline{J}^1_C)(g-k) \to P_k h(\overline{J}^d_C).$$

In other words, the following composition is zero:

$$h(\overline{J}^1_C)(g-k) \xrightarrow{(\mathfrak{F}_{1,d})_k} h(\overline{J}^d_C) \to Q_{k+1} h(\overline{J}^d_C).$$

(b) *The motivic perverse filtration is multiplicative:*

$$\cup : P_k h(\overline{J}^d_C) \times P_l h(\overline{J}^d_C) \to P_{k+l} h(\overline{J}^d_C).$$

In other words, the following composition is zero:

$$P_k h(\overline{J}^d_C) \times P_l h(\overline{J}^d_C) \xrightarrow{\cup} h(\overline{J}^d_C) \to Q_{k+l+1} h(\overline{J}^d_C).$$

Statement (a) was proven in [47, Corollary 4.6(iv)] by setting $e = 1$. However, (b) does not follow directly from the multiplicativity result [47, Corollary 4.6(iii)] (since we cannot make $e_1 = e_2 = e_1 + e_2 = 1$); we postpone the proof to Section 3.3. The following result is an immediate consequence of the multiplicativity.

COROLLARY 3.7. – *Assume that* $\alpha \in \mathrm{CH}^l(\overline{J}^d_C)$ *lies in the* k*-th piece of the perverse filtration, i.e.,*

$$\alpha \in \mathrm{CH}^l(P_k h(\overline{J}^d_C)).$$

Then cupping with α *interacts with the motivic perverse filtration as follows:*

$$\alpha : P_i h(\overline{J}^d_C) \to P_{i+k} h(\overline{J}^d_C)(l).$$

Proof. – It suffices to prove the relation

$$\mathfrak{q}_{i+k+1} \circ \Delta_* \alpha \circ \mathfrak{p}_i = 0$$

on the relative product $\overline{J}_C^d \times_B \overline{J}_C^d$, where $\Delta : \overline{J}_C^d \hookrightarrow \overline{J}_C^d \times_B \overline{J}_C^d$ is the diagonal embedding. Theorem 3.6(b) reads

$$\mathfrak{q}_{i+k+1} \circ [\Delta^{\mathrm{sm}}_{\overline{J}_C^d/B}] \circ (\mathfrak{p}_k \times \mathfrak{p}_i) = 0,$$

which is a relation on the relative triple product. Therefore, we have

$$\begin{aligned} 0 &= p_{23*}\left(p_1^*\alpha \cap \left(\mathfrak{q}_{i+k+1} \circ [\Delta^{\mathrm{sm}}_{\overline{J}_C^d/B}] \circ (\mathfrak{p}_k \times \mathfrak{p}_i)\right)\right) \\ &= \mathfrak{q}_{i+k+1} \circ p_{23*}\left(p_1^*\alpha \cap \left([\Delta^{\mathrm{sm}}_{\overline{J}_C^d/B}] \circ (\mathfrak{p}_k \times \mathfrak{p}_i)\right)\right) \\ &= \mathfrak{q}_{i+k+1} \circ \Delta_* \mathfrak{p}_k(\alpha) \circ \mathfrak{p}_i \\ &= \mathfrak{q}_{i+k+1} \circ \Delta_* \alpha \circ \mathfrak{p}_i . \end{aligned}$$

Here p_{ij}, p_i are the natural projections from the relative triple product $\overline{J}_C^d \times_B \overline{J}_C^d \times_B \overline{J}_C^d$ to the corresponding factors. This completes the proof. □

Next, we discuss the motivic χ-independence phenomenon for compactified Jacobian fibrations. For two integers d, d', we consider the correspondence

$$\mathfrak{C}_{d,d'} := \sum_{i=0}^{2g} (\mathfrak{F}_{1,d'})_i \circ (\mathfrak{F}_{1,d}^{-1})_{2g-i} \in \mathrm{Corr}^0_B(\overline{J}_C^d, \overline{J}_C^{d'}).$$

Recall that r is the degree of the multisection $D \subset C$ we fixed at the beginning.

THEOREM 3.8. – *Assume d, d' are two integers coprime to r. Then $\mathfrak{C}_{d,d'}$ induces an isomorphism*

$$\mathfrak{C}_{d,d'} : h(\overline{J}_C^d) \xrightarrow{\simeq} h(\overline{J}_C^{d'})$$

preserving the motivic perverse filtrations

$$\mathfrak{C}_{d,d'} : P_k h(\overline{J}_C^d) \xrightarrow{\simeq} P_k h(\overline{J}_C^{d'}).$$

Moreover, the inverse of $\mathfrak{C}_{d,d'}$ is given by $\mathfrak{C}_{d',d}$,

$$\mathfrak{C}_{d',d} \circ \mathfrak{C}_{d,d'} = [\Delta_{\overline{J}_C^d/B}], \quad \mathfrak{C}_{d,d'} \circ \mathfrak{C}_{d',d} = [\Delta_{\overline{J}_C^{d'}/B}]. \tag{27}$$

Both Theorem 3.6(b) and Theorem 3.8 are proven in the next section, which relies on a generalized version of the Fourier vanishing established in [47].

3.3. Proofs of Theorem 3.6(b) and Theorem 3.8

In [47, Section 4.4], we establish a series of relations between the Fourier components, known as Fourier vanishing (FV). For two integers d, e, this vanishing states that

$$(\mathrm{FV}) \qquad (\mathfrak{F}_{e,d}^{-1})_i \circ (\mathfrak{F}_{e,d})_j = 0, \quad i + j < 2g.$$

In this section, we prove a generalized Fourier vanishing given as follows:

PROPOSITION 3.9. – *Let d, d', e, e' be integers. Then if e is coprime to r, we have*

$$(\mathrm{FV1}) \qquad (\mathfrak{F}_{e',d}^{-1})_i \circ (\mathfrak{F}_{e,d})_j = 0, \quad i + j < 2g.$$

If d is coprime to r, we have

$$(\mathrm{FV2}) \qquad (\mathfrak{F}_{e,d})_i \circ (\mathfrak{F}_{e,d'}^{-1})_j = 0, \quad i + j < 2g.$$

Proof. – The proof of (FV1) follows closely that of (FV) in [47, Proof of Corollary 4.6(i)], where we used the Adams operations $\psi^N(-)$ of [2, Section 4] to scale the Poincaré sheaf $\mathcal{P}_{e,d}$ in K-theory. By applying the tau-class $\tau(-)$ to the K-theory classes supported in codimension g:

$$\mathcal{P}_{e,d}^{-1} \circ \psi^N(\mathcal{P}_{e,d}) \in K_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^e)$$

for infinitely many N, we were able to obtain (FV) for each pair of i, j satisfying $i + j < 2g$. The crucial point here is that the functor $\tau(-)$ of [25] is only covariant under proper *representable* morphisms of quotient stacks. By choosing

$$N \equiv 1 \pmod{r}$$

we made sure that the class

$$p_{12}^* \psi^N(\mathcal{P}_{e,d}) \otimes p_{23}^* \mathcal{P}_{e,d}^{-1} \in K_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^d \times_B \overline{\mathcal{J}}_C^e)_{(d,0,-d)}$$

descends to a class in $K_*(\overline{\mathcal{J}}_C^e \times_B \overline{J}_C^d \times_B \overline{\mathcal{J}}_C^e)$ in order to apply $\tau(-)$ to the proper representable morphism

$$p_{13} : \overline{\mathcal{J}}_C^e \times_B \overline{J}_C^d \times_B \overline{\mathcal{J}}_C^e \to \overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^e.$$

Here p_{ij} are the natural projections to the corresponding factors.

Now, under the assumptions of (FV1), we consider the K-theory classes supported in codimension g:

$$\mathcal{P}_{e',d}^{-1} \circ \psi^N(\mathcal{P}_{e,d}) \in K_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^{e'}).$$

Since $(e, r) = 1$, we have $ae \equiv 1 \pmod{r}$ for some integer a. Then by taking

$$N = aN', \quad N' \equiv e' \pmod{r},$$

we get an infinite sequence of classes

$$p_{12}^* \psi^N(\mathcal{P}_{e,d}) \otimes p_{23}^* \mathcal{P}_{e',d}^{-1} \in K_*(\overline{\mathcal{J}}_C^e \times_B \overline{\mathcal{J}}_C^d \times_B \overline{\mathcal{J}}_C^{e'})_{(ae'd,0,-d)},$$

which descend to $K_*(\overline{\mathcal{J}}_C^e \times_B \overline{J}_C^d \times_B \overline{\mathcal{J}}_C^{e'})$. The argument of [47, Proof of Corollary 4.6(i)] then applies *verbatim* to yield (FV1).

For the proof of (FV2), we can either repeat the whole argument by switching the composition order of the Fourier transform and its inverse, or simply observe that the transpose of

$$(\mathfrak{F}_{e,d})_i \circ (\mathfrak{F}_{e,d'}^{-1})_j$$

is precisely

$$(\mathfrak{F}_{d',e}^{-1})_j \circ (\mathfrak{F}_{d,e})_i. \tag{28}$$

This is an immediate consequence of the symmetry in Arinkin's formula [3, (1.1)] for the Poincaré sheaf; see also [47, Footnote 12]. Therefore, (FV2) follows from (FV1) and (28) by replacing e with d, e' with d', d with e, and by switching i and j. □

We are ready to prove Theorems 3.6(b) and 3.8. We begin with the latter.

Proof of Theorem 3.8. – We first prove the identities (27). We expand

$$\begin{aligned}[\Delta_{\overline{J}_C^d/B}] &= \mathfrak{F}_{1,d} \circ \mathfrak{F}_{1,d'}^{-1} \circ \mathfrak{F}_{1,d'} \circ \mathfrak{F}_{1,d}^{-1} \\ &= \sum_{i,j,l,m} (\mathfrak{F}_{1,d})_i \circ (\mathfrak{F}_{1,d'}^{-1})_j \circ (\mathfrak{F}_{1,d'})_l \circ (\mathfrak{F}_{1,d}^{-1})_m. \end{aligned} \tag{29}$$

For degree reasons the terms in (29) vanish unless $i + j + l + m = 4g$. Applying the Fourier vanishing (FV2) with d, d' coprime to r, we find that the nonvanishing terms in (29) must also satisfy $i + j \geq 2g$, $l + m \geq 2g$. Hence both inequalities are actual equalities and we have

$$[\Delta_{\overline{J}_C^d/B}] = \left(\sum_{i=0}^{2g} (\mathfrak{F}_{1,d})_i \circ (\mathfrak{F}_{1,d'}^{-1})_{2g-i}\right) \circ \left(\sum_{l=0}^{2g} (\mathfrak{F}_{1,d'})_l \circ (\mathfrak{F}_{1,d}^{-1})_{2g-l}\right) = \mathfrak{C}_{d',d} \circ \mathfrak{C}_{d,d'},$$

which proves the first identity in (27). The second identity follows by symmetry.

It remains to verify that $\mathfrak{C}_{d,d'}, \mathfrak{C}_{d',d}$ preserve the motivic perverse filtrations. This amounts to showing the relations (21) where $\mathfrak{p}_k, \mathfrak{q}_{k+1}$ (resp. $\mathfrak{p}'_k, \mathfrak{q}'_{k+1}$) denote the projectors for $\overline{J}_C^d$ (resp. $\overline{J}_C^{d'}$). By symmetry it suffices to prove the first identity:

$$\mathfrak{q}'_{k+1} \circ \mathfrak{C}_{d,d'} \circ \mathfrak{p}_k = 0.$$

The left-hand side can be expressed as

$$\left(\sum_{i \geq k+1} (\mathfrak{F}_{1,d'})_i \circ (\mathfrak{F}_{1,d'}^{-1})_{2g-i}\right) \circ \left(\sum_{j=0}^{2g} (\mathfrak{F}_{1,d'})_j \circ (\mathfrak{F}_{1,d}^{-1})_{2g-j}\right) \circ \left(\sum_{m \leq k} (\mathfrak{F}_{1,d})_m \circ (\mathfrak{F}_{1,d}^{-1})_{2g-m}\right). \tag{30}$$

This time applying (FV1) with $e = 1$ to (30), we see that the nonvanishing terms must satisfy

$$2g - i + j \geq 2g, \quad 2g - j + m \geq 2g, \quad i \geq k+1, \quad m \leq k,$$

or, equivalently,

$$k \geq m \geq j \geq i \geq k+1.$$

This is absurd, and hence (30) vanishes. □

The proof of Theorem 3.6(b) is slightly more complicated but follows a similar pattern. For two integers e_1, e_2, recall from [47, Proof of Corollary 4.6(iii)] the Chow-theoretic convolution kernel (with a different notation to avoid confusion with the correspondence $\mathfrak{C}_{d,d'}$)

$$\mathfrak{K}_{e_1,e_2,d} \in \mathrm{Corr}_B^{\geq -g}(\overline{J}_C^{e_1}, \overline{J}_C^{e_2}; \overline{J}_C^{e_1+e_2}),$$

which satisfies

$$\mathfrak{F}_{e_1+e_2,d} \circ \mathfrak{K}_{e_1,e_2,d} \circ \left(\mathfrak{F}_{e_1,d}^{-1} \times \mathfrak{F}_{e_2,d}^{-1}\right) = [\Delta^{\mathrm{sm}}_{\overline{J}^d_C/B}]. \tag{31}$$

Proof of Theorem 3.6(b). – We need to show the vanishing

$$\mathfrak{q}_{k+l+1} \circ [\Delta^{\mathrm{sm}}_{\overline{J}^d_C/B}] \circ (\mathfrak{p}_k \times \mathfrak{p}_l) = 0 \in \mathrm{Corr}^0_B(\overline{J}^d_C, \overline{J}^d_C; \overline{J}^d_C). \tag{32}$$

By (31) we have

$$\begin{aligned}
\mathfrak{q}_{k+l+1} \circ [\Delta^{\mathrm{sm}}_{\overline{J}^d_C/B}] \circ (\mathfrak{p}_k \times \mathfrak{p}_l) &= \mathfrak{q}_{k+l+1} \circ \mathfrak{F}_{e_1+e_2,d} \circ \mathfrak{K}_{e_1,e_2,d} \circ \left((\mathfrak{F}_{e_1,d}^{-1} \circ \mathfrak{p}_k) \times (\mathfrak{F}_{e_2,d}^{-1} \circ \mathfrak{p}_l)\right) \\
&= \left(\sum_{i_3 \geq k+l+1} (\mathfrak{F}_{1,d})_{i_3} \circ (\mathfrak{F}_{1,d}^{-1})_{2g-i_3}\right) \circ \left(\sum_{j_3} (\mathfrak{F}_{e_1+e_2,d})_{j_3}\right) \circ \mathfrak{K}_{e_1,e_2,d} \\
&\quad \circ \left(\left(\left(\sum_{j_1} (\mathfrak{F}_{e_1,d}^{-1})_{j_1}\right) \circ \left(\sum_{i_1 \leq k} (\mathfrak{F}_{1,d})_{i_1} \circ (\mathfrak{F}_{1,d}^{-1})_{2g-i_1}\right)\right)\right. \\
&\quad \left.\times \left(\left(\sum_{j_2} (\mathfrak{F}_{e_2,d}^{-1})_{j_2}\right) \circ \left(\sum_{i_2 \leq l} (\mathfrak{F}_{1,d})_{i_2} \circ (\mathfrak{F}_{1,d}^{-1})_{2g-i_2}\right)\right)\right).
\end{aligned} \tag{33}$$

So far the choices of e_1, e_2 have been arbitrary. But if we choose e_1, e_2 such that $e_1 + e_2$ is coprime to r, then by applying (FV1) with $e = 1$ or $e = e_1 + e_2$ we see that the nonvanishing terms in (33) must satisfy

$$j_1 + i_1 \geq 2g, \qquad j_2 + i_2 \geq 2g, \qquad 2g - i_3 + j_3 \geq 2g.$$

This implies

$$j_1 \geq 2g - k, \qquad j_2 \geq 2g - l, \qquad j_3 \geq k + l + 1.$$

Therefore we have

$$\begin{aligned}
\mathfrak{q}_{k+l+1} \circ [\Delta^{\mathrm{sm}}_{\overline{J}^d_C/B}] \circ (\mathfrak{p}_k \times \mathfrak{p}_l) &= \mathfrak{q}_{k+l+1} \circ \left(\sum_{j_3 \geq k+l+1} (\mathfrak{F}_{e_1+e_2,d})_{j_3}\right) \circ \mathfrak{K}_{e_1,e_2,d} \\
&\quad \circ \left(\left(\left(\sum_{j_1 \geq 2g-k} (\mathfrak{F}_{e_1,d}^{-1})_{j_1}\right) \circ \mathfrak{p}_k\right) \times \left(\left(\sum_{j_2 \geq 2g-l} (\mathfrak{F}_{e_2,d}^{-1})_{j_2}\right) \circ \mathfrak{p}_l\right)\right).
\end{aligned} \tag{34}$$

Comparing degrees, we find that the right-hand side of (34) lies in

$$\mathrm{Corr}^{\geq -g+(k+l+1-g)+(g-k)+(g-l)}_B(\overline{J}^d_C, \overline{J}^d_C; \overline{J}^d_C) = \mathrm{Corr}^{\geq 1}_B(\overline{J}^d_C, \overline{J}^d_C; \overline{J}^d_C),$$

while the left-hand side lies in $\mathrm{Corr}^0_B(\overline{J}^d_C, \overline{J}^d_C; \overline{J}^d_C)$. This shows the desired vanishing (32). □

3.4. Parabolic Higgs bundles: general residue

Theorems 3.5, 3.6, and 3.8 in Section 3.2 are analogues of our main theorems in the general setting of compactified Jacobians. For our purpose, we specialize to the case where $C \to B$ arises as a family of spectral curves. The main results of this section are Theorems 3.11, 3.12, and 3.13 which treat the case of the Hitchin system associated with parabolic Higgs bundles with general residue.

3.4.1. *Normalized Chern characters.* – For the curve Σ, we consider the nonsingular surface

$$S := \mathrm{Tot}_\Sigma(\omega_\Sigma(p)).$$

Now we work with a family of curves $C \to B$ as in Section 3.2, imposing the extra condition that they lie in the linear system $|n\Sigma|$ of the surface S; in particular, every curve $C_b \subset S$ is a degree n finite cover over the zero-section Σ. We have a natural evaluation map

$$\mathrm{ev} : C \to S \times B.$$

The composition of ev with the projection $S \times B \to \Sigma \times B$ gives the spectral cover together with a (degree n) multisection:

$$\pi : C \to \Sigma \times B, \quad D := \pi^{-1}(p \times B).$$

For each curve C_b with a degree n spectral cover $\pi_b : C_b \to \Sigma$, the multisection over this fiber is $D_b = \pi_b^{-1}(p)$.

Next, we introduce the normalized Chern character of Section 1.2. Let M^{mero} be the moduli space of stable meromorphic Higgs bundles on the curve Σ, which parameterizes slope stable (meromorphic) Higgs bundles:

$$(\mathcal{E}, \theta), \quad \theta : \mathcal{E} \to \mathcal{E} \otimes \omega_\Sigma(p), \quad \mathrm{rk}(\mathcal{E}) = n, \quad \deg(\mathcal{E}) = d.$$

Here we call them meromorphic Higgs bundles since they can be viewed as Higgs bundles which are allowed to have a simple pole at p. The variety M^{mero} is nonsingular, and the coprime condition $(n, d) = 1$ ensures the existence of a universal bundle $\mathcal{U}$ on $\Sigma \times M^{\mathrm{mero}}$. The reason we work first with this moduli space is that all the moduli spaces in Section 2.2, including the original moduli space $M_{n,d}$, admit natural morphisms to M^{mero}, and the pullback of the universal bundle gives a universal bundle on each moduli space. So the normalized Chern character

$$\widetilde{\mathrm{ch}}(\mathcal{U}) \in \mathrm{CH}^*(\Sigma \times M^{\mathrm{mero}}),$$

yields the normalized Chern character for each moduli space of Section 2.2. This strategy was taken in [47, Section 5.4].

In cohomology, the normalization is given by a class

$$\delta := \pi_\Sigma^* \delta_\Sigma + \pi_M^* \delta_M \in H^2(\Sigma \times M^{\mathrm{mero}}, \mathbb{Q}),$$

with $\pi_\bullet$ the natural projections, satisfying that

$$c_1(\mathcal{U}) + \delta \in H^1(\Sigma, \mathbb{Q}) \otimes H^1(M^{\mathrm{mero}}, \mathbb{Q}) \subset H^2(\Sigma \times M^{\mathrm{mero}}, \mathbb{Q});$$

this determines δ completely. Then the normalized Chern character is defined to be

$$\widetilde{\mathrm{ch}}(\mathcal{U}) := \mathrm{ch}(\mathcal{U}) \cup \exp(\delta) \in H^*(\Sigma \times M^{\mathrm{mero}}, \mathbb{Q}),$$

which is canonical and independent of the choice of $\mathcal{U}$. We refer to [9, Section 0.3] for more details.

For our purpose, we need to lift δ (associated with a universal bundle $\mathcal{U}$) canonically to a divisor

$$\delta := \pi_\Sigma^* D_\Sigma + \pi_M^* D_M \in \mathrm{CH}^1(\Sigma \times M^{\mathrm{mero}}). \tag{35}$$

We first determine D_M using the marked point $p \in \Sigma$:

$$D_M := c_1(\mathcal{U})|_{p \times M^{\mathrm{mero}}} \in \mathrm{CH}^1(M^{\mathrm{mero}}).$$

Now we determine D_Σ, which is eventually written as a linear combination of the point class of p and the canonical divisor K_Σ on the curve Σ. Pick a nonsingular spectral curve $C_b \subset \mathrm{Tot}_\Sigma(\omega_\Sigma(p))$; a Higgs bundle in M^{mero} associated with the spectral curve C_b can be represented by a line bundle $\mathcal{L}_{C_b}$ on C_b of degree $l(n,d)$, where $l(n,d)$ is given by a Riemann-Roch calculation

$$l(n,d) = \frac{n(n-1)(2g-1)}{2} + d.$$

In particular, we have

$$c_1(\mathcal{L}_{C_b}) - \frac{l(n,d)}{n} \cdot D_b = c_1(\mathcal{L}_{C_b}) - \frac{l(n,d)}{n} \cdot \pi_b^*[p] = 0 \in H^2(C_b, \mathbb{Q}).$$

Finally, there is a unique constant $\lambda_0 \in \frac{1}{2n}\mathbb{Z}$ such that

$$c_1(\mathcal{U}) - \frac{l(n,d)}{n} \cdot \pi_b^*[p] + \lambda_0 \cdot K_\Sigma = 0 \in H^2(\Sigma, \mathbb{Q}).$$

We set

$$D_\Sigma := -\frac{l(n,d)}{n} \cdot [p] + \lambda_0 \cdot K_\Sigma,$$

in (35), which defines the normalized Chern character

$$\widetilde{\mathrm{ch}}(\mathcal{U}) := \mathrm{ch}(\mathcal{U}) \cup \exp(\delta) \in \mathrm{CH}^*(\Sigma \times M^{\mathrm{mero}}). \tag{36}$$

We denote by $\widetilde{\mathrm{ch}}_k(\mathcal{U})$ its degree k part:

$$\widetilde{\mathrm{ch}}(\mathcal{U}) = \sum_k \widetilde{\mathrm{ch}}_k(\mathcal{U}), \quad \widetilde{\mathrm{ch}}_k(\mathcal{U}) \in \mathrm{CH}^k(\Sigma \times M^{\mathrm{mero}}).$$

The normalized Chern character (36) does not depend on the choice of the universal bundle $\mathcal{U}$, and its image under the cycle class map recovers the normalized Chern character in cohomology. Indeed, two universal bundles $\mathcal{U}, \mathcal{U}'$ differ by a line bundle pulled back from M^{mero}. Therefore we have $\mathrm{ch}(\mathcal{U}') = \mathrm{ch}(\mathcal{U}) \cup \exp(\ell_M)$ with $\ell_M \in \mathrm{CH}^1(M^{\mathrm{mero}})$. By definition, $\mathcal{U}, \mathcal{U}'$ share the same normalized Chern character.

3.4.2. *Normalized Chern characters via Fourier transforms.* – Next, we give an explicit description of the normalized Chern character for the moduli space M_η^{par} for a general $\eta \in T$ as in Section 2.2(a); it admits a compactified Jacobian fibration associated with the family of spectral curves $C_\eta \to W_\eta$. In other words, we have

$$M_\eta^{\mathrm{par}} \simeq \overline{J}^e_{C_\eta} \to W_\eta, \quad e = l(n,d).$$

We consider the diagram

$$\begin{array}{ccc} C_\eta & \xrightarrow{\mathrm{AJ}} & \overline{J}^1_{C_\eta} \\ \downarrow{\scriptstyle \pi} & & \\ \Sigma \times W_\eta, & & \end{array} \tag{37}$$

where the horizontal map is the Abel-Jacobi map

$$\mathrm{AJ} : C_\eta \to \overline{J}^1_{C_\eta}, \qquad (x \in C_b) \mapsto \mathfrak{m}^\vee_{x/C_b}.$$

The diagram (37) yields a relative correspondence over W_η:

$$[C_\eta] \in \mathrm{CH}_{\dim C_\eta}(\Sigma \times \overline{J}^1_{C_\eta}).$$

PROPOSITION 3.10 ([47]). – *Let $\mathcal{U}$ be a universal bundle over $\Sigma \times M^{\mathrm{par}}_\eta \simeq \Sigma \times \overline{J}^e_{C_\eta}$. Then we have*

$$\widetilde{\mathrm{ch}}(\mathcal{U}) = \mathfrak{F}_{1,e} \circ [C_\eta] \in \mathrm{CH}^*(\Sigma \times M^{\mathrm{par}}_\eta).$$

Proof. – Applying the Grothendieck-Riemann-Roch (GRR) formula to [47, Proposition 4.7] as in the proof of [47, Proposition 5.1(a)], we express the Chern character of the universal 1-dimensional torsion free sheaf over the universal spectral curve C_η in terms of $\mathfrak{F}_{1,e}$. Then the result follows from applying further the GRR formula along $\pi : C_\eta \to \Sigma \times W_\eta$, expressing the Chern character of a universal bundle over Σ in terms of the Chern character of a universal torsion free sheaf over C_η [47, Section 5].

Note that the term $-\frac{l(n,d)}{n} \cdot [p]$ in the normalization comes from the line bundle $\mathcal{N}$ of [47, Proposition 4.7], and the second term $\lambda_0 \cdot K_\Sigma$ arises from the Todd contribution of the GRR calculation along $\pi : C_\eta \to \Sigma \times W_\eta$. □

3.4.3. *Main results for Higgs bundles with general residue*. – We deduce the following three theorems for the moduli space M^{par}_η, the first of which is an immediate consequence of Theorem 3.5.

THEOREM 3.11. – *The morphism $h_\eta : M^{\mathrm{par}}_\eta \to W_\eta$ admits a motivic perverse filtration in the sense of Definition 3.1, where the projectors are given in Theorem 3.5.*

The next theorem determines the perversity of the normalized Chern character.

THEOREM 3.12. – *Taking cup-product with the normalized Chern character satisfies*

$$\widetilde{\mathrm{ch}}_k(\mathcal{U}) : P_i h(\Sigma \times M^{\mathrm{par}}_\eta) \to P_{i+k} h(\Sigma \times M^{\mathrm{par}}_\eta)(k) \in \mathrm{CHM}(\Sigma \times W_\eta).$$

Here the motivic perverse filtration for $\Sigma \times M^{\mathrm{par}}_\eta \to \Sigma \times W_\eta$ is given by the pullback of the motivic perverse filtration for $h_\eta : M^{\mathrm{par}}_\eta \to W_\eta$.

Proof. – Recall the perverse filtration on $\mathrm{CH}_*(M^{\mathrm{par}}_\eta)$ induced by the motivic perverse filtration associated with the compactified Jacobian fibration $M^{\mathrm{par}}_\eta \to W_\eta$. We first note that

$$\widetilde{\mathrm{ch}}_k(\mathcal{U}) \in P_k \mathrm{CH}^k(\Sigma \times M^{\mathrm{par}}_\eta).$$

This is a direct consequence of Proposition 3.10 combined with Theorem 3.6(a):

$$\widetilde{\mathrm{ch}}_k(\mathcal{U}) = (\mathfrak{F}_{1,e})_k \circ [C_\eta].$$

Then applying Corollary 3.7 to

$$\Sigma \times M_\eta^{\mathrm{par}} \to \Sigma \times W_\eta$$

completes the proof of the theorem. □

The third theorem concerns the χ-independence phenomenon and is immediately deduced from Theorem 3.8.

THEOREM 3.13. – *Assume d, d' are two integers coprime to n with $e = l(n,d), e' = l(n,d')$. Let $M_\eta^{\mathrm{par}}, M_\eta^{\mathrm{par}'}$ be the parabolic moduli spaces associated with the degrees d, d' respectively. Then $\mathfrak{C}_{e,e'}$ from Theorem 3.8 induces an isomorphism*

$$\mathfrak{C}_{e,e'} : h(M_\eta^{\mathrm{par}}) \xrightarrow{\simeq} h(M_\eta^{\mathrm{par}'})$$

preserving the motivic perverse filtrations

$$\mathfrak{C}_{e,e'} : P_k h(M_\eta^{\mathrm{par}}) \xrightarrow{\simeq} P_k h(M_\eta^{\mathrm{par}'}).$$

Moreover, the inverse of $\mathfrak{C}_{e,e'}$ is given by $\mathfrak{C}_{e',e}$.

3.5. Parabolic Higgs bundles: nilpotent residue

In the previous section, we have established the desired results for

$$h_\eta : M_\eta^{\mathrm{par}} \to W_\eta$$

associated with parabolic Higgs bundles with general residue. The proofs in fact give the results for the family

$$h^\circ : M_{T^\circ}^{\mathrm{par}} \to W_{T^\circ}$$

over T° introduced in Section 2.2(a), which puts together the fibrations h_η for $\eta \in T^\circ$. Since $\chi_p : M_T^{\mathrm{par}} \to T$ is a smooth morphism, by applying the specialization map for Chow groups via (13), all the results concerning relations of algebraic cycles hold for

$$h_0 : M_0^{\mathrm{par}} \to W_0.$$

THEOREM 3.14. – *The morphism $h_0 : M_0^{\mathrm{par}} \to W_0$ admits a motivic perverse filtration in the sense of Definition 3.1, where the projectors are induced by specializing the projectors of $h_\eta : M_\eta^{\mathrm{par}} \to W_\eta$ with $\eta \in T^\circ$ given in Theorem 3.11.*

As we discussed at the beginning of this Section 3.5, the projectors we obtain from specialization automatically satisfy the conditions (a) and (c) in Definition 3.1. It remains to prove (b), i.e., the homological realization recovers the perverse truncation functor. This will be proven in Section 5.1.

THEOREM 3.15. – *Let $\mathcal{U}$ be a universal bundle over $\Sigma \times M_0^{\mathrm{par}}$. Then we have*

$$\widetilde{\mathrm{ch}}_k(\mathcal{U}) : P_i h(\Sigma \times M_0^{\mathrm{par}}) \to P_{i+k} h(\Sigma \times M_0^{\mathrm{par}})(k) \in \mathrm{CHM}(\Sigma \times W_0).$$

Proof. – Since the statement can be expressed in terms of a relation of algebraic cycles:

$$\mathfrak{q}_{i+k+1} \circ \Delta_* \widetilde{\mathrm{ch}}_k(\mathcal{U}) \circ \mathfrak{p}_i = 0$$

with $\Delta : M_0^{\mathrm{par}} \hookrightarrow M_0^{\mathrm{par}} \times_{W_0} M_0^{\mathrm{par}}$ the relative diagonal, it follows from specializing the corresponding relation for $h_\eta : M_\eta^{\mathrm{par}} \to W_\eta$ established in Theorem 3.12. □

Similarly, by a straightforward specialization argument, we obtain from Theorem 3.13 the following theorem concerning the χ-independence.

THEOREM 3.16. – *Assume d, d' are two integers coprime to n. Let $M_0^{\mathrm{par}}, M_0^{\mathrm{par}'}$ be the parabolic moduli spaces associated with the degrees d, d' respectively. Then there is an isomorphism*

$$h(M_0^{\mathrm{par}}) \simeq h(M_0^{\mathrm{par}'})$$

preserving the motivic perverse filtrations

$$P_k h(M_0^{\mathrm{par}}) \simeq P_k h(M_0^{\mathrm{par}'}).$$

REMARK 3.17. – We conclude this section by noting that the motivic perverse filtration we introduced in Theorem 3.14 is independent of the choice of the 1-dimensional family T. Indeed, we consider the Hitchin map associated with the total space of the moduli of stable parabolic Higgs bundles

$$h : M^{\mathrm{par}} \to W$$

in Section 2.2. The Fourier transform can be defined over a Zariski open subset W° of the base W formed by integral spectral curves, and the projectors $\mathfrak{p}_k$ we obtained in Theorem 3.15 is the specialization of the projectors over W°. Here we choose to work with a 1-dimensional family T for convenience (e.g., in the sheaf-theoretic argument of Section 5).

4. Projectors and Springer theory

In this section, we study the correspondences given by (16) connecting M_0^{par} and $M_{n,d}$. Our main purpose is to prove Proposition 4.4, which yields a candidate for the motivic perverse filtration (47) for $f : M_{n,d} \to B$. For the approach, we reduce the Higgs case to a *local model* where the correspondence (16) is understood via classical Springer theory. This lifts several cohomological results of [30, Section 8] motivically. Springer-theoretic interpretations of parabolic Hitchin moduli spaces have previously been used in [23] in the study of Kac polynomials.

4.1. Springer theory

For our purpose, we consider the case $G = \mathrm{PGL}_n$. We use $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ to denote the Lie algebras of G, the Borel subgroup given by upper triangular matrices, and the maximal torus given by diagonal matrices, respectively. The Weyl group is the permutation group $\mathfrak{S}_n$ which acts on $\mathfrak{t}$ naturally. Let

$$\mathfrak{c} := \mathfrak{g} /\!\!/ G \simeq \mathfrak{t} /\!\!/ \mathfrak{S}_n$$

be the categorical quotient, and we have a natural morphism

$$\chi : \mathfrak{g} \to \mathfrak{c},$$

which can be viewed as calculating the characteristic polynomial of a matrix. We consider the Grothendieck-Springer resolution over the base $\mathfrak{c}$:

$$\begin{array}{ccc} \widetilde{\mathfrak{g}} & \longrightarrow & \mathfrak{t} \\ \downarrow{\scriptstyle q} & & \downarrow \\ \mathfrak{g} & \xrightarrow{\chi} & \mathfrak{c}. \end{array}$$

The fibers over $0 \in \mathfrak{c}$ recover the symplectic resolution of the nilpotent cone

$$q_0 : \widetilde{\mathcal{N}}_0 \to \mathcal{N}_0$$

by the total cotangent bundle of the flag variety $\widetilde{\mathcal{N}}_0 = T^*\mathrm{Fl}$, while the fibers over a general $\zeta \in \mathfrak{c}$ (e.g., the corresponding matrix has distinct eigenvalues) are a natural quotient by the Weyl group:

$$q_\zeta : \widetilde{\mathcal{N}}_\zeta \to \mathcal{N}_\zeta = \widetilde{\mathcal{N}}_\zeta / \mathfrak{S}_n.$$

From this description, we see that any element $w \in \mathfrak{S}_n$ induces a self-correspondence of $\widetilde{\mathcal{N}}_\zeta$. As we will illustrate in the next section, the local analogue of the correspondence (16) is (the G-equivariant version of) the following:

$$\begin{array}{ccc} \mathrm{Fl} & \xrightarrow{\iota} & \widetilde{\mathcal{N}}_0 = T^*\mathrm{Fl} \\ \downarrow{\scriptstyle \pi} & & \\ \mathrm{pt}, & & \end{array} \tag{38}$$

where ι is the closed embedding of the 0-section. The following lemma collects a few standard results in Springer theory, which concerns the (G-equivariant) self-correspondences

$$[\mathrm{Fl}] \circ {}^t[\mathrm{Fl}] \in \mathrm{CH}_*(\widetilde{\mathcal{N}}_0 \times \widetilde{\mathcal{N}}_0) = \mathrm{Corr}^*(\widetilde{\mathcal{N}}_0, \widetilde{\mathcal{N}}_0) \tag{39}$$

and

$${}^t[\mathrm{Fl}] \circ [\mathrm{Fl}] \in \mathrm{CH}_*(\mathrm{pt}) = \mathrm{Corr}^*(\mathrm{pt}, \mathrm{pt}). \tag{40}$$

See [17, Section 3] and [58, Section 1.5.10] for references.

LEMMA 4.1. – *The following statements hold for the G-equivariant correspondences* $[\mathrm{Fl}]$, ${}^t[\mathrm{Fl}]$.

(a) *The self-correspondence* (39) *of $\widetilde{\mathcal{N}}_0$ is the specialization of the self-correspondence of $\widetilde{\mathcal{N}}_\zeta$ induced by the longest element w_0 in the Weyl group $\mathfrak{S}_n$.*

(b) *The self-correspondence of* (40) *of a point is*

$$\varepsilon n! \cdot \mathrm{id} \in \mathrm{Corr}^0(\mathrm{pt}, \mathrm{pt}).$$

Here $\varepsilon = \pm 1$ is a sign dependent on n.

(c) *There is a natural Weyl group $\mathfrak{S}_n$ action on the derived push-forward*

$$Rq_{0*}\mathbb{Q}_{\widetilde{\mathcal{N}}_0} \in D^{\mathrm{b}}_{\mathrm{c}}(\mathcal{N}_0)$$

with the anti-invariant part a shifted skyscraper sheaf

$$\left(Rq_{0*}\mathbb{Q}_{\widetilde{\mathcal{N}}_0}\right)^{\mathrm{sgn}} = \mathbb{Q}_0\left[-2\binom{n}{2}\right].$$

Moreover, the correspondences

$$[\mathrm{Fl}] : Rq_{0*}\mathbb{Q}_{\widetilde{\mathcal{N}}_0} \to \mathbb{Q}_0\left[-2\binom{n}{2}\right], \qquad {}^{\mathrm{t}}[\mathrm{Fl}] : \mathbb{Q}_0\left[-2\binom{n}{2}\right] \to Rq_{0*}\mathbb{Q}_{\widetilde{\mathcal{N}}_0} \in D^{\mathrm{b}}_{\mathrm{c}}(\mathcal{N}_0)$$

are projecting to the anti-invariant part and the natural inclusion from the anti-invariant part respectively.

Proof. – Part (b) is the self-intersection of the Lagrangian zero-section

$$\iota : \mathrm{Fl} \hookrightarrow \widetilde{\mathcal{N}}_0 = T^*\mathrm{Fl},$$

where the constrant $n!$ is given by the Euler characteristic of the flag variety Fl.

To see (a) and (c), we recall the Grothendieck-Springer resolution

$$q : \widetilde{\mathfrak{g}} \to \mathfrak{g},$$

which is G-equivariant and small; the restriction of q to the regular semisimple locus is a torsor under the Weyl group $\mathfrak{S}_n$. The derived pushforward

$$Rq_*\mathbb{Q}_{\widetilde{\mathfrak{g}}} \in D^b_c(\mathfrak{g})$$

is the intermediate extension of the local systems obtained from the regular semisimple locus, which naturally carries an $\mathfrak{S}_n$-action by the functoriality of intermediate extension. In particular, this induces the $\mathfrak{S}_n$-action on every Springer fiber. The isotypic component with respect to the longest element w_0 for the Springer fiber over $0 \in \mathcal{N}_0 \subset \mathfrak{g}$ is given as in (c). For our purpose, it suffices to show that the generalized eigenspace with respect to w_0 is realized as the correspondence (39). As explained in [58, Section 1.5.10] and [57, Remark 3.3.4], the Springer action of $\mathfrak{S}_n$ can be realized by correspondences of the Steinberg variety $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$. After restricting over $\mathcal{N}_0$, the irreducible components of the Steinberg variety are parameterized by elements $w \in \mathfrak{S}_n$. More precisely, the Steinberg variety over $\mathcal{N}_0$ admits a stratification into conormal bundles of G-orbits in $\mathrm{Fl} \times \mathrm{Fl}$ (see [17, Corollary 3.3.5]), and the element w_0 corresponds to the largest open stratum whose conormal bundle has closure

$$\mathrm{Fl} \times \mathrm{Fl} \subset \widetilde{\mathcal{N}}_0 \times_{\mathcal{N}_0} \widetilde{\mathcal{N}}_0.$$

Its induced correspondence is exactly (39), which proves the lemma. □

In the next section, we will discuss a global version of Lemma 4.1 and use it to analyze the correspondence

$$[\widetilde{M_0}] \circ {}^{\mathrm{t}}[\widetilde{M_0}] \in \mathrm{Corr}^*_{W_0}(M_0^{\mathrm{par}}, M_0^{\mathrm{par}}). \tag{41}$$

4.2. Hitchin moduli spaces

Recall the moduli space M^{mero} of stable meromorphic Higgs bundles of Section 3.4. It admits a natural evaluation map at the point $p \in \Sigma$ to the stack quotient $\mathfrak{g}/G$:

$$\mathrm{ev}_p : M^{\mathrm{mero}} \to \mathfrak{g}/G \tag{42}$$

as we introduce below.

We denote by M^{ab} the *abelian* Hitchin moduli space

$$M^{\mathrm{ab}} = \mathrm{Pic}^0(\Sigma) \times H^0(\Sigma, \omega_\Sigma(p)),$$

which parameterizes rank 1 Higgs bundles. There is a natural action of M^{ab} on M^{mero} by

$$(\mathcal{L},\sigma)\cdot(\mathcal{E},\theta)=(\mathcal{L}\otimes\mathcal{E},\sigma+\theta),\qquad (\mathcal{L},\sigma)\in M^{\mathrm{ab}},\qquad (\mathcal{E},\theta)\in M^{\mathrm{mero}},$$

where the sum $\sigma+\theta$ is defined in the sense that we view both σ,θ as morphisms

$$\mathcal{L}\otimes\mathcal{E}\to\mathcal{L}\otimes\mathcal{E}\otimes\omega_\Sigma(p).$$

The quotient space

$$\widehat{M}^{\mathrm{mero}}:=M^{\mathrm{mero}}/M^{\mathrm{ab}}$$

is a nonsingular Deligne-Mumford stack, known as the moduli space of stable meromorphic PGL_n-Higgs bundles, which admits a natural evaluation map at the point p; see [32, Section 2] and [12, Section 2.4]. The evaluation map (42) for the original moduli space is the composition of the quotient map and the PGL_n-evaluation map:

$$M^{\mathrm{mero}}\twoheadrightarrow\widehat{M}^{\mathrm{mero}}\to\mathfrak{g}/G.$$

PROPOSITION 4.2. – *The morphism* ev_p *in* (42) *is smooth and surjective.*

Proof. – The quotient map $M^{\mathrm{mero}}\twoheadrightarrow\widehat{M}^{\mathrm{mero}}$ is smooth. Therefore, the smoothness of the composition follows from the smoothness of the second evaluation map which has been proven in [44, Proposition 4.1] via deformation theory. □

REMARK 4.3. – We do not work directly with the evaluation map associated with GL_n for the following reasons. First, Lemma 2.2 implies that the image of ev_p for a GL_n-Higgs bundle has trace 0; in particular the trace lies naturally in

$$\mathfrak{g}=\{\mathrm{tr}=0\}\subset\mathfrak{gl}_n.$$

Second, when we pass from the moduli stack of GL_n-stable Higgs bundle to the corresponding moduli space, we need to rigidify the automorphism group $\mathbb{C}^*\subset\mathrm{GL}_n$. Working with the evaluation map associated with $G=\mathrm{PGL}_n$ is more convenient which resolves both issues.

Now we may use the evaluation map to pull back natural morphisms in Springer theory. As a first example, the pullback of (42) via the (G-equivariant) Grothendieck-Springer resolution recovers the parabolic moduli space

$$\begin{array}{ccc} M^{\mathrm{par}} & \xrightarrow{\mathrm{ev}_p} & \widetilde{\mathfrak{g}}/G \\ \downarrow & & \downarrow q \\ M^{\mathrm{mero}} & \xrightarrow{\mathrm{ev}_p} & \mathfrak{g}/G; \end{array}$$

see [57, Section 2.1]. This further yields the specialization

$$\mathrm{ev}_p: M^{\mathrm{par}}_\xi\to\widetilde{\mathcal{N}}_\xi/G\quad\rightsquigarrow\quad\mathrm{ev}_p: M^{\mathrm{par}}_0\to\widetilde{\mathcal{N}}_0/G. \tag{43}$$

Finally, the pullback along ev_p of the G-equivariant version of the diagram (38)

$$\begin{array}{ccc} \mathrm{Fl}/G & \xrightarrow{\iota} & \widetilde{\mathcal{N}}_0/G \\ \downarrow\pi & & \\ \mathrm{pt}/G & & \end{array} \tag{44}$$

recovers the diagram (16). Since all subsequent maps ev_p are obtained via base change from (42), they are all smooth and surjective by Proposition 4.2.

Recall from Section 2.2(a) that a general point $\eta \in T$ maps to a point $\zeta \in \overline{B}^{\mathrm{par}}$ (by abuse of notation), which forgets the order of the prescribed eigenvalues at p. The universal spectral curve $C_\eta \to W_\eta$ is obtained via the base change

$$\begin{array}{ccc} C_\eta & \longrightarrow & W_\eta \\ \downarrow\simeq & & \downarrow\simeq \\ C_\zeta & \longrightarrow & B_\zeta^{\mathrm{par}}, \end{array}$$

which further induces the base change

$$\begin{array}{ccc} M_\eta^{\mathrm{par}} & \longrightarrow & W_\eta \\ \downarrow\simeq & & \downarrow\simeq \\ \overline{J}^e_{C_\zeta} & \longrightarrow & B_\zeta^{\mathrm{par}}. \end{array}$$

We also note that the fiber M_ζ^{par} over $\zeta \in \overline{B}^{\mathrm{par}}$ consists of $n!$ copies of $\overline{J}^e_{C_\zeta}$ corresponding to the different orders of the eigenvalues at p, and that M_η^{par} is one of them.

PROPOSITION 4.4. – *Assume that the self-correspondence*

$$\mathfrak{Z}_\eta \in \mathrm{Corr}^*_{W_\eta}(M_\eta^{\mathrm{par}}, M_\eta^{\mathrm{par}})$$

is pulled back from $\overline{J}^e_{C_\zeta} \times_{B_\zeta^{\mathrm{par}}} \overline{J}^e_{C_\zeta}$. *Let*

$$\mathfrak{Z}_0 \in \mathrm{Corr}^*_{W_0}(M_0^{\mathrm{par}}, M_0^{\mathrm{par}})$$

be the specialization of $\mathfrak{Z}_\eta$. *Then* (41) *commutes with* $\mathfrak{Z}_0$, *i.e., we have*

$$\big([\widetilde{M}_0] \circ {}^t[\widetilde{M}_0]\big) \circ \mathfrak{Z}_0 = \mathfrak{Z}_0 \circ \big([\widetilde{M}_0] \circ {}^t[\widetilde{M}_0]\big) \in \mathrm{Corr}^*_{W_0}(M_0^{\mathrm{par}}, M_0^{\mathrm{par}}).$$

Proof. – Since the diagram (16) is induced by the pullback of the diagram (44) via the evaluation maps, we obtain that

$$\mathrm{ev}^*\big([\mathrm{Fl}/G] \circ {}^t[\mathrm{Fl}/G]\big) = [\widetilde{M}_0] \circ {}^t[\widetilde{M}_0]. \tag{45}$$

Here the evaluation map is given by

$$\mathrm{ev} : M_0^{\mathrm{par}} \times_{W_0} M_0^{\mathrm{par}} \to \widetilde{\mathcal{N}}_0/G \times \widetilde{\mathcal{N}}_0/G.$$

Lemma 4.1(a) then implies that the correspondence (45) is the specialization via (43) of a self-correspondence of M_ζ^{par} induced by $w_0 \in \mathfrak{S}_n$.

For our purpose we need to compare the specialization via (13) which is used to define the correspondence $\mathcal{Z}_0$, to the specialization via (43) which yields (45). The former is the specialization coming from the family $\chi_p : M^{\mathrm{par}} \to \mathcal{A}$ as in (12), which is smooth and compatible with intersection-theoretic operations. The latter can be rephrased as coming from the family given by the composition

$$M^{\mathrm{par}} \xrightarrow{\chi_p} \mathcal{A} \to \mathcal{A} /\!\!/ \mathfrak{S}_n \simeq \overline{B}^{\mathrm{par}}.$$

This family is smooth in the following *stacky* sense: we consider instead the composition of χ_p with the map to the stack quotient

$$M^{\mathrm{par}} \xrightarrow{\chi_p} \mathcal{A} \to \mathcal{A}/\mathfrak{S}_n. \tag{46}$$

The specialization map for Chow groups with respect to the family (46) is again compatible with intersection-theoretic operations. Note that since we are dealing with quotient stacks, the required intersection theory, including the functorialities of the first six chapters of [26] needed to define the specialization map, has been established in [24]. Further, by identifying the Chow groups/rings of a Deligne-Mumford stack and its coarse moduli space, we see that for cycles pulled back from correspondences for $\overline{J}^e_{C_\zeta}$ relative over B^{par}_ζ, the two specializations via (13) and (43) only differ by a nonzero constant.

Finally, the pullback of any self-correspondence of $\overline{J}^e_{C_\zeta}$ to M^{par}_ζ is invariant under the $\mathfrak{S}_n$-action permuting the $n!$ copies of $\overline{J}^e_{C_\zeta}$, and hence commutes with the aforementioned self-correspondence of M^{par}_ζ induced by $\omega_0 \in \mathfrak{S}_n$. The commutativity is preserved by specializing via the family (46), which proves the proposition. □

Recall the morphism

$$\Gamma : \mathrm{Corr}^0_{W_0}(M^{\mathrm{par}}_0, M^{\mathrm{par}}_0) \to \mathrm{Corr}^0_B(M_{n,d}, M_{n,d}),$$

from (17). From now on, we use

$$\mathfrak{p}^{\mathrm{par}}_k \in \mathrm{Corr}^0_{W_0}(M^{\mathrm{par}}_0, M^{\mathrm{par}}_0), \quad k = 0, 1, \ldots, 2(\dim M^{\mathrm{par}}_0 - \dim W_0)$$

to denote the projectors associated with the motivic perverse filtration of Theorem 3.14.

PROPOSITION 4.5. – *The following statements hold.*

(a) *We have*

$$\Gamma([\Delta_{M^{\mathrm{par}}_0/W_0}]) = \varepsilon n! \cdot [\Delta_{M_{n,d}/B}].$$

Here $\varepsilon = \pm 1$ is as in Lemma 4.1(b).

(b) *We have the vanishing*

$$\Gamma(\mathfrak{p}^{\mathrm{par}}_k) = 0, \quad k < \binom{n}{2}.$$

(c) *We have the stabilization*

$$\Gamma(\mathfrak{p}^{\mathrm{par}}_k) = [\Delta_{M_{n,d}/B}], \quad k \geq 2(\dim M^{\mathrm{par}}_0 - \dim W_0) - \binom{n}{2}.$$

(d) *We have the semi-orthogonality relation*

$$\Gamma(\mathfrak{p}^{\mathrm{par}}_l) \circ \Gamma(\mathfrak{p}^{\mathrm{par}}_k) = \varepsilon n! \cdot \Gamma(\mathfrak{p}^{\mathrm{par}}_k), \quad k < l.$$

Proof. – (a) follows from pulling back the local statement of Lemma 4.1(b). (b) and (c) follow from Lemma 2.3 and the construction of $\mathfrak{p}_k^{\mathrm{par}}$ in terms of the specialization of algebraic cycles of the form (25). (d) is a consequence of part (a) and Proposition 4.4:

$$\begin{aligned}\Gamma(\mathfrak{p}_l^{\mathrm{par}}) \circ \Gamma(\mathfrak{p}_k^{\mathrm{par}}) &= {}^{\mathrm{t}}[\widetilde{M}_0] \circ \mathfrak{p}_l^{\mathrm{par}} \circ \left([\widetilde{M}_0] \circ {}^{\mathrm{t}}[\widetilde{M}_0]\right) \circ \mathfrak{p}_k^{\mathrm{par}} \circ [\widetilde{M}_0] \\ &= \left({}^{\mathrm{t}}[\widetilde{M}_0] \circ [\widetilde{M}_0]\right) \circ {}^{\mathrm{t}}[\widetilde{M}_0] \circ \left(\mathfrak{p}_l^{\mathrm{par}} \circ \mathfrak{p}_k^{\mathrm{par}}\right) \circ [\widetilde{M}_0] \\ &= \varepsilon n! \cdot {}^{\mathrm{t}}[\widetilde{M}_0] \circ \mathfrak{p}_k^{\mathrm{par}} \circ [\widetilde{M}_0] \\ &= \varepsilon n! \cdot \Gamma(\mathfrak{p}_k^{\mathrm{par}}).\end{aligned}$$

Here we have used that every $\mathfrak{p}_k^{\mathrm{par}}$ satisfies the assumption of $\mathfrak{Z}_0$ in Proposition 4.4, and we applied (a) to deduce the third identity. □

By a direct dimension calculation, we have

$$(\dim M_0^{\mathrm{par}} - \dim W_0) - \binom{n}{2} = d_f.$$

Therefore, after rescaling and relabeling as

$$\mathfrak{p}_k := \frac{\varepsilon}{n!} \cdot \Gamma\left(\mathfrak{p}_{k-\binom{n}{2}}^{\mathrm{par}}\right),$$

we obtain the projectors

$$\mathfrak{p}_k \in \mathrm{Corr}_B^0(M_{n,d}, M_{n,d}), \quad k = 0, 1, \ldots, 2d_f. \tag{47}$$

In Section 5, we will prove that (47) forms a motivic perverse filtration; as for Theorem 3.14, we only need to show that the homological realization of $\mathfrak{p}_k$ recovers the perverse truncation functor.

5. Sheaf-theoretic operations

The purpose of this section is to prove a general specialization result for the decomposition theorem (Proposition 5.1). Then we apply it to the Hitchin systems we studied in the previous sections and complete the proofs of all the main theorems.

5.1. Specializations

In this section, we finish the proof of Theorem 3.14 by addressing the compatibility of the Chow and sheaf-theoretic specialization maps, and their relationship with the perverse truncation. Our setup is the following: let $T \simeq \mathbb{A}^1$ and let $\pi : B \to T$ be a smooth morphism. Consider two proper morphisms $f : X \to B$, $g : Y \to B$ such that the compositions $\pi \circ f : X \to T$, $\pi \circ g : Y \to T$ are smooth:

$$\begin{array}{ccccc} X & \xrightarrow{f} & B & \xleftarrow{g} & Y \\ & \searrow & \downarrow{\scriptstyle \pi} & \swarrow & \\ & & T. & & \end{array}$$

We think of $f : X \to B$, $g : Y \to B$ as two families of proper morphisms over T.

Suppose we are given over a nonempty open subset $T^\circ \subset T$ a correspondence

$$\mathfrak{Z} \in \mathrm{Corr}^*_{B_{T^\circ}}(X_{T^\circ}, Y_{T^\circ}) = \mathrm{CH}_*(X_{T^\circ} \times_{B_{T^\circ}} Y_{T^\circ}), \tag{48}$$

which puts together a family of correspondences $\mathfrak{Z}_\eta$ for $\eta \in T^\circ$:

$$\mathfrak{Z}_\eta \in \mathrm{Corr}^*_{B_\eta}(X_\eta, Y_\eta) = \mathrm{CH}_*(X_\eta \times_{B_\eta} Y_\eta).$$

Let $0 \in T$ be the special point. Then by the specialization map for Chow groups via

$$X_\eta \to B_\eta \quad \rightsquigarrow \quad X_0 \to B_0,$$

we obtain a correspondence

$$\mathfrak{Z}_0 \in \mathrm{Corr}^*_{B_\eta}(X_\eta, Y_\eta) = \mathrm{CH}_*(X_0 \times_{B_0} Y_0).$$

Also recall from [19, Lemma 2.21] that the cycle class $\mathrm{cl}(\mathfrak{Z}_\eta) \in H^{\mathrm{BM}}_{2*}(X_\eta \times_{B_\eta} Y_\eta, \mathbb{Q})$ induces a morphism in $D^{\mathrm{b}}_{\mathrm{c}}(B_\eta)$:

$$\mathrm{cl}(\mathfrak{Z}_\eta) : Rf_{\eta *}\mathbb{Q}_{X_\eta} \to Rg_{\eta *}\mathbb{Q}_{Y_\eta}[*], \tag{49}$$

where we ignore the precise shifting index. Similarly, we have a morphism in $D^{\mathrm{b}}_{\mathrm{c}}(B_0)$:

$$\mathrm{cl}(\mathfrak{Z}_0) : Rf_{0*}\mathbb{Q}_{X_0} \to Rg_{0*}\mathbb{Q}_{Y_0}[*]. \tag{50}$$

PROPOSITION 5.1. – *Let $\mathfrak{Z}_\eta, \mathfrak{Z}_0$ be as above. For $k \in \mathbb{Z}$, if*

$${}^{\mathfrak{p}}\mathcal{H}^k(\mathrm{cl}(\mathfrak{Z}_\eta)) : {}^{\mathfrak{p}}\mathcal{H}^k(Rf_{\eta *}\mathbb{Q}_{X_\eta}) \to {}^{\mathfrak{p}}\mathcal{H}^k(Rg_{\eta *}\mathbb{Q}_{Y_\eta}[*]) \tag{51}$$

is an isomorphism (resp. zero), then the same holds for

$${}^{\mathfrak{p}}\mathcal{H}^k(\mathrm{cl}(\mathfrak{Z}_0)) : {}^{\mathfrak{p}}\mathcal{H}^k(Rf_{0*}\mathbb{Q}_{X_0}) \to {}^{\mathfrak{p}}\mathcal{H}^k(Rg_{0*}\mathbb{Q}_{Y_0}[*]). \tag{52}$$

Proof. – Let $\overline{\mathfrak{Z}} \in \mathrm{CH}_*(X \times_B Y)$ be any extension of $\mathfrak{Z} \in \mathrm{CH}_*(X_{T^\circ} \times_{B_{T^\circ}} Y_{T^\circ})$ in (48), e.g., by taking the Zariski closure. The cycle class $\mathrm{cl}(\overline{\mathfrak{Z}}) \in H^{\mathrm{BM}}_{2*}(X \times_B Y, \mathbb{Q})$ induces a morphism in $D^{\mathrm{b}}_{\mathrm{c}}(B)$:

$$\mathrm{cl}(\overline{\mathfrak{Z}}) : Rf_*\mathbb{Q}_X \to Rg_*\mathbb{Q}_Y[*]. \tag{53}$$

Then by construction and proper base change, the morphism $\mathrm{cl}(\mathfrak{Z}_0)$ in (50) is obtained by applying i^* to (53) for the inclusion $i : B_0 \hookrightarrow B$:

$$\mathrm{cl}(\mathfrak{Z}_0) = i^*\mathrm{cl}(\overline{\mathfrak{Z}}) : Rf_{0*}\mathbb{Q}_{X_0} \to Rg_{0*}\mathbb{Q}_{Y_0}[*].$$

We give an alternative construction of (50) via nearby and vanishing cycles following [14, 11]. Recall the distinguished triangle

$$i^*[-1] \to \psi_B[-1] \xrightarrow{\mathrm{can}} \phi_B \xrightarrow{+1}, \tag{54}$$

where $\psi_B, \phi_B : D^{\mathrm{b}}_{\mathrm{c}}(B) \to D^{\mathrm{b}}_{\mathrm{c}}(B_0)$ are the nearby and vanishing cycle functors respectively. Since X is smooth over T, we have $\phi_X\mathbb{Q}_X \simeq 0 \in D^{\mathrm{b}}_{\mathrm{c}}(X_0)$ and by proper base change for ϕ,

$$\phi_B Rf_*\mathbb{Q}_X \simeq Rf_{0*}\phi_X\mathbb{Q}_X \simeq 0 \in D^{\mathrm{b}}_{\mathrm{c}}(B_0).$$

Similarly, we have

$$\phi_B Rg_*\mathbb{Q}_Y \simeq Rg_{0*}\phi_Y\mathbb{Q}_Y \simeq 0 \in D^{\mathrm{b}}_{\mathrm{c}}(B_0).$$

We now apply the distinguished triangle (54) to the morphism $\mathrm{cl}(\overline{\mathfrak{Z}})$ in (53), which yields a commutative diagram

$$\begin{array}{ccc} Rf_{0*}\mathbb{Q}_{X_0} & \xrightarrow{\mathrm{cl}(\mathfrak{Z}_0)} & Rg_{0*}\mathbb{Q}_{Y_0}[*] \\ \downarrow\simeq & & \downarrow\simeq \\ \psi_B Rf_*\mathbb{Q}_X & \xrightarrow{\psi_B \mathrm{cl}(\overline{\mathfrak{Z}})} & \psi_B Rg_*\mathbb{Q}_Y[*]. \end{array} \tag{55}$$

Here both columns are isomorphisms since the vanishing cycles are trivial. Moreover, since ψ_B is the *nearby cycle* functor, the bottom morphism $\psi_B \mathrm{cl}(\overline{\mathfrak{Z}})$ is fully determined by the morphism $\mathrm{cl}(\mathfrak{Z}_\eta)$ in (49).

Finally, we recall the fact that $\psi_B[-1]$ is perverse t-exact, and hence commutes with taking perverse cohomology. Therefore, if (51) is an isomorphism (resp. zero), then

$${}^{\mathfrak{p}}\mathcal{H}^k(\psi_B \mathrm{cl}(\overline{\mathfrak{Z}})) : {}^{\mathfrak{p}}\mathcal{H}^k(\psi_B Rf_*\mathbb{Q}_X) \to {}^{\mathfrak{p}}\mathcal{H}^k(\psi_B Rg_*\mathbb{Q}_Y[*])$$

is also an isomorphism (resp. zero). By the diagram (55) the same holds for (52). □

A useful consequence of Proposition 5.1 is that the validity of the Corti-Hanamura motivic decomposition conjecture [19] is preserved under specialization. This also completes the proof of Theorem 3.14.

COROLLARY 5.2. – *Let $X \xrightarrow{f} B \xrightarrow{\pi} T$ be as above, with f proper and $\pi, \pi \circ f$ smooth. If the motivic decomposition conjecture holds for $X_\eta \to B_\eta$, then the specializations of the projectors provide a motivic decomposition for $X_0 \to B_0$.*

Proof. – The (semi-)orthogonal projectors of $X_\eta \to B_\eta$ specialize immediately to (semi-)orthogonal projectors of $X_0 \to B_0$, while the relative diagonal class $[\Delta_{X_\eta/B_\eta}]$ specializes to $[\Delta_{X_0/B_0}]$. Now Proposition 5.1 guarantees that the homological realization of the specialized projectors recovers the decomposition theorem for $X_0 \to B_0$. □

5.2. Correspondences

For our purpose, we connect the decomposition theorem associated with the Hitchin maps:

$$h_0 : M_0^{\mathrm{par}} \to W_0, \quad f : M_{n,d} \to B \subset W_0.$$

The diagram (16) yields sheaf-theoretic correspondences

$$[\widetilde{M_0}] : Rh_{0*}\mathbb{Q}_{M_0^{\mathrm{par}}} \to Rf_*\mathbb{Q}_{M_{n,d}}\left[-2\binom{n}{2}\right], \quad {}^{\mathrm{t}}[\widetilde{M_0}] : Rf_*\mathbb{Q}_{M_{n,d}}\left[-2\binom{n}{2}\right] \to Rh_{0*}\mathbb{Q}_{M_0^{\mathrm{par}}}, \tag{56}$$

which take places in the derived category $D_{\mathrm{c}}^{\mathrm{b}}(W_0)$.

The following lemma is a global version of Lemma 4.1(c).

LEMMA 5.3. – *There is a natural Weyl group $\mathfrak{S}_n$-action on the object*

$$Rh_{0*}\mathbb{Q}_{M_0^{\mathrm{par}}} \in D_{\mathrm{c}}^{\mathrm{b}}(W_0);$$

the anti-invariant part is

$$\left(Rh_{0*}\mathbb{Q}_{M_0^{\mathrm{par}}}\right)^{\mathrm{sgn}} = Rf_*\mathbb{Q}_{M_{n,d}}\left[-2\binom{n}{2}\right] \in D_{\mathrm{c}}^{\mathrm{b}}(W_0).$$

Furthermore, under this isomorphism the correspondences (56) *are projecting to the anti-invariant part and the natural inclusion from the anti-invariant part respectively.*

Proof. – By Proposition 4.2, the diagram (16) is the pullback of the local diagram (38) along smooth morphisms ev_p. Therefore, Lemma 5.3 follows from pulling back the G-equivariant version of Lemma 4.1(c). □

5.3. Proofs of Theorems 1.1, 1.2, 1.4, 1.5

In this section, we complete the proofs of all the main theorems.

5.3.1. *Proof of Theorem 1.1*. – We start with the proof of Theorem 1.1. By Proposition 3.2, it suffices to show that the correspondences we constructed in (47) form a motivic perverse filtration. By Proposition 4.5, it remains to check that the homological realization

$$\mathfrak{p}_k(Rf_*\mathbb{Q}_{M_{n,d}}) \to Rf_*\mathbb{Q}_{M_{n,d}}$$

recovers the perverse truncation

$${}^{\mathfrak{p}}\tau_{\leq k+\dim B}Rf_*\mathbb{Q}_{M_{n,d}} \to Rf_*\mathbb{Q}_{M_{n,d}}.$$

We know that this statement holds for $h_0 : M_0^{\mathrm{par}} \to W_0$ and $\mathfrak{p}_k^{\mathrm{par}}$ by Theorem 3.14 (proven in Section 5.1). Now, we consider the correspondence of $\mathfrak{p}_k$ given by the composition:

$$\mathfrak{p}_k : Rf_*\mathbb{Q}_{M_{n,d}} \xrightarrow{[\widetilde{M}_0]} Rh_{0*}\mathbb{Q}_{M_0^{\mathrm{par}}}\left[2\binom{n}{2}\right] \xrightarrow{\mathfrak{p}_k^{\mathrm{par}}} Rh_{0*}\mathbb{Q}_{M_0^{\mathrm{par}}}\left[2\binom{n}{2}\right] \xrightarrow{{}^{\mathrm{t}}[\widetilde{M}_0]} Rf_*\mathbb{Q}_{M_{n,d}}.$$

By Lemma 5.3, we have a direct sum decomposition induced by the $\mathfrak{S}_n$-action:

$$Rh_{0*}\mathbb{Q}_{M_0^{\mathrm{par}}}\left[2\binom{n}{2}\right] = (Rh_{0*}\mathbb{Q}_{M_0^{\mathrm{par}}})^{\mathrm{sgn}}\left[2\binom{n}{2}\right] \oplus \text{others}, \tag{57}$$

and applying $\mathfrak{p}_k$ to $Rf_*\mathbb{Q}_{M_{n,d}}$ is induced by applying $\mathfrak{p}_k^{\mathrm{par}}$ on the first factor above. In particular, it coincides with the perverse truncation. This completes the proof. □

5.3.2. *Proof of Theorem 1.2*. – We first note the following result of [1, Proposition 6.4].

Fact 1. – The compactified Jacobian fibration

$$h_\eta : M_\eta^{\mathrm{par}} \to W_\eta$$

of Section 2.2(a) satisfies the relative Lefschetz standard conjecture.

More precisely, this is due to the resolutions of the Lefschetz standard conjecture for abelian varieties [40, 37] and the fact that the decomposition theorem associated with a compactified Jacobian fibration associated with integral locally planar curves has *full support* by the Ngô support theorem [54].

Our strategy is to reduce the case of $f : M_{n,d} \to B$ to $h_\eta : M_\eta^{\mathrm{par}} \to W_\eta$.

Let $\sigma \in H^2(M_{n,d}, \mathbb{Q})$ be a relative ample class. By [43], we can write σ in terms of tautological classes as follows. We first recall that we have a canonical isomorphism

$$H^2(M_{n,d}, \mathbb{Q}) = H^2(\widehat{M}_{n,d}, \mathbb{Q}) \oplus H^2(J_\Sigma, \mathbb{Q}), \tag{58}$$

where $\widehat{M}_{n,d}$ is the PGL_n-Hitchin moduli space and J_Σ is the Jacobian variety of the curve Σ; see [12, Section 2.4]. The first summand on the right-hand side of (58) spans a 1-dimensional subspace, generated by the relative ample tautological class given by the Künneth component of normalized second Chern character $\widetilde{\mathrm{ch}}_2(\mathcal{U})$ with respect to the point class $[\mathrm{pt}] \in H^2(\Sigma, \mathbb{Q})$ on the curve. This class can be lifted to the ($\mathbb{Q}$-)divisor class

$$c_2([\Sigma]) := \pi_{M*}\left(\pi_\Sigma^*[\Sigma] \cup \widetilde{\mathrm{ch}}_2(\mathcal{U})\right) \in \mathrm{Pic}(M_{n,d})_\mathbb{Q}.$$

Therefore, under the decomposition (58), we may present the relative ample class σ as

$$\sigma = c_2([\Sigma]) + \theta, \tag{59}$$

where θ is induced by an ample divisor on J_Σ. The right-hand side is the pullback of a divisor on M^{mero} with the same tautological presentation (59). As in Section 3.4, pulling back this class yields relative ample classes on each variety M_η^{par} with $\eta \in T$.

In conclusion, we obtain the following fact.

Fact 2. – There exists a divisor $\sigma_0^{\mathrm{par}} \in \mathrm{Pic}(M_0^{\mathrm{par}})_\mathbb{Q}$ satisfying

$$\pi^*\sigma = \iota^*\sigma_0^{\mathrm{par}}, \tag{60}$$

where ι, π are defined in the diagram (16). Moreover, this divisor σ_0^{par} is obtained as the specialization of a relative ample divisor

$$\sigma_\eta^{\mathrm{par}} \in \mathrm{Pic}(M_\eta^{\mathrm{par}})_\mathbb{Q}$$

for general η.

Note that in general the cup-product with respect to an algebraic cycle E is induced by a correspondence given by the $\Delta_* E$, where Δ is the (relative) diagonal. Now we consider the correspondences

$$\Delta_*\sigma^i \in \mathrm{Corr}^*_B(M_{n,d}, M_{n,d}), \quad \Delta_*(\sigma_0^{\mathrm{par}})^i \in \mathrm{Corr}^*_{W_0}(M_0^{\mathrm{par}}, M_0^{\mathrm{par}}).$$

Fact 3. – We have

$$\Gamma\left(\Delta_*(\sigma_0^{\mathrm{par}})^i\right) = \varepsilon n! \cdot \Delta_*\sigma^i \in \mathrm{Corr}^*_B(M_{n,d}, M_{n,d}).$$

In particular, after the homological realization we obtain the commutative diagram

$$\begin{array}{ccc} Rf_*\mathbb{Q}_{M_{n,d}} & \xrightarrow{\varepsilon n!\cdot(\cup\sigma^i)} & Rf_*\mathbb{Q}_{M_{n,d}}[2i] \\ \downarrow{\scriptstyle [\widetilde{M}_0]} & & {\scriptstyle {}^t[\widetilde{M}_0]}\uparrow \\ Rh_{0*}\mathbb{Q}_{M_0^{\mathrm{par}}}\left[2\binom{n}{2}\right] & \xrightarrow{\cup(\sigma_0^{\mathrm{par}})^i} & Rh_{0*}\mathbb{Q}_{M_0^{\mathrm{par}}}\left[2\binom{n}{2} + 2i\right]. \end{array}$$

Proof of Fact 3. – It suffices to show the following relation of correspondences

$${}^t[\widetilde{M}_0] \circ \Delta_*(\sigma_0^{\mathrm{par}})^i = \Delta_*\sigma^i \circ {}^t[\widetilde{M}_0] \in \mathrm{Corr}^*_{W_0}(M_0^{\mathrm{par}}, M_{n,d}). \tag{61}$$

Then by Proposition 4.5(a), we have

$$\Gamma(\Delta_*(\sigma_0^{\mathrm{par}})^i) = {}^t[\widetilde{M}_0] \circ \Delta_*(\sigma_0^{\mathrm{par}})^i \circ [\widetilde{M}_0] = \Delta_*\sigma^i \circ {}^t[\widetilde{M}_0] \circ [\widetilde{M}_0] = \varepsilon n! \cdot \Delta_*\sigma^i,$$

which proves Fact 3. To see the relation (61), we expand the left-hand side

$$\begin{aligned}
{}^{t}[\widetilde{M}_0] \circ \Delta_*(\sigma_0^{\mathrm{par}})^i &= p_{13*}\delta^! \left(\Delta_*(\sigma_0^{\mathrm{par}})^i \times {}^{t}[\widetilde{M}_0]\right) \\
&= p_{13*}\delta^! \left(\left(q_2^*(\sigma_0^{\mathrm{par}})^i \cap [\Delta_{M_0^{\mathrm{par}}/W_0}]\right) \times {}^{t}[\widetilde{M}_0]\right) \\
&= p_{13*}\delta^! \left([\Delta_{M_0^{\mathrm{par}}/W_0}] \times \left(r_1^*(\sigma_0^{\mathrm{par}})^i \cap {}^{t}[\widetilde{M}_0]\right)\right) \\
&= r_1^*(\sigma_0^{\mathrm{par}})^i \cap {}^{t}[\widetilde{M}_0] = {}^{t}(\iota^*(\sigma_0^{\mathrm{par}})^i \cap [\widetilde{M}_0]).
\end{aligned}$$

Here $\delta^!$ is the refined Gysin pullback with respect to the regular (absolute diagonal) embedding $\Delta : M_0^{\mathrm{par}} \to M_0^{\mathrm{par}} \times M_0^{\mathrm{par}}$, and $p_{13} : M_0^{\mathrm{par}} \times_{W_0} M_0^{\mathrm{par}} \times_{W_0} M_{n,d} \to M_0^{\mathrm{par}} \times_{W_0} M_{n,d}$, $q_2 : M_0^{\mathrm{par}} \times_{W_0} M_0^{\mathrm{par}} \to M_0^{\mathrm{par}}$, and $r_1 : M_0^{\mathrm{par}} \times_{W_0} M_{n,d} \to M_0^{\mathrm{par}}$ are the natural projections. Similarly, we compute the right-hand side of (61):

$$\begin{aligned}
\Delta_*\sigma^i \circ {}^{t}[\widetilde{M}_0] &= p_{13*}\delta^! \left({}^{t}[\widetilde{M}_0] \times \Delta_*\sigma^i\right) \\
&= p_{13*}\delta^! \left({}^{t}[\widetilde{M}_0] \times \left(r_1^*\sigma^i \cap [\Delta_{M_{n,d}/B}]\right)\right) \\
&= p_{13*}\delta^! \left(\left(q_2^*\sigma^i \cap {}^{t}[\widetilde{M}_0]\right) \times [\Delta_{M_{n,d}/B}]\right) \\
&= q_2^*\sigma^i \cap {}^{t}[\widetilde{M}_0] = {}^{t}(\pi^*\sigma^i \cap [\widetilde{M}_0]).
\end{aligned}$$

This time $\delta^!$ is the refined Gysin pullback with respect to $\Delta : M_{n,d} \to M_{n,d} \times M_{n,d}$, and p_{13}, q_2, r_1 are the corresponding natural projections. Hence (61) follows from (60). □

Now we construct the cycle $\mathfrak{Z}_{\sigma,i}$ for the relative ample class σ on $M_{n,d}$. We start with the correspondence as in Fact 1 for M_η^{par}:

$$\mathfrak{Z}_{\sigma_\eta^{\mathrm{par}},i} \in \mathrm{Corr}^*_{W_\eta}(M_\eta^{\mathrm{par}}, M_\eta^{\mathrm{par}}), \tag{62}$$

which gives the inverse of the Lefschetz symmetry induced by $(\sigma_\eta^{\mathrm{par}})^i$ for general η.

By Proposition 5.1, its specialization

$$\mathfrak{Z}_{\sigma_0^{\mathrm{par}},i} \in \mathrm{Corr}^*_{W_0}(M_0^{\mathrm{par}}, M_0^{\mathrm{par}}) \tag{63}$$

gives the inverse of the Lefschetz symmetry induced by $(\sigma_0^{\mathrm{par}})^i$, which respects the decomposition given by the $\mathfrak{S}_n$-structure since (62) is pulled back from a self-correspondence of $\overline{J}^e_{C_\zeta}$ relative over B_ζ^{par}. In particular, the correspondence (63) induces an isomorphism on the perverse cohomology sheaves of the first summand of (57).

Finally, we set

$$\mathfrak{Z}_{\sigma,i} := \Gamma\left(\mathfrak{Z}_{\sigma_0^{\mathrm{par}},i}\right) \in \mathrm{Corr}^*_B(M_{n,d}, M_{n,d}).$$

As in the proof of Theorem 1.1 in Section 5.3.1, the diagram of Fact 3 together with Lemma 5.3 implies that the sheaf-theoretic operator $\cup\sigma^i$ coincides with the restriction of the operator $\cup(\sigma_0^{\mathrm{par}})^i$ to the anti-invariant component with respect to the $\mathfrak{S}_n$-equivariant structure. Since the inverse operator (63) of $\cup(\sigma_0^{\mathrm{par}})^i$ also respects the decomposition (57), we conclude that the homological realization of $\mathfrak{Z}_{\sigma,i}$ gives the inverse of $\cup\sigma^i$ as desired. □

5.3.3. *Proof of Theorem 1.4.* – We consider the diagram

$$\begin{array}{ccc} \Sigma \times \widetilde{M}_0 & \xrightarrow{\iota} & \Sigma \times M_0^{\mathrm{par}} \\ \downarrow{\scriptstyle \pi} & & \\ \Sigma \times M_{n,d} & & \end{array} \tag{64}$$

given by the base change of (16). For convenience, we use the same notation for this diagram as for (16); e.g., we still use $[\widetilde{M}_0]$ to denote the correspondence from $\Sigma \times M_{n,d}$ to $\Sigma \times M_0^{\mathrm{par}}$.

By the construction of the normalized Chern character, we have

$$\pi^* \widetilde{\mathrm{ch}}(\mathcal{U}) = \iota^* \widetilde{\mathrm{ch}}(\mathcal{U}).$$

Here the normalized Chern character on the left-hand side lies over $\Sigma \times M_{n,d}$ and the normalized Chern character on the right-hand side lies over $\Sigma \times M_0^{\mathrm{par}}$; their match after pulling back to $\Sigma \times \widetilde{M}_0$ follows from the fact that both normalized Chern characters are pulled back from $\Sigma \times M^{\mathrm{mero}}$. By the same argument as for Fact 3 in Section 5.3.2, we have

$$\Gamma\left(\Delta_* \widetilde{\mathrm{ch}}_k(\mathcal{U})\right) = \varepsilon n! \cdot \Delta_* \widetilde{\mathrm{ch}}_k(\mathcal{U}) \in \mathrm{Corr}^*_{\Sigma \times B}(\Sigma \times M_{n,d}, \Sigma \times M_{n,d}), \tag{65}$$

where the two sides concern the normalized Chern characters on $\Sigma \times M_0^{\mathrm{par}}$ and $\Sigma \times M_{n,d}$ respectively. Recall the projectors $\mathfrak{p}_k$ from (16) and we use the same notation to denote the pullback of these projectors to the varieties in (64). We define

$$\mathfrak{q}_{k+1} := \Delta_{\Sigma \times M_{n,d}/\Sigma \times B} - \mathfrak{p}_k = \frac{\varepsilon}{n!} \cdot \Gamma\left(\Delta_{\Sigma \times M_0^{\mathrm{par}}/\Sigma \times B} - \mathfrak{p}^{\mathrm{par}}_{k-\binom{n}{2}}\right),$$

where the second identity is a consequence of Proposition 4.5(a).

For our purpose, we want to prove the relation:

$$\mathfrak{q}_{i+k+1} \circ \Delta_* \widetilde{\mathrm{ch}}_k(\mathcal{U}) \circ \mathfrak{p}_i = 0;$$

using (65) this can be rewritten as

$$\Gamma\left(\mathfrak{q}^{\mathrm{par}}_{i+k+1-\binom{n}{2}}\right) \circ \Gamma\left(\Delta_* \widetilde{\mathrm{ch}}_k(\mathcal{U})\right) \circ \Gamma\left(\mathfrak{p}^{\mathrm{par}}_{i-\binom{n}{2}}\right) = 0. \tag{66}$$

Since all the correspondences

$$\mathfrak{q}^{\mathrm{par}}_{i+k+1-\binom{n}{2}}, \quad \Delta_* \widetilde{\mathrm{ch}}_k(\mathcal{U}), \quad \mathfrak{p}^{\mathrm{par}}_{i-\binom{n}{2}} \in \mathrm{Corr}^*_{\Sigma \times W_0}(\Sigma \times M_0^{\mathrm{par}}, \Sigma \times M_0^{\mathrm{par}})$$

are specializations of cycles pulled back from correspondences for $\overline{J}^e_{C_\zeta}$ relative over B^{par}_ζ, by Proposition 4.4 the relation (66) follows from the relation

$$\mathfrak{q}^{\mathrm{par}}_{i+k+1-\binom{n}{2}} \circ \Delta_* \widetilde{\mathrm{ch}}_k(\mathcal{U}) \circ \mathfrak{p}^{\mathrm{par}}_{i-\binom{n}{2}} = 0 \in \mathrm{Corr}^*_{\Sigma \times W_0}(\Sigma \times M_0^{\mathrm{par}}, \Sigma \times M_0^{\mathrm{par}}).$$

This is given by Theorem 3.15. The proof is completed. □

5.3.4. *Proof of Theorem 1.5.* – We denote by

$$\mathfrak{C} \in \mathrm{Corr}^0_{W_0}(M_0^{\mathrm{par}}, M_0^{\mathrm{par}\prime}), \quad \mathfrak{C}' \in \mathrm{Corr}^0_{W_0}(M_0^{\mathrm{par}\prime}, M_0^{\mathrm{par}})$$

the correspondences constructed in Theorem 3.16 inducing an isomorphism

$$h(M_0^{\mathrm{par}}) \simeq h(M_0^{\mathrm{par}\prime}).$$

Then an identical argument as above (using Proposition 4.4) implies that

$$\overline{\mathfrak{C}} := \frac{\varepsilon}{n!} \cdot \mathfrak{C} \in \mathrm{Corr}^0_B(M_{n,d}, M_{n,d'}), \quad \overline{\mathfrak{C}}' := \frac{\varepsilon}{n!} \cdot \mathfrak{C}' \in \mathrm{Corr}^0_B(M_{n,d'}, M_{n,d})$$

induces an isomorphism as desired. □

5.3.5. *Remarks on the twisted case.* – As in Remark 2.1, one can also consider the $\mathfrak{D}$-twisted case with $\mathfrak{D} > 0$. Here we discuss briefly the possible strategies to prove the main theorems in the $\mathfrak{D}$-twisted case.

The decomposition theorem associated with $f : M^{\mathfrak{D}}_{n,d} \to B^{\mathfrak{D}}$ has full support; see Remark 2.1 and [16]. In this case, Theorem 1.2 follows directly from [1, Proposition 6.4] as Fact 1 above, and the argument of Section 5.3.3 is not needed.

On the other hand, due to the failure of the $\mathfrak{D}$-twisted version of Lemma 2.2, for general $\eta \in T$ we can only guarantee that all the spectral curves are reduced, but there may exist non-integral ones. Therefore, the extension of Arinkin's Fourier-Mukai transform over reduced locally planar curves [50, 51] may be needed to construct the projectors for the motivic decomposition.

6. Questions and conjectures

In this section, we discuss some conjectures and open questions concerning algebraic cycles for the Hitchin system.

6.1. Multiplicativity

A central question raised by de Cataldo, Hausel, and Migliorini in accompany with their discovery of the P=W conjecture (see [12, Introduction]) is the *multiplicativity* of the perverse filtration:

$$\cup : P_k H^*(M_{n,d}, \mathbb{Q}) \times P_l H^*(M_{n,d}, \mathbb{Q}) \to P_{k+l} H^*(M_{n,d}, \mathbb{Q}). \tag{67}$$

This phenomenon now is a consequence of the resolutions of the P=W conjecture, whose proofs rely heavily on the tautological generation result of Markman [43].

In general, the multiplicativity holds for Leray filtrations associated with proper morphisms, but fails for the perverse filtration. The main reason is that the (sheaf-theoretic) cup-product does not interact nicely with the perverse truncation functor in general.

For the Hitchin system, we expect that the sheaf-theoretic enhancement of (67) holds.

CONJECTURE 6.1. – *The perverse truncation functor interacts with cup-product as follows*

$$\cup : {}^{\mathfrak{p}}\tau_{\leq k} Rf_* \mathbb{Q}_{M_{n,d}} \times {}^{\mathfrak{p}}\tau_{\leq l} Rf_* \mathbb{Q}_{M_{n,d}} \to {}^{\mathfrak{p}}\tau_{\leq k+l-(\dim M_{n,d} - d_f)} Rf_* \mathbb{Q}_{M_{n,d}}.$$

By the proof of Theorem 1.1, the perverse filtration and its sheaf-theoretic enhancement are governed by the algebraic cycles $\mathfrak{p}_k, \mathfrak{q}_k$. We propose the following conjecture which further strengthens Conjecture 6.1; this also provides an explanation of the multiplicativity phenomenon in terms of *tautological relations* concerning $\mathfrak{p}_k, \mathfrak{q}_k$.

CONJECTURE 6.2. – *The motivic perverse filtration* (6) *is multiplicative with respect to cup-product:*

$$\cup : P_k h(M_{n,d}) \times P_l h(M_{n,d}) \to P_{k+l} h(M_{n,d}),$$

i.e., the following relations hold

$$\mathfrak{q}_{k+l+1} \circ [\Delta^{\mathrm{sm}}_{M_{n,d}/B}] \circ (\mathfrak{p}_k \times \mathfrak{p}_l) = 0 \in \mathrm{CH}_*(M_{n,d} \times_B M_{n,d} \times_B M_{n,d}).$$

Here $\Delta^{\mathrm{sm}}_{M_{n,d}/B}$ is the relative small diagonal in the relative triple product.

By Theorem 3.6(b), Conjecture 6.2 holds for the parabolic Hitchin system $h_\eta : M_\eta^{\mathrm{par}} \to W_\eta$ for general η or the elliptic part of the Hitchin system $f^{\mathrm{ell}} : M_{n,d}^{\mathrm{ell}} \to B^{\mathrm{ell}}$. From the perspective of this paper, the main difficulty in approaching Conjecture 6.2 is to understand the small diagonal cycle for $M_{n,d}$ using parabolic moduli spaces.

Conjectures 6.1 and 6.2, which concern multiplicative filtrations at the sheaf-theoretic and the motivic levels respectively, are further expected to be refined into multiplicative decompositions; see Remark 1.3(b).

6.2. Refined motivic decomposition

Support theorems yield rich structures for the decomposition theorem associated with the Hitchin system. For example, by studying the decomposition theorem over the locus $B^{\mathrm{red}} \subset B$ of the Hitchin base formed by reduced spectral curves, de Cataldo, Heinloth, and Migliorini showed in [13] that every partition of n contributes a support of $Rf_*\mathbb{Q}_{M_{n,d}}$. Therefore, following the philosophy of Corti-Hanamura [19], we expect to have the motives which lift the contribution of the support indexed by each partition.

QUESTION 6.3. – *Can we find algebraic cycles which yield a refinement of the motivic decomposition* (5) *based on supports of the decomposition theorem for $f : M_{n,d} \to B$?*

The conjectural refinement of Question 6.3 is mysterious for the following reasons. As we commented in Remark 2.1, if we consider the decomposition theorem for the $\mathfrak{D}$-twisted Hitchin system, the support theorem behaves very differently for $\mathfrak{D} = 0$ and $\mathfrak{D} > 0$. In particular, the decomposition theorem has full support when $\mathfrak{D} > 0$. Therefore, the conjectural motives of Question 6.3, which lift the contribution of smaller supports of the decomposition theorem, should only exist for $\mathfrak{D} = 0$.

6.3. Singular moduli spaces

When the degree d is not coprime to the rank n, the Hitchin moduli space $M_{n,d}$ is singular. Many properties concerning the decomposition theorem and the perverse filtration in the coprime setting are expected to be generalized to the non-coprime setting. We expect that there should exist algebraic cycles which induce these sheaf-theoretic structures.

A challenging question in this direction is to construct relative *BPS motives* for all $M_{n,d}$ without the coprime assumption. More precisely, from the perspective of cohomological Donaldson-Thomas theory the role played by the constant sheaf $\mathbb{Q}_{n,d}$ should be replaced by a mysterious perverse sheaf, called the BPS sheaf

$$\Phi_{\mathrm{BPS},d} \in \mathrm{Perv}(M_{n,d}),$$

when $M_{n,d}$ is possibly singular. This perverse sheaf recovers the (shifted) constant sheaf $\mathbb{Q}_{M_{n,d}}[\dim M_{n,d}]$ when $(n,d)=1$, but is in general more complicated than the intersection cohomology complex $\mathrm{IC}_{M_{n,d}}$; see [56] or [45, Section 0.4]. The sheaf-theoretic χ-independence result (8) is generalized by Kinjo-Koseki [36] using the BPS perverse sheaf:

$$Rf_{d*}\Phi_{\mathrm{BPS},d} \simeq Rf_{d'*}\Phi_{\mathrm{BPS},d'}. \tag{68}$$

QUESTION 6.4. – *Can we lift the sheaf-theoretic χ-independence identity* (68) *to relative Chow motives over the Hitchin base B?*

When $(n,d) \neq 1$, we have

$$\Phi_{\mathrm{BPS},d} = \mathrm{IC}_{M_{n,d}} \oplus \varphi_d,$$

where φ_d is a perverse sheaf supported on the singular locus of $M_{n,d}$. It would be interesting to see how the extra summand φ_d is detected by algebraic cycles. The BPS motive which lifts $Rf_{d*}\Phi_{\mathrm{BPS},d}$ may be viewed as the motivic Gopakumar-Vafa invariant associated with the local Calabi-Yau 3-fold $X := \mathrm{Tot}_\Sigma(\omega_\Sigma \oplus \mathcal{O}_\Sigma)$.

BIBLIOGRAPHY

[1] G. ANCONA, M. CAVICCHI, R. LATERVEER, G. SACCÀ, Relative and absolute Lefschetz standard conjectures for some Lagrangian fibrations, *J. Lond. Math. Soc.* **111** (2025), Paper No. e70133, 27.

[2] D. ANDERSON, R. GONZALES, S. PAYNE, Equivariant Grothendieck-Riemann-Roch and localization in operational K-theory, *Algebra Number Theory* **15** (2021), 341–385.

[3] D. ARINKIN, Autoduality of compactified Jacobians for curves with plane singularities, *J. Algebraic Geom.* **22** (2013), 363–388.

[4] Y. BAE, D. MAULIK, J. SHEN, Q. YIN, On generalized Beauville decompositions, preprint arXiv:2402.08861.

[5] P. BAUM, W. FULTON, R. MACPHERSON, Riemann-Roch for singular varieties, *Inst. Hautes Études Sci. Publ. Math.* **45** (1975), 101–145.

[6] A. BEAUVILLE, Sur l'anneau de Chow d'une variété abélienne, *Math. Ann.* **273** (1986), 647–651.

[7] A. Beauville, M. S. Narasimhan, S. Ramanan, Spectral curves and the generalised theta divisor, *J. reine angew. Math.* **398** (1989), 169–179.

[8] A. A. Beĭlinson, J. Bernstein, P. Deligne, Faisceaux pervers, in *Analysis and topology on singular spaces, I (Luminy, 1981)*, Astérisque **100**, Soc. Math. France, 1982, 5–171.

[9] M. A. de Cataldo, D. Maulik, J. Shen, Hitchin fibrations, abelian surfaces, and the $P = W$ conjecture, *J. Amer. Math. Soc.* **35** (2022), 911–953.

[10] M. A. A. de Cataldo, Hodge-theoretic splitting mechanisms for projective maps, *J. Singul.* **7** (2013), 134–156.

[11] M. A. A. de Cataldo, Perverse Leray filtration and specialisation with applications to the Hitchin morphism, *Math. Proc. Cambridge Philos. Soc.* **172** (2022), 443–487.

[12] M. A. A. de Cataldo, T. Hausel, L. Migliorini, Topology of Hitchin systems and Hodge theory of character varieties: the case A_1, *Ann. of Math.* **175** (2012), 1329–1407.

[13] M. A. A. de Cataldo, J. Heinloth, L. Migliorini, A support theorem for the Hitchin fibration: the case of GL_n and K_C, *J. reine angew. Math.* **780** (2021), 41–77.

[14] M. A. A. de Cataldo, D. Maulik, The perverse filtration for the Hitchin fibration is locally constant, *Pure Appl. Math. Q.* **16** (2020), 1441–1464.

[15] M. Cavicchi, F. Déglise, J. Nagel, Motivic decompositions of families with Tate fibers: smooth and singular cases, *Int. Math. Res. Not.* **2023** (2023), 14239–14289.

[16] P.-H. Chaudouard, G. Laumon, Un théorème du support pour la fibration de Hitchin, *Ann. Inst. Fourier* **66** (2016), 711–727.

[17] N. Chriss, V. Ginzburg, *Representation theory and complex geometry*, 1997 ed., Modern Birkhäuser Classics, Birkhäuser, 2010.

[18] W.-Y. Chuang, D.-E. Diaconescu, G. Pan, BPS states and the $P = W$ conjecture, in *Moduli spaces*, London Math. Soc. Lecture Note Ser. **411**, Cambridge Univ. Press, 2014, 132–150.

[19] A. Corti, M. Hanamura, Motivic decomposition and intersection Chow groups. I, *Duke Math. J.* **103** (2000), 459–522.

[20] B. Davison, L. Hennecart, S. S. Mejia, BPS Lie algebras for totally negative 2-Calabi-Yau categories and nonabelian Hodge theory for stacks, preprint arXiv:2212.07668.

[21] P. Deligne, Décompositions dans la catégorie dérivée, in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., 1994, 115–128.

[22] C. Deninger, J. Murre, Motivic decomposition of abelian schemes and the Fourier transform, *J. reine angew. Math.* **422** (1991), 201–219.

[23] G. Dobrovolska, V. Ginzburg, R. Travkin, Moduli spaces, indecomposable objects and potentials over a finite field, preprint arXiv:1612.01733.

[24] D. Edidin, W. Graham, Equivariant intersection theory, *Invent. math.* **131** (1998), 595–634.

[25] D. Edidin, W. Graham, Riemann-Roch for equivariant Chow groups, *Duke Math. J.* **102** (2000), 567–594.

[26] W. Fulton, *Intersection theory*, Ergebn. Math. Grenzg. **2**, Springer, 1998.

[27] B. B. Gordon, M. Hanamura, J. P. Murre, Relative Chow-Künneth projectors for modular varieties, *J. reine angew. Math.* **558** (2003), 1–14.

[28] M. Groechenig, S. Shen, Complex K-theory of moduli spaces of Higgs bundles, *J. Eur. Math. Soc. (JEMS)* (2025).

[29] M. Groechenig, D. Wyss, P. Ziegler, Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration, *Invent. math.* **221** (2020), 505–596.

[30] T. Hausel, A. Mellit, A. Minets, O. Schiffmann, $P = W$ via $\mathcal{H}_2$, preprint arXiv:2209.05429.

[31] T. Hausel, F. Rodriguez-Villegas, Mixed Hodge polynomials of character varieties, *Invent. math.* **174** (2008), 555–624.

[32] T. Hausel, M. Thaddeus, Mirror symmetry, Langlands duality, and the Hitchin system, *Invent. math.* **153** (2003), 197–229.

[33] N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc.* **55** (1987), 59–126.

[34] N. Hitchin, Stable bundles and integrable systems, *Duke Math. J.* **54** (1987), 91–114.

[35] V. Hoskins, S. P. Lehalleur, Motivic mirror symmetry and χ-independence for Higgs bundles in arbitrary characteristic, preprint arXiv:2205.15393.

[36] T. Kinjo, N. Koseki, Cohomological χ-independence for Higgs bundles and Gopakumar-Vafa invariants, *J. Eur. Math. Soc. (JEMS)* **28** (2026), 619–671.

[37] K. Künnemann, A Lefschetz decomposition for Chow motives of abelian schemes, *Invent. math.* **113** (1993), 85–102.

[38] J. Le Potier, Faisceaux semi-stables de dimension 1 sur le plan projectif, *Rev. Roumaine Math. Pures Appl.* **38** (1993), 635–678.

[39] J. C. Lee, S. Lee, Generators for the cohomology of the moduli space of irregular parabolic Higgs bundles, preprint arXiv:2402.05380.

[40] D. I. Lieberman, Numerical and homological equivalence of algebraic cycles on Hodge manifolds, *Amer. J. Math.* **90** (1968), 366–374.

[41] F. Loeser, D. Wyss, Motivic integration on the Hitchin fibration, *Algebr. Geom.* **8** (2021), 196–230.

[42] M. Logares, J. Martens, Moduli of parabolic Higgs bundles and Atiyah algebroids, *J. reine angew. Math.* **649** (2010), 89–116.

[43] E. Markman, Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces, *J. reine angew. Math.* **544** (2002), 61–82.

[44] D. Maulik, J. Shen, Endoscopic decompositions and the Hausel-Thaddeus conjecture, *Forum Math. Pi* **9** (2021), Paper No. e8, 49.

[45] D. Maulik, J. Shen, Cohomological χ-independence for moduli of one-dimensional sheaves and moduli of Higgs bundles, *Geom. Topol.* **27** (2023), 1539–1586.

[46] D. Maulik, J. Shen, The $P = W$ conjecture for GL_n, *Ann. of Math.* **200** (2024), 529–556.

[47] D. Maulik, J. Shen, Q. Yin, Perverse filtrations and Fourier transforms, *Acta Math.* **234** (2025), 1–69.

[48] M. Mauri, L. Migliorini, Hodge-to-singular correspondence for reduced curves, *J. Eur. Math. Soc. (JEMS)* **28** (2026), 2151–2196.

[49] A. Mellit, Toric stratifications of character varieties, *Publ. Math. Inst. Hautes Études Sci.* **142** (2025), 153–240.

[50] M. Melo, A. Rapagnetta, F. Viviani, Fourier-Mukai and autoduality for compactified Jacobians. I, *J. reine angew. Math.* **755** (2019), 1–65.

[51] M. Melo, A. Rapagnetta, F. Viviani, Fourier-Mukai and autoduality for compactified Jacobians, II, *Geom. Topol.* **23** (2019), 2335–2395.

[52] S. Mukai, Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, *Nagoya Math. J.* **81** (1981), 153–175.

[53] J. P. Murre, On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples, *Indag. Math.* **4** (1993), 177–188.

[54] B. C. Ngô, Le lemme fondamental pour les algèbres de Lie, *Publ. Math. Inst. Hautes Études Sci.* **111** (2010), 1–169.

[55] J. Shen, Topology of abelian fibrations, in *Proceedings of the International Congress of Mathematicians*, 2026, to appear.

[56] Y. Toda, Gopakumar-Vafa invariants and wall-crossing, *J. Differential Geom.* **123** (2023), 141–193.

[57] Z. Yun, Global Springer theory, *Adv. Math.* **228** (2011), 266–328.

[58] Z. Yun, Lectures on Springer theories and orbital integrals, in *Geometry of moduli spaces and representation theory*, IAS/Park City Math. Ser. **24**, Amer. Math. Soc., 2017, 155–215.

(Manuscrit reçu le 22 décembre 2024 ;
accepté le 9 décembre 2025.)

Davesh Maulik
Massachusetts Institute of Technology
E-mail: maulik@mit.edu

Junliang Shen
Yale University
E-mail: junliang.shen@yale.edu

Qizheng Yin
Peking University
E-mail: qizheng@math.pku.edu.cn